A Portfolio Optimization Problem with Stochastic Interest Rate and a Defaultable Bond

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The free-default market

Consider a probability space $(\Omega, \mathcal{F}, P)$ endowded a filtration $(\mathcal{F}_t)$, which is the default-free market filtration (it is also called the *reference filtration*); it stands for the natural filtration generated by a two-dimensional standard Brownian motion $W(t) := (W_1(t), W_2(t))$.

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1. $W_1(t)$ stands for the source of randomness of the default-free market;
2. $W_2(t)$ stands for the source of randomness generated by the defaultable asset.

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The dynamics of the money market account are given by

$$dR(t) = R(t)r(t)dt, \tag{1}$$

Assume that the short rate $r(t)$ is stochastic and follows a Hull-White process

$$dr(t) = (a_1(t) - b_1(t)r(t))dt + \sigma_1(t)dW_1(t).$$

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The setting

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We adopt the reduced form approach for the defaultable asset, i.e. the bond may default at some random time τ which is not a stopping time with respect to the default-free market filtration $(\mathcal{F}_t)$.
It satisfies

- $P(\tau = 0) = 0$ (the default cannot arrive at the initial time);
- For any $0 < t < T$, $P(\tau > t) > 0$ (default can arrive at any time till maturity).

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Define

- the default indicator process $H_t := 1_{(\tau \leq t)}$;
- the filtration generated by the *default indicator* process, $\mathcal{H}_t := \sigma(H_s; 0 \leq s \leq t)$ – the minimal filtration with respect to which τ is a stopping time;
- the enlarged filtration (called also the *full filtration*) $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$.

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Let Q be a risk-neutral probability (it will be specified later).
Set $F_t := Q(\tau \leq t | \mathcal{F}_t)$.
Then F_t is clearly a bounded non-negative $\mathcal{F}_t$ - submartingale.
According to the Doob-Meyer decomposition it can be written as the sum of a martingale and an increasing process. Assume that the martingale part is 0.

The defaultable asset

If F_t is absolutely continuous with respect to the Lebesque measure, then

$$F_t = Q(\tau \leq t|\mathcal{F}_t) = \int_0^t f_s ds,$$

where (f_t) is a non-negative $\mathcal{F}_t$-adapted process.

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Define $\Gamma_t := -\ln(1 - F_t)$. Γ_t is called the hazard process of τ under Q, conditionally on $\mathcal{F}_t$.
Since F_t is increasing, then Γ_t is also increasing and $\Gamma_t = \int_0^t \lambda_s ds$. λ_t is called the conditional hazard rate of τ given the free default filtration.

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$$\lambda_t = \frac{1 - F_t}{f_t}.$$

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We assume that $\lambda(t)$ follows a Hull-White process

$$d\lambda_t = (a_2(t) - b_2(t)\lambda_t)dt + \tilde{\sigma}_1(t)dW_1(t) + \tilde{\sigma}_2(t)dW_2(t). \tag{2}$$

Remark

It is usually assumed that market risk and default risk are correlated, so it would be convenient to represent $f(t)$ as

$$d\lambda_t = (a_2(t) - b_2(t)\lambda_t)dt + \sigma_1(t)dB(t), \tag{3}$$

where the Brownian motions $W_1(t)$ and $B(t)$ are correlated and let $\rho(t) := E(W_1(t)B(t)) = \langle W_1, B\rangle_t$ their cross variation process which it is assumed deterministic in the subsequent.
Notice that this leads to the same formulation, since if we define

$$\tilde{W}_1 := W_1(t)$$

and

$$\tilde{W}_2(t) := \int_0^t \frac{1}{\sqrt{1-\rho^2(s)}} dB(s) - \int_0^t \frac{\rho(s)}{\sqrt{1-\rho^2(s)}} dW_1(s),$$

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At the time of occurence of the default (if it occurs) the bond cesses to exist and the holder of the bond receives a compensation $z(t)$ given by a proportion of the pre-default value of the bond

$$z(t) = (1 - L(t))D(t-, T),$$

where

- $D(t, T)$ stands for the value of the bond at time t (it has a jump at the default time τ) and $D(\tau-)$ stands for the value prior to default;
- $L(t)$ stands for the loss-rate.

