Dynamics for noninvertible chaotic economic models

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In this paper we study the dynamics and ergodic theory of certain economic models which are **implicitly defined**. We consider:

- 1-dimensional and 2-dimensional overlapping generations models;
- a cash-in-advance model;
- and a cobweb model with adaptive adjustment.

We consider the **inverse limit spaces** of the associated chaotic invariant fractal sets and their metric, ergodic and stability properties.
The inverse limits give the set of intertemporal perfect foresight equilibria for the economic problem considered.
The common feature of all these models is that they are given by non-invertible dynamical systems and present **chaotic behavior.**

In some of these models, we have *hyperbolic horseshoes* (as in the cobweb model, see Onozaki, Zhang), in others *transversal homoclinic/heteroclinic orbits from saddle points* (see the heterogeneous market model, as in Foroni), or yet in others there exist *snap-back repellers* as in the 1-dimensional and 2-dimensional overlapping generations models for certain offer curves (see Gardini, Tramontana).
Also in the case of unimodal maps modelling some overlapping generations scenarios, we have chaotic behavior on *repelling invariant Cantor sets* (as for the logistic map F_ν with $\nu > 4$, see Medio, Robinson).

For such noninvertible dynamical systems, the inverse limits are very important since they provide a natural framework in which the system "unfolds" and they give sequences of intertemporal equilibria.

Definition

Given a continuous map $f : X \to X$ on a metric space (X, d), we form the **inverse limit** $(\hat{X}, \hat{f})$, where $\hat{X} := \{\hat{x} = (x, x_{-1}, x_{-2}, \dots), f(x_{-i}) = x_{-i+1}, i \geq 1\}$ and $\hat{f} : \hat{X} \to \hat{X}, f(x, x_{-1}, \dots) = (f(x), x, x_{-1}, \dots), \hat{x} \in \hat{X}$. We consider the topology induced on $\hat{X}$ from the infinite product of X with itself.

In fact $\hat{X}$ is a metric space with the metric

$$d(\hat{x}, \hat{y}) = \sum_{i \geq 0} \frac{d(x_{-i}, y_{-i})}{2^i}, \hat{x}, \hat{y} \in \hat{X}$$

Another important feature for economic dynamical systems is that of **stability**. We are interested if a certain model is *stable* on invariant sets at small fluctuations. In our case, since we work with infinite sequences of intertemporal equilibria, one would like to have stability of the shifts on the inverse limit spaces.

The standard method of studying evolution of a system in economics is to use dynamical systems which transfer exogeneous "shocks" to the system. However a system which presents chaotic behavior, has also complicated *endogeneous* fluctuations.
Even if the system is defined deterministically, still it may be impossible to describe quantitatively precisely its evolution, due to sensitivity to initial conditions.

We will use the notion of **chaotic map** several times. We say that f is **chaotic** on an invariant set X if f is topologically transitive on X and f has sensitive dependence on initial conditions.

We study then utility functions on inverse limits for noninvertible economic systems. Invariant measures for a dynamical system are important since they preserve the ergodic and dynamical properties of the system in time; in fact from any measure one can form canonically an invariant measure by a well-known procedure.

For a central government it is important to know or to estimate the average value of a certain utility function for a non-invertible economic model.

We rank utility functions of systems given by certain unimodal maps according to their average values with respect to invariant borelian measures $\hat{\mu}$ on the inverse limits, especially with respect to measures of maximal entropy. For certain expanding systems, namely for logistic maps $F_\nu, \nu > 4$ we are able to compare the **average utility values** with respect to the corresponding measures of maximal entropy when perturbing both the discount factor β of the utility W, as well as the system parameter $\nu > 4$.

Let us remind several examples of economic dynamical systems, which are non-invertible:

1. The 1-dimensional overlapping generations model.
This model was proposed initially by Grandmont. In this model we have an economy with constant population divided into young and old agents, and with a household sector and a production sector. A typical agent lives for the 2 periods, works when young and consumes when old and receives a salary for his work in the first period. There is a perishable consumption good and one unit of it is produced with one unit of labour.

