Weak transportation cost inequalities on metric measure spaces

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The L_2-Wasserstein space

(M, d, m) normalized metric measure space

- (M, d) is a complete separable metric space
- m is a probability measure on $(M, \mathcal{B}(M))$

The L_2-Wasserstein distance between two measures μ and ν on M is defined as

$$\mathrm{d}_W(\mu,\nu) = \inf\left\{ \left(\int_{M\times M} \mathrm{d}^2(x,y)dq(x,y) \right)^{1/2} : q \text{ coupling of } \mu,\ \nu \right\},$$

with the convention $\inf \emptyset = \infty$.

$$\mathcal{P}_2(M, \mathrm{d}) := \left\{ \nu : \int_M \mathrm{d}^2(o,x)d\nu(x) < \infty \text{ for some } o \in M \right\}.$$

The relative entropy

$$\mathrm{Ent}(\nu|m) := \begin{cases} \int_M \rho \log \rho \, dm & \text{, for } \nu = \rho \cdot m \\ +\infty & \text{, otherwise} \end{cases}$$

We denote by $\mathcal{P}_2(M, \mathrm{d}, m)$ the subspace of measures $\nu \in \mathcal{P}_2(M, \mathrm{d})$ of finite entropy $\mathrm{Ent}(\nu|m) < \infty$.

Transportation cost inequality

The probability measure m satisfies a Talagrand inequality (or transportation cost inequality) with constant K iff for all $\nu \in \mathcal{P}_2(M, \mathrm{d})$

$$\mathrm{d}_W(\nu, m) \leq \sqrt{\frac{2\,\mathrm{Ent}(\nu|m)}{K}}.$$

Ricci curvature bounds - the Riemannian case

Theorem (v.Renesse-Sturm 2005)

For any smooth connected Riemannian manifold M with intrinsic metric d *and volume measure m and any* $K \in \mathbb{R}$ *the following properties are equivalent :*

1. $\mathrm{Ric}_x(v, v) \geq K|v|^2$ *for* $x \in M$ *and* $v \in T_x(M)$.
2. *The entropy* $Ent(\cdot|m)$ *is displacement K-convex on* $\mathcal{P}_2(M, \mathrm{d})$ *in the sense that for each geodesic* $\gamma : [0, 1] \to \mathcal{P}_2(M, \mathrm{d})$ *and for each* $t \in [0, 1]$

$$\begin{aligned} \mathrm{Ent}(\gamma(t)|m) \quad &\leq \quad (1-t)\mathrm{Ent}(\gamma(0)|m) + t\mathrm{Ent}(\gamma(1)|m) \\ &\qquad - \frac{K}{2} t(1-t)\, \mathrm{d}_W^2(\gamma(0), \gamma(1)). \end{aligned}$$

Curvature bounds for metric measure spaces

Definition (Sturm, Acta Math. 2006)

A metric measure space (M, d, m) has curvature $\geq K$ for some number $K \in \mathbb{R}$ iff the relative entropy $\mathrm{Ent}(\cdot|m)$ is weakly K-convex on $\mathcal{P}_2(M, \mathrm{d}, m)$ in the sense that for each pair $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathrm{d}, m)$ there exists a geodesic $\Gamma : [0, 1] \to \mathcal{P}_2(M, \mathrm{d}, m)$ connecting ν_0 and ν_1 with

$$\begin{aligned} \mathrm{Ent}(\Gamma(t)|m) \quad &\leq \quad (1-t)\mathrm{Ent}(\Gamma(0)|m) + t\mathrm{Ent}(\Gamma(1)|m) \\ &\qquad - \frac{K}{2} t(1-t)\, \mathrm{d}_W^2(\Gamma(0), \Gamma(1)) \end{aligned}$$

for all $t \in [0, 1]$.

L_2-transportation distance $\mathbb{D}$

$$\mathbb{D}((M, \mathrm{d}, m), (M', \mathrm{d}', m')) = \inf \left(\int_{M \sqcup M'} \hat{\mathrm{d}}^2(x, y) dq(x, y) \right)^{1/2},$$

where $\hat{\mathrm{d}}$ ranges over all couplings of d and d' and q ranges over all couplings of m and m'.

A pseudo-metric $\hat{\mathrm{d}}$ on the disjoint union $M \sqcup M'$ is a coupling of d and d' if $\hat{\mathrm{d}}(x, y) = \mathrm{d}(x, y)$ and $\hat{\mathrm{d}}(x', y') = \mathrm{d}'(x', y')$ for all $x, y \in \mathrm{supp}[m] \subset M$ and all $x', y' \in \mathrm{supp}[m'] \subset M'$.

$\mathbb{D}$ defines a complete separable length metric on the family of all isomorphism classes of normalized metric measure spaces (M, d, m) with $m \in \mathcal{P}_2(M, \mathrm{d})$.

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h-Geodesic spaces

Let $h > 0$ be given. We say that a metric space (M, d) is h-rough geodesic iff for each pair of points $x_0, x_1 \in M$ and each $t \in [0, 1]$ there exists a point $x_t \in M$ satisfying

$$\begin{cases} \mathrm{d}(x_0, x_t) & \leq & t\,\mathrm{d}(x_0, x_1) + h \\ \mathrm{d}(x_t, x_1) & \leq & (1-t)\,\mathrm{d}(x_0, x_1) + h \end{cases}$$

The point x_t will be referred to as the h-rough t-intermediate point between x_0 and x_1.