Price of the defaultable bond

The price at time t of a defaultable zero-coupon bond with maturity T and recovery $z(t)$ is given by (see Bielecki and Rutkovski (2004))

$$
\begin{aligned}
D(t,T) &= E^Q\left(1_{(\tau>T)}e^{-\int_t^T r_u du}X + 1_{(t<\tau\le T)}e^{-\int_t^\tau r_u ds} z_\tau \middle| \mathcal{G}_t\right) \\
&= 1_{(\tau>t)}E^Q\left(e^{-\int_t^T (r_u+\lambda_u)du}X + \int_t^T e^{-\int_t^s (r_u+\lambda_u)du} z_s \lambda_s ds \middle| \mathcal{F}_t\right),
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where E^Q stands for the expectation with respect to the probability measure Q.

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In the case of *recovery of the market value* at default

$$D(t,T) = 1_{(\tau>t)}E^Q\left(e^{-\int_t^T (r_s+\lambda_s L_s)ds)}X|\mathcal{F}_t\right) := 1_{(\tau>t)}B(t,T), \tag{5}$$

where $B(t,T)$ can be viewed as the *pre-default value of the bond* and may be seen as the value of a *non-defaultable bond* with

- *default-risk adjusted interest rate* $\hat{r}_t := r_t + \lambda_t L_t$;
- *credit spread* given by $\hat{r}_t - r_t = \lambda_t L_t$.

The wealth process

Consider an investor who can invest in the assets specified from above. We denote by $N_R(t)$, $N_S(t)$ and $N_D(t)$ the quantity of each asset (money market, stock respectively defaultable bond) detained by the investor at time t. $N_R(t)$ and $N_S(t)$ are assumed $(\mathcal{F}_t)$ predictable processes and $N_D(t)$ a $(\mathcal{G}_t)$ predictable process.

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The wealth process is given by

$$X(t) = N_R(t)R(t) + N_S(t)S(t) + N_D(t)D(t,T)$$

and is assumed self-financed, which means that

$$dX(t) = N_R(t)dR(t) + N_S(t)dS(t) + N_D(t)dD(t,T). \tag{6}$$

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Set $\pi_R(t)$, $\pi_S(t)$ and $\pi_D(t)$ the corresponding fractions of wealth, i.e.

$$\pi_R(t) := \frac{N_R(t)R(t)}{X(t-)}, \pi_S(t) := \frac{N_S(t)S(t)}{X(t-)}, \pi_D(t) := \frac{N_D(t)D(t-,T)}{X(t-)}.$$

Decomposition of the investment strategy

The self-financing condition imposed on the portfolio reads

$$dX^{\pi}(t) = X^{\pi}(t-)\left(\pi_R(t)\frac{dR(t)}{R(t)} + \pi_S(t)\frac{dS(t)}{S(t)} + \pi_D(t)\frac{dD(t,T)}{D(t-,T)}\right). \qquad (7)$$

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The strategy $\pi(t)$ adopted by the investor can be decomposed in a *pre-default strategy* $\underline{\pi}(t)$ (for $t < \tau$) and a *post-default strategy* $\overline{\pi}(t)$ (for $t > \tau$), according to which the wealth process evolves as

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Admissible portfolios

The jump of the wealth process in the default time τ is

$$\begin{aligned}\Delta X^{\pi}(\tau) &= X^{\pi}(\tau) - X^{\pi}(\tau-) = N_D(\tau)z(\tau) - N_D(\tau-)B(\tau-,T) \\ &= -N_D(\tau)L(\tau)B(\tau,T) = -X^{\pi}(\tau-)\pi_D(\tau)L(\tau),\end{aligned}$$

by the left-continuity of $\pi_D(t)$ and the continuity of $B(t,T)$.
Then

$$X^{\pi}(\tau) = \Delta X^{\pi}(\tau) + X^{\pi}(\tau-) = X^{\pi}(\tau-)\left(1 - \pi_D(\tau)L(\tau)\right).$$

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The set $\mathcal{A}(x)$ of the admissible portfolios is determined by the bounded and left-continuous portfolio processes $(\pi(t))_{0\le t\le T}$ such that

- all the integrals appearing in the formulas (8) and (9) are well defined;
- the initial endowenement is given by the positive amount x, $X^{\pi}(0) = x$;
- the wealth remains positive during the investment process, i.e. for each $t \in [0,T]$, $X^{\pi}(t) \ge 0$, P a.s.

The optimization problem

Our interest $\rightarrow$ to maximize the expected utility (under the historical probability P) of the investor from the final wealth over the class $\mathcal{A}(x)$ of admissible portfolios.