If money is supplied in a fixed amount, say M, then we have at time t, that $w_t \ell_t = M$, where w_t is the wage rate and ℓ_t is the labour. At the same time, $M = p_{t+1} c_{t+1}$ where p_{t+1} is the expected price of the consumption good at time $t+1$ and c_{t+1} is the amount of future consumption. Now agents have an utility function of type $U = V_1(\ell_* - \ell_t) + V_2(c_{t+1})$ where ℓ_* is the fixed labour endowment of the young and $\ell_* - \ell_t$ is the leisure at time t.

Agents would like to have both as much leisure currently as well as consumption when old. Thus under the budget constraint from above $M = w_y \ell_t = p_{t+1} c_{t+1}$ the optimization problem above gives, by the method of Lagrange multipliers, an implicit difference equation: $\ell_t = \chi(c_{t+1})$, where $\chi(\cdot)$ is the *offer curve*.

Since by assumption one unit of labour produces one unit of consumption good, we have $\ell_t = c_t$, hence by denoting ℓ_t by y_t, we obtain

$$y_t = \chi(y_{t+1}) \tag{1}$$

As Grandmont showed, in many cases the offer curve is not given by a monotonic/injective function, making (1) a non-invertible difference equation. Thus for a level of consumption at time t there may be several levels of optimal consumption at time $t+1$. In this case we study the *backward dynamics* of the system, i.e the sequences of future consumption levels allowed by (1).

The backward dynamics given by relation (1) is chaotic in certain cases. For instance a condition was given by Mitra in order to guarantee the existence of a *snap-back repeller*. Let us first recall the definition of snap-back repeller and that of one-sided shift:

Definition

Let a smooth function $f : U \to U$, where U is an open set in $\mathbb{R}^n, n \geq 1$. Suppose that p is a fixed repelling point of f, i.e all the eigenvalues of $Df(p)$ are larger than 1 in absolute value, and assume that there exists another point $x_0 \neq p$ in a repelling neighbourhood of p, so that $f^m(x_0) = p$ and $\det Df(f^i(x_0)) \neq 0, 1 \leq i \leq m$. Then p is called a *snap-back repeller* of f.

Definition

We will denote by Σ_m^+ (where $m \geq 2$) the space of 1-sided infinite sequences formed with m symbols, i.e $\Sigma_m^+ = \{(i_0, i_1, i_2, \ldots), i_j \in \{1, \ldots, m\}, j \geq 0\}$. We have the *shift map* on Σ_m^+, namely $\sigma_m : \Sigma_m^+ \to \Sigma_m^+, \ \sigma_m(i_0, i_1, \ldots) = (i_1, i_2, \ldots)$. The space Σ_m^+ is compact with the product topology.

Snap-back repellers appear only for non-invertible maps, and are important since they are similar to transverse homoclinic orbits.

Theorem (Marotto)

Let p a snap-back repeller for a smooth non-invertible map f and $\mathcal{O}(x_0)$ a homoclinic orbit of x_0 towards the repelling fixed point p, i.e $\mathcal{O}(x_0) = \{\dots, x_{-i}, \dots, x_0, f(x_0), \dots, p\}$, with $f(x_{-i}) = x_{-i+1}, i \geq 1$. Then in any neighbourhood of the orbit $\mathcal{O}(x_0)$ there exists a Cantor set Λ on which some iterate of f is topologically conjugated to the shift on the space Σ_2^+ of one-sided infinite sequences on 2 symbols. Hence f itself is chaotic on Λ.

For many economic models, the offer curve $\chi(\cdot)$ is given by a smooth (or piecewise smooth) *unimodal map*.

A continuous map $f : [a, b] \to [a, b]$ is called **unimodal** if f is not monotone and there exists a point $c \in (a, b)$ so that $f(c) \in [a, b]$ and f is increasing on $[a, c)$ and decreasing on $(c, b]$.

