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CLASSICAL AND QUANTUM SYMMETRIES IN OPTION PRICING; A THEORETICAL APPROACH TO RISK AND RANDOMNESS IN FINANCE

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Abstract

We provide answers to some questions raised by recent papers in the field of Econophysics *Haven* (2008), where ideas from quantum field theory try to fill the gap between empirical studies on option pricing and the economic forecasting based on diffusion processes. We indicate where these theoretical tools were already applied in Finance; we found Hopf algebraic structures, we wrote down a PDE characterizing certain local volatility models and a class of hypergeometric solutions and we have a simple test to check if a basket option is heat-solvable.

Keywords: ***Black-Scholes option model, local volatility, derivatives, Hopf algebra, differential geometry, Ito calculus, stochastic Taylor expansions, econophysics***

JEL Classification: C02, C15, C65, C63, C53, G12

I. Introduction

(possible applications) The stochastic behavior of certain asset prices (energy, market indexes) which influence the prices of-for example- plane tickets, suggests that the discounted airfares could be studied with the methodology and mathematical tools of European call options. These topics made me review the following basic problems:

- examples where there is a closed formula for the price of an European call option with fixed interest rate and
- examples of stochastic processes where Monte Carlo simulations were successfully applied.

We would like to review certain above mentioned mathematical keywords recently connected with financial mathematics.

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1. Malham and Wiese (2009) use the Hopf algebra structure of iterated Stratonovich integrals to prove that the error generated by a numerical method of approximation of the stochastic flow associated with sinh-log series is smaller than the stochastic Taylor error; they apply the method to simulate the Heston model for derivative pricing.

2.Hudson and Parthasarathy are the creators of the first quantum stochastic calculus. In their lines, a quantum Black-Scholes formula was obtained by Boukas and Accardi (2007). Hudson(2009) generalized Ito calculus for differentials introducing the concept of *Ito Hopf algebra* associated with an arbitrary associative algebra. Hudson analized quasitriangular structures associated with these Hopf algebras, as well as a concrete formula for its antipode. The nice formula of Hudson and Pulmannova gives a hint of a formula for the antipode of a very important Hopf algebra met in Schneps (2004, p.38).

3. The Hopf algebra structure of multiple stochastic integrals was used by Henry-Labordere (2009, p. 353), who studied abelian and 2-step nilpotent local volatility models for option pricing. He also use a differential geometry approach: a heat kernel approximation in his study of Heston and SABR stochastic volatility models.

4. A path integral approach to option pricing was studied by Linetzky (1998) and Taddei (1999). They associate to any stochastic differential equation a Lagrangian functional, as well as a *Van Vleck determinant* required to compute a path integral, which is computable in very few cases: Gaussian models and *models that can be reduced to Gaussians by changes of variables*, reparametrizations of time and projections (the Black–Scholes model, Ornstein–Uhlenbeck, Cox–Ingersoll–Ross model, Bessel process. Linetsky (1998, p.146) . For other cases there are well developed approximation techiques. We did not find cases where we don't know to compute the theoretical option prices using partial differential equation techniques but we know to compute them using path integral approaches.

we study some questions raised by one and multi-dimensional Black-Scholes local volatility models, concerning solvability and changes of variables using Hopf algebraic and differential geometry tools (traditionally connected with the concepts of quantum/background independent and Lie/classical symmetries).

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2. Classical and quantum symmetries. Local volatility models

2.1.0 We consider a risky asset driven by a Brownian motion; the dynamics of its prices is described by the following Ito process:

$dx_t = a(x_t,t)dt + b(x_t,t)dW$

W is a Brownian motion. The bond price is a deterministic process:

$dx^0 = rx^0 dt \Rightarrow x^0(t) = e^{r(t-T)}$; The Black-Scholes equation is the following partial differential equation in the unknown function f(x,t):

$$\frac{\partial f}{\partial t} + rx\frac{\partial f}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial x^2} = rf \qquad \textbf{(3)}$$

Under reasonable assuptions for the drift and volatility functions a(x,t) and b(x,t), there is a unique strong solution on [0,T] for the stochastic process and for f, with f(x,T) =h(x) a given function. Usually h(x) = max (x-K,0); K is the strike price. f(x,0) is the price of an European call option with maturity T and strike price K, for a given asset price x at t=0.