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The optimization problem is stated as

$$V(x) := \sup_{\pi \in \mathcal{A}(x)} E[U(X^{\pi}(T))] = \sup_{\pi \in \mathcal{A}(x)} J(\pi), \tag{10}$$

where the utility function $U : (0, \infty) \to \mathbb{R}$ describing the preferences of the investor

- is a strictly increasing, strictly concave and continuously differentiable function;
- satisfies the *Inada conditions*: $\lim_{x \to 0} U'(x) = +\infty$ and $\lim_{x \to \infty} U'(x) = 0$.

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Related citations

Continuous-time portfolio optimization problems were studied starting with the papers of Merton ((1969), (1971), (1973)).
Other significant contributions

- in the case of *complete* financial markets: Karatzas, Lehoczky and Shreve (1987); Korn and Kraft (2001) - where the interest rate is stochastic; Blanchet-Scaillet, El Karoui, Jeanblanc and Martellini (2008) - where the time-horizon is random.
- in the case of *incomplete* financial markets: Kramkov and Schachermayer (1999), Jiao and Pham (2010) and Lim and Quenez (2010) - in a market with counterparty default risk.

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Approaches

There are two possible approaches

1. the dynamic programming approach (as a tool of *stochastic control theory*), leading to, in the case of
   - *complete markets* to some nonlinear PDE $\rightarrow$ the Hamilton-Jacobi-Bellman equation (which generally is not easy to solve);
   - *incomplete markets* to some BSDE;
2. the martingale approach $\rightarrow$ using *convex duality arguments* (using the properties of the convex dual of U).

The convex dual of U

In this spirit of the martingale approach, define the *convex dual function* of U

$$U^*(x) := \sup_{y>0}(U(y) - xy). \tag{11}$$

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Under the assumptions imposed on U,

- U' is invertible;
- if $I := (U')^{-1}$ then $(U^*)' = -I$;
- the supremum in the formula (11) is attained for $y = I(x)$, which leads to

$$U(y) - xy \leq U(I(x)) - xI(x),$$

for any $x, y > 0$.

A result of Kramkov and Schachermayer

Using Theorem 2.2 from Kramkov and Schachermayer (1999), we know that the optimization problem (10) admitts a solution under the assumptions

(i) The asymptotic elasticity of the utility function $U(x)$ satisfies

$$AE(U) := \lim\sup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1;$$

(ii) There exist at least an equivalent local martingale measure, i.e. a probability measure Q equivalent with P under which the discounted wealth process is a (local) martingale;

(iii) for some $x > 0$ the value function $V(x)$ is finite.

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The assumption (i) is obviously satisfied for our choices of CARA utility functions (*power utility* and *logarithmic utility*).

Next, it can be shown that the price of the defaultable bond satisfies

$$D(t,T) = \tilde{H}_t B(t,T) = D(t,T)(r_t - \lambda_t(1-L_t))dt + \tilde{H}_t e^{\int_0^t \hat{r}(s)ds} p(t)dW(t) \\ - B(t,T)dM_t, \tag{12}$$

where

- $\tilde{H}_t := 1_{(\tau > t)} = 1 - H_t$;
- recall that $\hat{r}_t := r_t + \lambda_t L_t$;
- $p(t)$ appears when we apply the *Representation of Brownian Martingales Theorem* for the process $m_t := E^Q[e^{\int_0^T \hat{r}(s)ds} X | \mathcal{F}_t]$, i.e. $dm_t = p(t)dW(t)$;
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An existence result

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An existence result

With a properly defined random variable $\mathcal{F}_T$ – measurable random variable Z (via a stochastic exponential), the probability Q which is absolutely continuous with respect to the historical probability P, having the Radon–Nikodym density Z is such that, under Q, the discounted price of the defaultable asset, $e^{-\int_0^t r(s)ds} D(t,T)$ becomes a local martingale. This leads clearly to (ii).

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Next, according to Theorem 2.2 from Kramkov and Schachermayer (1999), $V(x)$ is finite for some positive x if the conjugate function of the value function V, denoted V^*, is finite at $y = V'(x)$.
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We are now in position to state an existence result

Theorem

Under our standing assumptions, the optimization problem (10) *has a solution.*

Remark

Theorem 2.2 from Kramkov and Schachermayer (1999) allows us to provide a dual characterization of the value function in (10) *and the associated optimal portfolio but not to obtain explicit formulas!*

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Remark

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Our next goal is to characterize the optimal portfolio for our choices of utility functions.