Type A unimodal maps are unimodal maps satisfying $f(a) = a$ and $f(c) < b$. Type B unimodal maps are those satisfying $f(a) > a$ and $f(b) = a$. Type C maps are of the form $f : [a, b] \to \mathbb{R}$ s. t f is not monotone, $f(a) = f(b) = a$ and $f(c) > b$. Type C maps are not strictly speaking unimodal as the map f does not take necessarily values inside the same interval $[a, b]$, but in general they are considered "unimodal" too.

In certain cases when the offer curve χ is unimodal, one can find snap-back repellers:

Theorem (Hommes, Tramontana)

Let $\chi : I \to I$ be a unimodal smooth function on the unit interval, with a maximum point at x_m and a fixed point at x^*. If $\chi^3(x_m) < x^*$, then x^* is a snap-back repeller and thus there exists an invariant Cantor set $\Lambda \subset I$ on which an iterate of χ is topologically conjugate to the shift; so χ is chaotic and has positive topological entropy.

2. The 2-dimensional overlapping generations model.

As before we have an economy with two sectors, a household and a production sector. The household sector is the same, hence with perfect foresight we have for the offer curve $\chi(\cdot)$: $\ell_t = \chi(c_{t+1})$. By comparison, output is now produced both from labour ℓ_t supplied at time t by the household sector, and by *capital stock* k_{t-1} from the previous period $t-1$, supplied by non-consuming companies which tend to maximize profits. The output y_t is the minimum between ℓ_t and k_{t-1}/a, where $1/a$ is the productivityy of the capital. We assume that the capital stock available at the begining of period $t+1$ is $k_t = (1-\delta)k_{t-1} + i_t$, where $0 < \delta < 1$ is the depreciation rate of the capital and i_t is the investment, i.e the portion of the output at time t which is invested in the next period. Thus the consumption at time t is $c_t = y_t - i_t$, and at equilibrium we have $y_t = \ell_t = \frac{k_{t-1}}{a}$.

One obtains then the second order difference equation:

$$y_t = \chi[a(1-\delta+\frac{1}{a})y_{t+1} - ay_{t+2}]$$

Hence by taking $z_t = y_t$ and $w_t = y_{t+1}$ we obtain the implicitly defined system of equations:

$$\begin{cases} z_t = \chi[a(1 - \delta + \frac{1}{a})z_{t+1} - aw_{t+1}] \\ w_t = z_{t+1} \end{cases} \tag{2}$$

In this model for certain parameter values, the fixed point x^* is a snap-back repeller, thus by the results of Marotto, in any neighbourhood of the orbit of the snap-back repeller there is an invariant set on which f is chaotic and conjugate to a 1-sided shift.

3. Cash-in-advance model.

There exists a government and an agent, where the government consumes nothing and sets monetary policy. There exists also a cash good and a credit good, and the agent has a utility function of type $\sum_{t=0}^{\infty} \beta^t U(c_{1t}, c_{2t})$, where $\beta \in (0,1)$ is the discount factor. The function U takes the form $U(x,y) = \frac{x^{1-\sigma}}{1-\sigma} + \frac{y^{1-\gamma}}{1-\gamma}$, with $\sigma > 0, \gamma > 0$. The cash good c_{1t} can be bought with money m_t, which is carried over from period $t-1$. The credit good c_{2t} does not require cash and can be bought on credit. Each period the agent has an endowment y and $c_{1t} + c_{2t} = y$. We assume also that the cash good costs the same price p_t as the credit good. The agent wants to maximize his utility function by a choice of $\{c_{1t}, c_{2t}, m_{t+1}\}_{t \geq 0}$ subject to constraints: $p_t c_{1t} \leq m_t$, and $m_{t+1} \leq p_t y + (m_t - p_t c_{1t}) + \theta M_t - p_t c_{2t}$, where M_t is the money supply controlled by the government for a constant growth, $M_{t+1} = (1+\theta) M_t$. Denote by $x_t = m_t / p_t$ the level of real money balance.