In general there is no closed formula for f(x,t), unless the above mentioned stochastic differential equation has a = constant and b(x,t)/x = constant.

When can we make *a change of variables* to transform the equation **(3)** to the heat equation $h_t + \frac{1}{2}h_{xx} = 0$ **(4)**? By a change of variables , we mean the existence of functions c(t) and H(x,t), such that for any f which is solution of equation **(3)** , c(t)f(H(x,t), t) will be a solution for the equation **4).**

We will discuss the same question if we have a basket or rainbow option (it depends on at least two assets) . We have in fact an equivalence relation on the set of PDE's defined by (a,b,r), if we require (x,t) → (H(x,t), t) to be a diffeomorfism.

2.1.1.If f satisfies (3) and $f(x,t) = e^{rt}g(e^{-rt}x,t), \Rightarrow \; f_t = rf + e^{rt}g_t - r_x g_x$,

$f_x = g_x \;\; f_x = e^{-rt}g_{xx} \;\; \Rightarrow g$ satisfies $g_t + \frac{1}{2}B^2 g_{xx} = 0$, where $\underline{B(x,t) = b(e^{rt}x,t)e^{-rt}}$

If $g(x,t) = u(H(x,t),t)$, where $u_t + \frac{1}{2}u_{xx} = 0 \Rightarrow$

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$$\left.\begin{array}{l} g_t = u_t + u_x H_t \\ g_x = u_x H_x \\ g_{xx} = u_{xx}\left(H_x\right)^2 + u_x H_{xx} \end{array}\right\} \Rightarrow g_t + \frac{1}{2}B^2 g_{xx} = u_t\left[1 - B^2 H_x^2\right] + u_x\left[H_t + \frac{1}{2}B^2 H_{xx}\right] = 0$$

For the particular case $u(y,t) = y \Rightarrow H$ is a solution of $H_t + \frac{1}{2}B^2 H_{xx} = 0$

$$\Rightarrow 1 - B^2 H_x^2 = 0 \Rightarrow H_x = \pm\frac{1}{B} \qquad H_t = -\frac{1}{2}\frac{H_{xx}}{H_x^2} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{H_x}\right) = \pm\frac{1}{2}B_x$$

$$\frac{\partial}{\partial t}H_x = \frac{\partial}{\partial x}H_t \Leftrightarrow \pm\frac{\partial}{\partial t}\left(\frac{1}{B}\right) = \pm\frac{1}{2}\frac{\partial}{\partial x}B_x \Leftrightarrow B_t + \frac{1}{2}B^2 B_{xx} = 0, \quad (5)$$

....

Also up to a sign $H = \int_a^x \frac{1}{B}(y,t)dy + C$. (8) The general solution for the heat equation solution u is, $u(x,t) = V\left(x/\sqrt{2}, T-t\right)$

where $V(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbf{R}} \mathbf{e}^{\frac{-|x-y|^2}{4t}} f(y)dy$ Jost (2002, p.87, Section 4.2 multidim heat eq.)

If B = 1/f, where $B_t + \frac{1}{2}B^2 B_{xx} = 0$, then $f_t + \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{1}{f^2}f_x\right] = 0$ **(5)**

This is the necessary and sufficient condition, **(5)** which has to be satisfied by the local volatility b, or equivalently B,or f, for the existence of a function H which transforms any solution of the heat equation in a solution of **(3).** c(t) is an exponential function. In this case, the volatility b itself will be *an option price* for its own Black-Scholes equation.

2.1.2 The Lie symmetries of this *nonlinear heat equation* were studied in Vaneeva (2008). In Nadjafikhah (2010) appears as a particular case of a general type of Burgers equation. Both papers study the infinitesimal generators of 1-dim symmetry groups. The following solution was extracted

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from Polyanin (2004, pag 43), having in this way a class of parameter-dependent solvable stochastic models to be calibrated against real data:

$$w(x,t)=\left[\frac{-2(x\ +\ A)^2}{\varphi(t)}+B\left|x+A\right|^2\left|\varphi(t)\right|\right]^{-\frac{1}{2}},\ \varphi(t)=C-2a(2+2)t,$$

2.2. We will show that the theory of **Hermite polynomials in two variables** provide solutions of the equation **5)** above ***„for free"***.