Characterization of the optimal portfolio

The problem (10) is equivalent with the problem

$$V(x) = \sup_{\pi} E[V(\tau, (X^{\pi}(\tau))] = \sup_{\pi} E[V(\tau, X^{\pi}(\tau-)(1 - \pi_D(\tau)L(\tau)))]. \quad (14)$$

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Remark

We thus have to solve first the post-default optimization problem and the solution of the pre-default problem will depend on the solution on the former.

Recall that the post-default value process satisfies

$$dX^{\overline{\pi}}(t) = X^{\overline{\pi}}(t)\left[r(t) + \overline{\pi}_S(t)(\mu(t) - r(t))\right] dt + \overline{\pi}_S(t)\sigma(t) dW_1(t), \quad (15)$$

for $t > \tau$. Then

$$X^{\overline{\pi}}(t) = X^{\overline{\pi}}(\tau) \exp\left(\int_\tau^t \left(r(s) + \overline{\pi}_S(s)(\mu(s) - r(s)) - \frac{1}{2}\overline{\pi}_S^2(s)\sigma^2(s)\right) ds\right) \exp\left(\int_\tau^t \overline{\pi}_S(s)\sigma(s) dW_1(s)\right). \quad (16)$$

Logarithmic utility

Let $U(x) = \ln(x)$. Then

$$U(X^{\pi}(T)) = \ln(X^{\underline{\pi}}(\tau)) + \int_{\tau}^{T} \Big(r(t) + \overline{\pi}_S(t)(\mu(t) - r(t)) - \frac{1}{2}\overline{\pi}_S^2(t)\sigma^2(t)\Big) dt + M_t, \tag{17}$$

where (M_t) is a (local) martingale (is a stochastic integral).

$$E\left(U(X^{\pi}(T))\right) = E\left(\ln(X^{\underline{\pi}}(\tau))\right) + E\Big(\int_{\tau}^{T} \Big(r(t) + \overline{\pi}_S(t)(\mu(t) - r(t)) - \frac{1}{2}\overline{\pi}_S^2(t)\sigma^2(t)\Big) dt\Big). \tag{18}$$

The second term on the r.h.s. of the last formula attains its maximum for

$$\overline{\pi}_S^*(t) = \frac{\mu(t) - r(t)}{\sigma^2(t)},$$

which is exactly the *Merton's optimal strategy*.

Logarithmic utility

Next, we need to maximize

$$E\left(\ln(X^{\underline{\pi}}(\tau))\right) = E\left(\ln\left(\tilde{X}^{\underline{\pi}}(\tau)(1-\underline{\pi}_D(\tau)L(\tau))\right)\right),$$

where $\tilde{X}(t)$ is the solution of

$$\begin{aligned} d\tilde{X}^{\pi}(t) = \tilde{X}^{\pi}(t)\Big((1-\pi_S(t)-\pi_D(t))r(t)dt + \pi_S(t)\mu(t)dt \\ + \pi_S(t)\sigma(t)dW_1(t) + \pi_D(t)\frac{1}{B(t,T)}dB(t,T)\Big), \text{ for } 0 \le t \le T. \end{aligned} \tag{19}$$

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This is a maximization problem with uncertain time-horizon, for which we may apply the results of Blanchet-Scaillet, El Karoui, Jeanblanc and Martellini (2008).

We look now at the case $U(x) = \frac{x^\gamma}{\gamma}$, with $0 < \gamma < 1$. We have

$$U(X^\pi(T)) = \frac{(X^{\underline{\pi}}(\tau))^\gamma}{\gamma} \exp\left(\gamma \int_\tau^T (r(t) + \overline{\pi}_S(t)(\mu(t) - r(t)) - \frac{1}{2}\overline{\pi}_S^2(t)\sigma^2(t))dt\right) \times \exp\left(\gamma \int_\tau^T \overline{\pi}_S(s)\sigma(s)dW_1(s)\right). \tag{20}$$

If the interest rate $r(t)$ is *deterministic*, by the so–called *Change-of-Measure Device* (see Theorem 4.1 in Korn and Seifried (2009)), we know how to compute the supremum of the first exponential term in the last formula, while for the first one we could still apply the results of Blanchet-Scaillet, El Karoui, Jeanblanc and Martellini (2008).
In the case of a stochastic interest rate, we think that the Change-of-Measure Device for semimartingales (see Section 3 in Seifried (2010)) could be a usefull tool.

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