Perturbed Wasserstein metric

Let (M, d) be a metric space. For each $h > 0$ and any pair of measures $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathrm{d})$ put

$$\mathrm{d}_W^{\pm h}(\nu_0, \nu_1) := \inf \left\{ \left(\int [(\mathrm{d}(x_0, x_1) \mp h)_+]^2 \, dq(x_0, x_1) \right)^{1/2} \right\},$$

where q ranges over all couplings of ν_0 and ν_1 and $(\cdot)_+$ denotes the positive part.

The infimum above is attained. A coupling q for which the infimum is attained in the definition of $\mathrm{d}_W^{\pm h}$ is called $\pm h$-optimal coupling.

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Rough curvature bounds

Definition

(M, d, m) has h-rough curvature $\geq K$ iff for each pair $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathrm{d}, m)$ and for any $t \in [0, 1]$ there exists an h-rough t-intermediate point $\eta_t \in \mathcal{P}_2(M, \mathrm{d}, m)$ between ν_0 and ν_1 satisfying

$$\mathrm{Ent}(\eta_t|m) \leq (1-t)\mathrm{Ent}(\nu_0|m) + t\mathrm{Ent}(\nu_1|m) - \frac{K}{2} t(1-t)\, \mathrm{d}_W^{\pm h}(\nu_0, \nu_1)^2,$$

where the sign in $\mathrm{d}_W^{\pm h}(\nu_0, \nu_1)$ is chosen '$+$' if $K > 0$ and '$-$' if $K < 0$.

Briefly, we write in this case h-$\mathbb{C}\mathrm{urv}(M, \mathrm{d}, m) \geq K$.

A weak Talagrand inequality

Proposition ("h-Talagrand Inequality")

Assume that (M, d, m) *is a metric measure space which has* $h\text{-}\mathbb{C}\mathrm{urv}(M, \mathsf{d}, m) \geq K$ *for some numbers* $h > 0$ *and* $K > 0$. *Then for each* $\nu \in \mathcal{P}_2(M, \mathsf{d})$ *we have*

$$\mathsf{d}_W^{+h}(\nu, m) \leq \sqrt{\frac{2\,\mathrm{Ent}(\nu|m)}{K}}.$$

We will call it *h*-Talagrand inequality of constant K.

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Concentration of measure

For a given $A \subset M$ measurable denote
$B_r(A) := \{x \in M : \mathrm{d}(x, A) < r\}$ for $r > 0$.

The concentration function of (M, d, m) is defined as

$$\alpha_{(M,\mathrm{d},m)}(r) := \sup\left\{1 - m(B_r(A)) : A \in \mathcal{B}(M), m(A) \geq \frac{1}{2}\right\},\ r > 0.$$

Theorem

Let (M, d, m) be a metric measure space with h-$\mathbb{C}\mathrm{urv}(M, \mathrm{d}, m) \geq K > 0$ for some $h > 0$. Then there exists an $r_0 > 0$ such that for all $r \geq r_0$

$$\alpha_{(M,\mathrm{d},m)}(r) \leq e^{-Kr^2/8}.$$

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Integrability of Lipschitz functions

Proposition

Assume that (M, d) *is a metric space and let* $h > 0$ *be given. If* m *is a probability measure on* (M, d) *that satisfies an* h*-Talagrand inequality of constant* $K > 0$ *then all Lipschitz functions are exponentially integrable. More precisely, for any Lipschitz function* φ *with* $\|\varphi\|_{\mathrm{Lip}} \leq 1$ *and* $\int \varphi \, dm = 0$ *we have*

$$\int_M e^{t\varphi} dm \leq e^{\frac{t^2}{2K} + ht}, \qquad \forall t > 0$$

or equivalently, for any Lipschitz function φ

$$\int_M e^{t\varphi} dm \leq \exp\left(t \int_M \varphi \, dm\right) \exp\left(\frac{t^2}{2K} \|\varphi\|_{\mathrm{Lip}}^2 + ht\|\varphi\|_{\mathrm{Lip}}\right), \quad \forall t > 0.$$

Stability under $\mathbb{D}$-convergence

Theorem

Let (M, d, m) be a compact normalized metric measure space and consider $\{(M_h, \mathsf{d}_h, m_h)\}_{h>0}$ a family of normalized metric measure spaces with uniformly bounded diameter and with (M_h, d_h, m_h) satisfying an h-Talagrand inequality of constant K_h, for $K_h \to K$ as $h \to 0$. If

$$(M_h, \mathsf{d}_h, m_h) \xrightarrow{\mathbb{D}} (M, \mathsf{d}, m)$$

as $h \to 0$ then (M, d, m) satisfies a Talagrand inequality of constant K.

Tensorization

Theorem

Let $\{(M_i, \mathrm{d}_i, m_i)\}_{i=1,\dots,n}$ be n normalized metric measure spaces that satisfy all an h-Talagrand inequality of constant K. Then the space $M = M_1 \times \dots \times M_n$, with the metric

$$\mathrm{d}(x, y) = \sqrt{\sum_{i=1}^{n} \mathrm{d}_i(x_i, y_i)}, \quad x, y \in M$$

and with the measure $m = m_1 \otimes \dots \otimes m_n$, satisfies also an h-Talagrand inequality of constant K.

Thank you for your attention