2.2.1 Theorem 2.2.1. Let $y_t=H_n(B_t,t)$ be the stochastic process defined by the n[th] Hermite polynomial in two variables. Then y_t satisfies a driftless Ito stochastic differential equation whose local volatility function satisfies the equation above.

There are several ways to define Hermite polynomials in one and two variables:

$H_0=1 \quad H_1=x \quad H_2=x^2-t \quad H_3(x,t)=x^3-3xt$

$H_4(x,t)=x^4-6x^2t+3t^2$

$$\tilde{H}_n(x)=(-1)^n e^{\frac{x^2}{2}}\frac{d^n}{dx^n}e^{-\frac{x^2}{2}} \qquad \sum_{n=0}^{\infty}H_n(x,t)\frac{y^n}{n!}=e^{xy-\frac{y^2t}{2}} \qquad H_n(x,t)=t^{\frac{n}{2}}\tilde{H}_n\left(\frac{x}{\sqrt{t}}\right),$$

where $\tilde{H}_n$ are above defined Hermite polinomials of one variable.

They are monic polinomials which satisfy $\int\tilde{H}_m(x)\tilde{H}_n(x)e^{\frac{x^2}{2}}=\sqrt{2\pi}n!\gamma_{mn}$ and

eq. 9) $H_t+\frac{1}{2}H_{xx}=0$ Widder (1975) p. 9.

In stochastic Ito integration theory, they satisfy the fundamental relation:

$$\int_0^t H_n(B_S,S)dB_S=\frac{H_{n+1}(B_t,t)}{n+1}$$

2.2.2. **Quasi-homogenity. Partial differential equations**.

We have saw that if $B_t+\frac{1}{2}B^2B_{xx}=0\Rightarrow$ there is *W*, solution for the heat equation $W_t+\frac{1}{2}W_{xx}=0$ such that $B(x,t)=W\left(\int_x\frac{1}{B},t\right)$. In the special case of the

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Hermite polynomials, $B=\frac{1}{f_n{}'}$ where $H_n(f_n,t)=x\Rightarrow B=W(f_n,t)$.For $x=H_n(f_n,t)\Rightarrow B(H_n(x,t),t)=W(x,t)$.

So B is determined by a solution W of the heat equation. ***(but not on any solution)***

$B\left(t^{n/2}h_n(x),t\right)=W\left(x\sqrt{t},t\right)$, where h is a one variable Hermite polynomial.

It is easy to see from the definition of H_n that $B(x,t)=t^{\alpha}q\left(xt^{\beta}\right)$, which forces $W(x,t)=\sqrt{t}a\left(\frac{x}{\sqrt{t}}\right)$. An easy computation show $a''-2xa'+a=0$ in order for w to satisfy the heat equation. For $a=e^f$

$\Rightarrow e^f\left(f''+(f')^2\right)-2xe^f f'+e^f=0\Rightarrow f''+(f')^2-2xf'+1=0\Leftrightarrow f''+(f'-x)^2=x^2-1$.

If $g=f'-x\Rightarrow g'=f''-1\Rightarrow g'+1+g^2=x^2-1\Rightarrow$ $g'+g^2=x^2-2$ **(2)**

If $g=h'\Rightarrow h''+(h')^2=x^2-2\Rightarrow\left(e^h\right)''=\left(x^2-2\right)e^h\Rightarrow$ $A''=\left(x^2-2\right)A$ **(1)**

These are the ordinary differential equations connected with *quasihomogenous* solutions of **eq. 5** The general solution of **eq. 1** is $y(x)=c_1D_{1/2}\left(\sqrt{2}x\right)+c_2D_{-3/2}\left(i\sqrt{2}x\right)$, where D_V is the parabolic cylinder function. $D_V(z)=2^{\frac{V}{2}}e^{-\frac{z^2}{4}}U\left(-\frac{1}{2}V,\frac{1}{2},\frac{1}{2}z^2\right)$, U is the confluent hypergeometric function of the first kind. The equation 2 is a Riccati equation, whose general solution is also written using D_V. $U(a,b,z)=\frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)}\int_0^1 e^t t^{a-1}(1-t)^{b-a-1}dt$, where $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$.