We obtain then an implicitly defined difference equation with a non-invertible map f, i.e

$$x_t = f(x_{t+1}) \tag{3}$$

For certain parameters, it can be shown that there exists an invariant interval $[x_l, x_r]$ such that the map f has a periodic cycle of period 3. Hence according to Li-Yorke classic result, the map f is chaotic on that interval. In fact it can be shown that there exists an invariant subset of $[x_l, x_r]$ on which the map is conjugate to a subshift of finite type.

4. Cobweb model with adaptive adjustment.

In this model the supplier adjusts his production x_t according to the realities of the market while keeping the intention to reach a profit maximum $\tilde{x}_{t+1}$. It is met for instance in agricultural markets where farmers who plant for example wheat cannot change their crop during the same year/period. This is a hedging rule $x_{t+1} = x_t + \alpha(\tilde{x}_{t+1} - x_t)$, with $\alpha \in (0,1)$ the speed of adjustment. The aggregate supply from n identical producers is $X_t = nx_t$, and the price is given by $p_t = \frac{c}{Y_t^{\beta}}$, where Y_t is the demand at period t and c is a fixed parameter. We assume the market clears at each period, i. e $X_t = Y_t$. After a change of variable we obtain:

$$z_{t+1} = f_{\alpha,\beta}(z_t) = (1-\alpha)z_t + \frac{\alpha}{z_t^{\beta}}, \ (\alpha,\beta) \in (0,1) \times (0,\infty) \quad (4)$$

This function has a unique fixed point $z = 1$, which is a repeller if $|f'_{\alpha,\beta}(1)| > 1$, i.e if $\beta > \frac{2-\alpha}{\alpha}$.

Then Onozaki et. al. showed that there exists a number $\bar{\beta} > \frac{2-\alpha}{\alpha}$ s.t for each $\beta > \bar{\beta}$, $f_{\cdot,\beta}(\cdot)$ has a hyperbolic horseshoe in the plane.

Conclusions:

In the examples above there exist parametrizations in which the noninvertible system is given implicitly as $z_t = f(z_{t+1})$ or directly as $z_{t+1} = f(z_t)$, and has some hyperbolic set Λ (in general without critical points) or a set where an iterate is conjugate to a 1-sided shift. The hyperbolic case includes also the case with no contracting directions, i.e the expanding case. The implicit difference equation gives the backward dynamics of the model. We notice that a point from the inverse limit $\hat{\Lambda}$ given by $\hat{x} = (x, x_{-1}, \dots)$ represents in fact a sequence of *future equilibria* which are *allowed by the backward dynamics*; so in the notation $\hat{x} = (x, x_{-1}, x_{-2}, \dots)$, we start from a level of consumption of x, then at time 1 we have a level of consumption x_{-1}, then x_{-2} at time 2, and so on.

For the implicitly defined economic models given before, we have seen that there exist invariant sets on which the function (or one of its iterates) is conjugated to a shift on a symbol space; this invariant limit set Λ is usually obtained from homoclinic/heteroclinic orbits or snap-back repellers and thus we have a hyperbolic structure on Λ.

Hyperbolicity is understood here in the sense of **endomorphisms**, in which the unstable directions and unstable manifolds depend on whole sequences of consecutive preimages (i.e elements of $\hat{\Lambda}$), not only on base points. We include in the hyperbolic case also the case of no contracting directions, i.e the expanding case. For a hyperbolic map f on a compact invariant set Λ and a small enough $\delta > 0$, we denote by $W^s_\delta(x)$ the local stable manifold at the point $x \in \Lambda$, and by $W^u_\delta(\hat{x})$ the local unstable manifold corresponding to the history $\hat{x} \in \hat{\Lambda}$.

Let us prove that in this non-invertible hyperbolic case we have stability of the inverse limits:

Theorem (M., 2011)

Let us consider one of the economic models from Section 1, given by a dynamical system f having a hyperbolic invariant set Λ. Then given any dynamical system g obtained by a small $\mathcal{C}^2$ perturbation of the parameters of f, there exists a g-invariant set Λ_g and a homeomorphism $H : \hat{\Lambda} \to \hat{\Lambda}_g$ such that $\hat{g} \circ H = H \circ \hat{f}$. Thus the dynamics of $\hat{g}$ on $\hat{\Lambda}_g$ is the same as the dynamics of $\hat{f}$ on $\hat{\Lambda}$.