3. In a totally different context, the equation 5) , and one of its solution which generate the hyperbolic model in finance is mentioned in the book of Henry-Labordere (p.367 eq. B.7).

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The hyperbolic model is given by $\sigma(t,f) = \sqrt{a(t)f^2 + b(t)f + c(t)}$, , $a(t) = a_0$, $b(t) = b_0$, $c(t) = c_0 e^{-a_0 t} + K\left(e^{-a_0 t}\right)$

He proves that if the forward satisfies Ito SDE $y_t = \sigma(y_t, t)$ dW and σ satisfies the equation 5), then y_t can be exactly simulated using Monte Carlo methods, because y_t is a functional of Brownian motion, according to Yamato theorem. We shortly review theoretical elements of his approach:

Let X_t n-dimensional solution of the Stratonovich SDE $dX_t = V_0 dt + \sum_{i=1}^{m} V_i \circ dW_t^i$

(**12)**

Notation: $dt = dW_t^0$. For any differentiable function f, we have the Stratonovich Taylor expansion

$$f(X_T) = \sum_{i_1,\ldots,i_k \le r} V_{i_1} \circ V_{i_2} \circ \ldots \circ V_{i_k} f(X_0) \underbrace{\int_{0 \le t_1 \le t_2 \ldots \le t_k \le T} dW_{t_1}^{i_1} \circ \ldots \circ dW_{t_k}^{i_k}}_{\text{Iterated Stratonovich Integrals}} + R_r(T, X_0, f)$$

(2) A Hopf algebra is a vector space *H* together with operations m, Δ, *E*, η and S such that (H,m) is an associative algebra $a(bc) = (ab)c$. (H,Δ) is a coalgebra $\Delta : H \to H \otimes H$, $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$, ε, η unit and counit such that $\Delta(xy) = \Delta(x)(y)$ (compatibility between algebra and coalgebra structures.

S is called the antipode of H. $S : H \to H$ such that $\sum S(a_1) a_2 = \sum a_1 S(a_2) = \varepsilon(a) 1_H$, where $\Delta(a) = \sum a_1 \otimes a_2$. The vector fields $V_0, V_1, \ldots, V_m$ form a Hopf algebra; multiplication is given by concatenation of vector based "words" $V_{i_1}, V_{i_2}, \ldots, V_{i_k}$, $\Delta(V) = V \otimes 1 + 1 \otimes V$

$S\left(V_{i_1}, V_{i_2}, \ldots, V_{i_k}\right) = (-1)^k V_{i_k}, V_{i_{k-1}}, \ldots, V_{i_1}$.

Definition. $L \in H$ is called primitive if $\Delta(L) = L \otimes 1 + 1 \otimes L$. $g \in H$ is called grouplike if $\Delta(g) = g \otimes g$ .The exponential map $\exp(x) = \sum_{k=0}^{\infty} \frac{1}{K!} X^k$ make sense if the series is finite or we work in a certain completion of *H*. If *L* is primitive, then $\exp(L)$ is grouplike.

Theorem (Chen)

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$X_{0,1}(W) = \sum_{i_1,\dots,i_k} V_{i_1}, V_{i_2}, \dots, V_{i_k} \int_{0\le t_1 \le t_2 \dots \le t_k \equiv 1} dW_{t_1}^{i_1} \circ \dots \circ dW_{t_k}^{i_k}$ is grouplike $X_{0,1}(W) = \exp(L)$ for L a primitive series.

Theorem (Yamoto). The solution X_t of the equation (12) is represented as

$X_t = \exp(L_t) X_0$, where $L_t = tV_0 + \sum_{i=1}^{m} W_t^i V_i + \sum_{r=2}^{\infty} \sum_{i_1,\dots,i_k \le r} c_j W_t^j V^j$,

W_t^j are iterated Stratonovich integrals

$V^j = \left[\left[\left[V_{j_1}, V_{j_2}\right] \dots V_{j_{n+1}}\right]\right]$ iterated Lie brackets of vector fields.

Corollary.(B1, pag. 366 Lobordere 2009) $df_t = \sigma(t, f_t) dW_t$. If σ satisfies equation 5), then V_0 and V_1 commute, they form a abelian Lie algebra and the SDE can be simulated exactly using Monte-Carlo methods.

3.1 Hopf algebra structures in the theory of the classical and quantum stochastic flow.

We review the appearance of Hopf algebras in the Hudson's approach on quantum stochastic calculus, an Ito differential calculus for quantum noises. (Belavkin 2008, pag 34)
Let A be an associative algebra over the complex numbers.

Over the tensor algebra $T(A) = \mathbf{C} \oplus \bigoplus_{k=1}^{\infty} A^{\otimes K}$ there are two structures of Hopf algebras, with the same comultiplication, but different multiplications

$\Delta(L_1 \otimes L_2 \otimes \dots \otimes L_n) = \sum_{j=0}^{n} (L_1 \otimes L_2 \otimes \dots \otimes L_j) \otimes (L_{j+1} \otimes \dots \otimes L_n)$

The shuffle product $\#$ between two homogenous elements is
$(a_1 \otimes a_2 \otimes \dots \otimes a_n) \# (b_1 \otimes b_2 \otimes \dots \otimes b_m) = \sum (c_1 \otimes c_2 \otimes \dots \otimes c_{m+n})$, where the sum is over all $\binom{m+n}{n}$ ways to "shuffle" $a's$ among $b's$: the elements $a_1, a_2, \dots, a_n$ will appear in the some order in $c's$. The some for $b\,'s$. Example:

$(a \otimes b) \# (x \otimes y) = x \otimes a \otimes b \otimes y + a \otimes x \otimes b \otimes y + a \otimes b \otimes x \otimes y +$
$+ x \otimes y \otimes a \otimes b + x \otimes a \otimes y \otimes b + a \otimes x \otimes y \otimes b$

$\#$ is called shuffle product; together with the comultiplication Δ, $T(A)$ is a Hopf algebra. The Ito product (called sometimes the "*Stuffle"* product) uses the algebra structure of A:

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$$\left(a_1 \otimes a_2 \otimes \ldots \otimes a_n\right) \# \left(b_1 \otimes b_2 \otimes \ldots \otimes b_m\right) = \sum_{j=0}^{\min(m,n)} \sum \left(c_1 \otimes c_2 \otimes \ldots \otimes c_{m+n-j}\right)$$

In the second sum, c_i is equal to on a_α, b_β or a product $a_i b_j$.

As in the $\#$ product, the order, the appearance of *a'* and *b'* respects the initial order of $a_1 \otimes a_2 \otimes \ldots \otimes a_n$ and $b_1 \otimes b_2 \otimes \ldots \otimes b_m$

The Shuffle product is in fact an Ito product for the trivial algebra $xy = 0, \quad \forall x$ and y. The antipode $S_{\#}\left(a_1 \otimes a_2 \otimes \ldots \otimes a_n\right) = (-1)^m a_n \otimes a_{n-1} \otimes \ldots \otimes a_1 +$ lower order terms It was described in Hudson (2009, section 4)

The algebra of Ito differentials of on n-dimensional quantum stochastic calculus is $A = \mathbf{C} < dA_\alpha^\beta, \alpha, \beta = 0,1,\ldots,n >$, $\quad dA_\alpha^\beta dA_x^\gamma = \rho_{\alpha\gamma} dA_x^\beta \quad \rho$ is Kronecker symbol.

In this case the Ito Hopf $T(A)$ acts weekley on the Fock space $L^2\left(\mathbf{R}_+\right) \otimes \mathbf{C}^n$.

There are also well defined iterated stochastic integrals $J_a^b\left(dA_{\alpha_1}^{\beta_1} \otimes \ldots \otimes dA_{\alpha_n}^{\beta_n}\right)$

$= \int_{a<s_1<s_2<\ldots<s_n<b} dA_{\alpha_1}^{\beta_1}\left(s_1\right) \otimes dA_{\alpha_2}^{\beta_2}\left(s_2\right) \otimes \ldots \otimes dA_{\alpha_n}^{\beta_n}\left(s_n\right)$ seen as operators in the Fock space. In this way, Fubini and integration by parts theorems in Ito calculus are retrievied using the concept of Ito Hopf algebra and its (weekly) representation.