Notice also that by perturbations and by lifting to the inverse limit, the topological entropy is not changed, i.e $h_{top}(g|\Lambda_g) = h_{top}(\hat{g}|_{\hat{\Lambda}_g})$ $= h_{top}(f|_\Lambda) = h_{top}(\hat{f}|_{\hat{\Lambda}})$.

Definition

Consider a continuous function $f : X \to X$ which is non-invertible on the compact set X contained in $\mathbb{R}$ or $\mathbb{R}^2$, and let $\hat{X}$ be the inverse limit. With $\beta \in (0,1)$ the *discount factor*, a **utility function** on $\hat{X}$ is a function $W : \hat{X} \to \mathbb{R}$,
$W(\hat{x}) = \sum_{i\geq 0} \beta^i U(x_{-i})$, where

a) in the case $X \subset (0,\infty)$ we have

$$U(x) := \frac{\min\{1,x\}^{1-\sigma}}{1-\sigma} + \frac{(2-\min\{1,x\})^{1-\gamma}}{1-\gamma}, \; x \in X, \text{ with } \sigma > 0, \gamma > 0.$$

b) in the case $X \subset (0,1) \times (0,1)$, we have

$$U(x,y) := \frac{x^{1-\sigma}}{1-\sigma} + \frac{y^{1-\gamma}}{1-\gamma}, \; (x,y) \in X, \text{ with } \sigma > 0, \gamma > 0.$$

The discount factor in the definition of W expresses the fact that future levels of consumption in intertemporal equilibria become less and less relevant to a representative consumer. In economic models with backward dynamics we form as before the set of intertemporal equilibria i.e the inverse limit $\hat{\Lambda}$, where $f|_\Lambda : \Lambda \to \Lambda$ is the restriction of the dynamical system f to a compact invariant set Λ. In general f is assumed hyperbolic on Λ or conjugated to a subshift of finite type of 1-sided sequences. The consumers/agents have a utility function W given on $\hat{\Lambda}$. A central government would like to know the average value of W over $\hat{\Lambda}$ with respect to certain invariant probability measures.

In general one uses probability measures which are preserved by the system; in fact from any arbitrary probability measure we can form an invariant one, according to Krylov-Bogolyubov procedure.

For instance in the cash-in-advance model, the government is controlling controls the money supply on the market by the growth rule $M_{t+1} = (1+\theta)M_t$, where $\theta > 0$ is the growth rate. For each θ there exists a different invariant interval $[x_l(\theta), x_r(\theta)]$ and inverse limit space $\hat{\Lambda}(\theta)$. For a utility function W like in Definition 7, economists are interested also in choosing the appropiate θ so that the average value $\int_{\hat{\Lambda}(\theta)} W d\hat{\mu}_\theta$ is largest, where $\hat{\mu}$ is an invariant probability on $\hat{\Lambda}(\theta)$. In this way given a certain utility function, we can adjust the money growth rate θ in such a way that the average utility value is largest.

Many times we want also to study systems from the point of view of the **measure of maximal entropy**, which best describes the chaotic nature of the model. Also one can be interested in adjusting the discount factor β of W in order to maximize the average utility value.

We will say below that a compact invariant set Λ is **basic** for f if there exists an open neighbourhood V of Λ s.t $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(V)$ and if f is topologically (forward) transitive on Λ; such a set is also called *locally maximal*. In general the invariant limit sets we have considered in the economic models so far, are basic by construction.

Theorem (M., 2011)

Let $f : \Lambda \to \Lambda$ be a continuous topologically transitive map on a basic set Λ as in the economic examples given, and let $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$ be its inverse limit space. Then there is a bijective correspondence $\mathcal{F}$ between f-invariant measures on Λ and $\hat{f}$-invariant measures on $\hat{\Lambda}$, given by $\mathcal{F}(\hat{\mu}) = \pi_*(\hat{\mu})$ (where $\pi : \hat{\Lambda} \to \Lambda, \pi(\hat{x}) = x$ is the canonical projection).
Moreover if in addition f is hyperbolic on the basic set Λ, then there exists a unique measure of maximal entropy $\hat{\mu}_0$ on $\hat{\Lambda}$ such that $\pi_*(\hat{\mu}_0) = \mu_0$, where μ_0 is the unique measure of maximal entropy on Λ; and also $h_{\hat{\mu}_0}(\hat{f}) = h_{\mu_0}(f) = h_{top}(f)$.

We give now a formula for the average value of the utility with respect to *any* invariant measure on the inverse limit.

Theorem (M. , 2011)

Consider a continuous non-invertible map f defined on an open set V in $\mathbb{R}^2$ or in $\mathbb{R}$, which has an invariant basic set Λ. Let also $W(\hat{x}) = \sum_{i \geq 0} \beta^i U(x_{-i})$ be a utility function on the inverse limit $\hat{\Lambda}$ as in Definition 7. Then for any $\hat{f}$-invariant borelian measure $\hat{\mu}$ on $\hat{\Lambda}$ we have that the average value

$$\int_{\hat{\Lambda}} W d\hat{\mu} = \frac{1}{1-\beta} \int_{\Lambda} U d\mu,$$

where $\mu = \pi_*(\hat{\mu})$. If in addition f is hyperbolic on Λ and if μ_0 is the unique f-invariant measure of maximal entropy on Λ and $\hat{\mu}_0$ is the unique measure of maximal entropy on $\hat{\Lambda}$, then $\mu_0 = \pi_*(\hat{\mu}_0)$ and $\int_{\hat{\Lambda}} W d\hat{\mu}_0 = \frac{1}{1-\beta} \int_{\Lambda} U d\mu_0$.

The average values of U on $\hat{\Lambda}_g$ with respect to the corresponding measures of maximal entropy, are easier to estimate than those on inverse limits. Economists can use this information to compare average utility values with respect to the corresponding measures of maximal entropy for various perturbations, which in reality are translated by adjustments of the money growth rates.

A case in which this average utility ranking can be applied nicely is for the 1-dimensional overlapping generations economic model in which the backward dynamics is given by a Type C unimodal map (typically the *logistic function* $F_\nu(x) = \nu x(1-x)$ with $\nu > 4$). In this case a central government can choose **both** the ν and the β which **maximize the average utility value** over the set of intertemporal equilibria, with respect to the **measure of maximal entropy** (i.e the invariant measure describing the chaotic distribution over time).

Corollary (M. , 2011)

Let a family of logistic maps given by $F_\nu(x) = \nu x(1-x), x \in [0,1]$ *with* $\nu > 4$*; then* F_ν *has an invariant expanding Cantor set* Λ_ν*. Consider also a utility function* $W_\beta(\hat{x}) = \sum_{i \geq 0} \beta^i U(x_{-i})$ *with* $U(x) := \frac{min\{1,x\}^{1-\sigma}}{1-\sigma} + \frac{(2-min\{1,x\})^{1-\gamma}}{1-\gamma}$, $x \in (0,1)$, *for some* $\sigma > 0, \gamma > 0$. *Then*

$$\int_{\hat{\Lambda}_\nu} W_\beta d\hat{\mu}_0 = \frac{1}{1-\beta} \int_{\Sigma_2^+} U \circ h_\nu^{-1} d\mu_{\frac{1}{2},\frac{1}{2}},$$

where $\hat{\mu}_0$ *is the measure of maximal entropy on* $\hat{\Lambda}_\nu$, $\mu_{\frac{1}{2},\frac{1}{2}}$ *is the measure of maximal entropy on* Σ_2^+ *and* $h_\nu : \Lambda_\nu \to \Sigma_2^+$ *is the itinerary map, i.e* $h_\nu(x) = (j_0, j_1, \ldots)$ *s.t* $F_\nu^k(x) \in I_{j_k}, k \geq 0$ *where* $F_\nu^{-1}([0,1]) = I_1 \cup I_2$, $I_1 \cap I_2 = \emptyset$.