Examples:Theorem 5.1. (Belavkin 2008, pag 42)

$< J_a^b(x)e(f), J_a^b(y)e(g) >=< e(f), J_a^b(xy)e(g) >$

Theorem 5.2. (Belavkin 2008, pag 43): $\quad F_a^c(b) J_a^c = \left(J_a^b \otimes J_b^c\right)\Delta$, where $F_a^c(b)$ is a splitting isomorphism for $a < b < c \in \mathbf{R}_+$. (Guta 2008, pag 42-43) Hudson obtained in this way a Fock space representation of Brownian motion and Poisson processes. A Fock space is the Hilbert space completion of the space of exponential vectors $\Psi(j)$, $f \in L^2\left(\mathbf{R}^+, \mathbf{C}\right)$ unde the inner product. $\langle f, g\rangle = \int_0^\infty \bar{f}(s) g(s) ds$.

We prove the following:

Theorem 3.2. There is a Hopf algebra isomorphism between the shuffle Hopf algebra $T_{\#}(A)$ and the Ito Hopf algebra $T(A)$, where *A* is the algebra of Ito differentials, for the special algebra A associated with the classical Ito calculus: $\quad A = \mathrm{R}\left\langle a_0, a_1, \ldots, a_n\right\rangle, \quad a_i a_j = 0$, for $i \neq j$, $a_i^2 = a_0$ for $i \neq 0$.

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Proof: The result is algebraic; but the proof is based on the interplay between Ito and Stratonovich iterated integrals. We define $f: T(A) \to T_{\#}(A)$ $f\left(a_{i_1} \otimes a_{i_2} \otimes \ldots \otimes a_{i_n}\right) = \sum \prod a_{i_1} \otimes a_{i_2} \otimes \ldots \otimes a_{i_k}$ is such a way that the Ito iterated integral with respect to the *n*-dim Wiener process $I = \int_0^t dW_{i_1} dW_{i_2} \ldots dW_{i_n}$ is a sum of products of Stratonovich integrals; to prove that *f* is bijective: using Corollary 5 and Proposition 6 from (Gaines 1995); which says that the Ito Hopf algebras $T(A)$ and $T_{\#}(A)$ have the same basis formed by Lyndon words;

Platen-Kloeden (1992, pag 174) $J_w = \sum_{u \in D(w)} \frac{1}{2^{n(u)}} I_n$ $D(w) =$ the set of word obtained from *W* by replacing two adjacent equal indices of W by 0.

$I_W(t) = \int_0^t \int_0^{t_1} \ldots \int_0^{t_{n-2}} dW_{t_n}^{a_k} dW_{t_{n-2}}^{a_2} \ldots dW_{t1}^{a_n}$ are Ito iterated integrals.

$J_W^{Ito} = \sum_{u \in D(W)} \frac{(-1)^{(u)}}{2^{n(u)}} J_u^{Strat}$.

references very helpful in describing the relations between multiple Ito and Stratonovich integrals, as well as Ito and Stratonovich stochastic Taylor expansions:Kim, Jong (2006), Kloeden, Platen (1991), Gaines (1994), Ben Arous (1989), Kallianpur (1997), Takanobu (1995), Hu (1988)

The Hopf algebras met in Mathematical Finance (that of Malham (2009, Sect 2b,c), Hudson (2005, Section 4) and Foissy (2010) are graded and connected bialgebras, so they have automatically an antipode according to Milnor-Moore Theorem. There is also a specific formula for the antipode of these Hopf algebras (called Takeuchi or Milnor-Moore formula), described in Mahajan & Aguiar (2010, pag 248, Prop. 8.13 and 8.14, and pag 36, formula 2.55 and remark 2.10) .

In Econophysics (quantum field theory) and the applications above, the antipode is used in the way iterated stochastic integrals are changing if we change the orientation of the simplex.

4.0. Multi-dimesional diffusion processes. Reduction to the heat equation.

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A similar question is : given an n-dimensional stochastic differential equation which describes the evolution of a basket of n asset prices, when does the option pricing Black-Scholes equation can be transformed, in a specific sense which is described below, to the heat equation in R^n **?** $\{W_i\}$ are n independent standard Brownian motions. The risk neutral dynamics of n stocks is given by: $dX_t^{\mu} = rX_t^{\mu}dt + \sum_i \sigma_i^{\mu}(X(t),t)dW_i$. ***r*** is a constant interest rate.

Define $\quad G^{\alpha\beta}(x,t) = \sum_i \sigma_i^{\alpha}\sigma_i^{\beta}$

Following Henry-Labordere (2009) (The reference contains the sufficient conditions for the existence and uniqueness of the strong solutions for a stochastic differential equation SDE (e.g.Kunita-Watanabe conditions, as well as the equivalence between 2 Ito forms of SDE using correlated or uncorrelated Brownian motions, via Choleski decomposition of the matrix $G^{\alpha\beta}(x,t)$), *the multidimensional Black-Scholes option pricing equation is :*

$\partial_t f(x,t) + \frac{1}{2}G^{\alpha\beta}f_{\alpha\beta} = r\left(f - x_i\partial_i f\right)$ (1) Then

$h\left(x_1, x_2, \ldots, x_n, t\right) = e^{-rt} f\left(x_1e^{rt}, x_2e^{rt}, \ldots, x_ne^{rt}, t\right)$ satisfies $\quad h_t + \frac{1}{2}g^{\alpha\beta}h_{\alpha\beta} = 0$ **(2)**

where $g^{\alpha\beta}(x,t) = G^{\alpha\beta}\left(xe^{rt}, t\right)e^{-2rt}$.

($h_{\alpha\beta}$ partial derivatives of h with respect to 2 variables)

Definition: We say that the eq. **2** is equivalent to the heat equation if there are *n* functions $H_1(x,t), \ldots, H_n(x,t)$, t time and x in R^n such that for any *W*, solution of $W_t + \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 W}{\partial x_i^2} = 0$ **(6),** $W\left(H_1(x,t), \ldots, H_n(x,t), t\right) = h(x,t)$ is a solution of **(2).** And for any *h* solution of (2), there is a *W* as above.

4.1 Econophysics. There is a path integral approach in computing certain financial and quantum mechanics observables. A path integral is a limit of a sequence of finite dimensional integrals. Also, Feynman –Kac formula connects the solution of the B.S. equation with probability theory . The transition probability function, or the Green kernel is computed as an integral over the *space of all paths*, where we assign a probability to each path.

$$f(t,x) = E^{t,x}\left[e^{-r(T-t)}h(X(T))\right] = \int dX(T)e^{-r(T-t)}h(X(T))p(X(T),T \mid x(T),t)$$

$$O(S(t),t) = E^{t,S(t)}\left[e^{-r\tau}F\ \left(e^{X(T)}\right)\right], \ \tau = T - t$$

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Taddei and Linetzki proposed a Lagrangian functional which can be used to approximate the Green kernel. *Instantons* are solutions of the Euler-Lagrange equation , $\left[\frac{d}{dt}\left(\frac{\partial}{\partial x(t)}\right)-\frac{\partial}{\partial x(t)}\right]L(\dot{x}(t),x(t),t)=0$ which are paths with extreme probabilities and which can be used to approximate p(a,b| x,y). Geodesics of a Riemannian manifold are also solutions of the the Euler-Lagrange equation associated with the energy functional, and this could be the first interplay with differential geometry. Batard and Henry-Labordere wrote down a covariant formulation of the Black-Scholes equation, involving a generalized Laplacian associated with a connection in a vector bundle. Roughly speaking, a *covariant* formulation implies to write equations and formulas using quantities which behave like covariant or contravariant tensors, as in a change of variables in a multiple integral.

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We go back to the equation 2 from this chapter:

$(g_{\alpha\beta})=(g^{ij})^{-1}$ is the inverse of the matrix given by $g^{\alpha\beta}(x,t)$. Taddei(1999) define a general Lagrangian, which for a driftless SDE is equal to

$L[x,\dot{x},t]=\frac{1}{2}g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}+\frac{1}{12}R$, where R is defined in the following way:

$R=\sum_{i,j,k}g^{ij}R^{k}_{ikj}$, $R^{a}_{bcd}=\frac{\partial P^{a}_{bc}}{\partial X_d}-\frac{\partial P^{a}_{ad}}{\partial X_c}+\sum_k P^{k}_{bc}P^{a}_{dk}-P^{k}_{bd}P^{a}_{ck}$, where P are defined below:

$P^{a}_{bc}=\frac{1}{2}\sum_k g^{ka}\left(\frac{\partial g_{ck}}{\partial X_b}+\frac{\partial g_{bk}}{\partial X_c}-\frac{\partial g_{bc}}{\partial X_k}\right)$.

If $R\equiv 0$ in a coordinate system (that of the heat equation) $\Rightarrow R\equiv 0$ in any coordinates $\Rightarrow$ we have an easy test to check if (2) could be equivalent to (6). For $n=2$, the condition R=0 is not only necessary, but also sufficient.

For $n\geq 3$ ***Eqs.*** $(2)\Leftrightarrow(6)$ if and only if all $R^{a}_{bcd}=0$. These conclusions are the consequences of Rosenberg (1997, Theorem 2.10 pag 57, Ex.5 pag 60) ***and*** the following proposition:

Proposition 4.1. If (2) $\Leftrightarrow$ the heat equation in $\mathbf{R}^n$, then H_i *and* (g^{ij}) do not depend on time.

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If the coefficients g's depend on time, a covariant formulation of the Black-Scholes equation is written as a *generalized Laplacian* which acts on the sections Sect(E) of a vector bundle over a Riemann manifold. Any generalized Laplacian is equal to a connection Laplacian plus a linear endomorfism of Sect(E), according to a theorem of Berline, Getzler, Vergne . *Batard (2011, Sections 3.1 ; 2.1).* The same problem, to simplify the form of the Kolmogorov (or Fokker-Planck) equations is sligthly more complicated; it implies that a sequence of scalar functions built using a metric of a differentiable manifold are invariant to coordinate-change; and we don't have a simple answer if the necessary conditions obtained using heat kernel expansion are also sufficient (Vassilievich 2003, pag.31).

5. Conclusions; possible applications and further directions.

- the Hopf algebra structure of Ito differentials is isomorphic with a shuffle algebra. These shuffle algebras play an important role in the theory of Rota-Baxter algebras which contain a lot of algebraic and combinatorial identities. (Spitzer identity)

The equation **(5)** which was written down in Section 2.1.1., related to 1-dim Ito processes for which we have a formula for the option pricing function, appears in different contexts: non-linear heat equations, solvable stochastic models from the point of view of Monte-Carlo simulation methods and computable path integrals. The solutions of this equation generated by *(the x-inverse of)* Hermite polynomials are quasi-homogenous and their differential equations are solvable using hypergeometric functions. Hermite polynomials appear in the theory of *Wiener chaos (or* polynomial chaos decomposition). Also, there are variants of multi-variables Hermite polynomials. We mention relevant references connected with orthogonal polynomials for Levy processes and Wiener chaos: Lawi (2008), Sole (2008), Peccati (2003), Wu (2010), Schoutens(1998), Debusschere (2004).

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We can begin with a given „basic" market model described by a stochastic differential equation, for which we have a formula for the Green kernel or Black-Scholes option pricing equation, and try to find, to write down, the partial differential equations satisfied by the local volatility function obtained using a change of variables, as above, where we start with the heat equation. Other

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solvable models to start with , which are classified according to their Lie symmetries, can be found in Carr(2006).

The applications of this approach goes back to the results of Dupire and Gatheral on local volatility surface and *implied volatility surface*. The local volatility fuction is equal to a functional of the *Black-Scholes implied volatility surface* and its one and second order derivatives Gatheral (2002, Section 2.3) Lee (2001, Section 2.2.1). So, a partial differential equation (pde) satisfied by the local volatility will generate a pde satisfied by the implied volatility surface, and we have a class of implied volatility surfaces, determined by pds's and boundary conditions, to be compared with real data. Dupire (1994), Armeanu (2009), Dumas (1998), Turinici (2009). Page 6

We have simple tests given by Prop. 4.1 and Riemann tensor $\{R^a_{bcd}\}$ to check if the Black-Scholes solutions are coordinate-changed solutions of the heat equations, for which we have a well –developed theory.

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