Nonlinear bifurcation problems

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Outline of the talk

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Main problems

Singular solutions

Bifurcation for singular L-E-F equations

Bifurcation for nonhomogeneous operators

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I. Singular (blow-up boundary) solutions of logistic-Malthusian problems

II. Bifurcation for Lane-Emden-Fowler singular equations

III. Bifurcation problems for non-homogeneous differential operators

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Main direction of the talk: perturbed singular, nonsingular or degenerate stationary problems.

Elementary example: Equation

$$\sin x = c \qquad c \in (-1,1),\ x \in \mathbb{R}$$

has infinitely many solutions, but the perturbed equation

$$\sin x = c + \varepsilon x \qquad c \in (-1,1),\ \varepsilon \in \mathbb{R} \setminus \{0\},\ x \in \mathbb{R}$$

has a finite number of solutions, which tends to $+\infty$ as $\varepsilon \to 0$.

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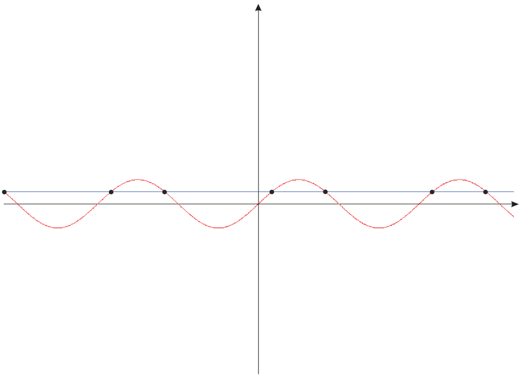

Figure: Equation $\sin x = 1/2$ has infinitely many solutions.

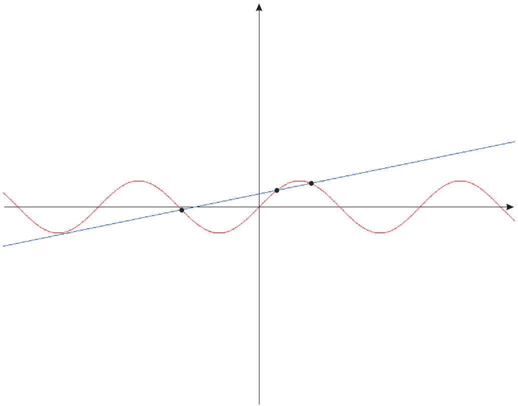

Figure: Solutions of the equation $\sin x = 1/2 + 0.2x$.

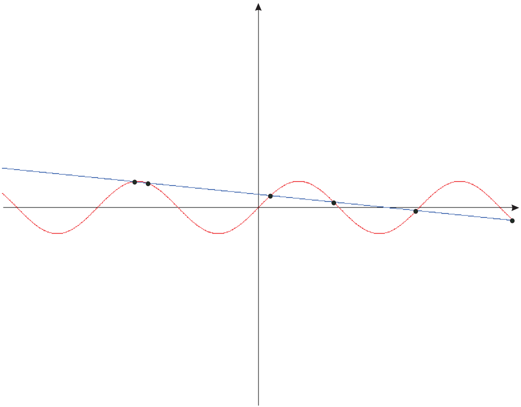

Figure: Solutions of the equation $\sin x = 1/2 - 0.1x$.

An example in PDEs: Problem

$$\begin{cases} -\Delta u = u^{-a} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution, provided that $a > 0$ (Crandall, Rabinowitz & Tartar, 1977).

Consider the perturbed problem

$$\begin{cases} -\Delta u = u^{-a} + \lambda|\nabla u|^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p \in (0, 2)$ and $\lambda > 0$.

Then (M. Ghergu, V.R., *J. Diff. Equations*, 2003):

(i) if $p \in (0, 1]$, existence of a solution for any $\lambda > 0$;

(ii) if $p \in (1, 2)$, there exists λ^* such that a solution exists if and only if $\lambda \in (0, \lambda^*)$.

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Singular solutions of logistic-Malthusian problems

Consider the problem (P):

$$\Delta u + \lambda u = b(x) f(u) \qquad \text{in } \Omega,$$

$\lambda \in \mathbb{R}$, $b \in C^{0,\mu}(\overline{\Omega})$, $0 < \mu < 1$, $b \geq 0$, $b \not\equiv 0$ in Ω. Assume that $f \in C^1[0, \infty)$ satisfies

(A_1) $\quad f \geq 0$ and $f(u)/u$ is increasing on $(0, \infty)$.

(A_2) $\quad \displaystyle\int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty\,, \quad F(t) = \int_0^t f(s)\, ds.$

Examples: (i) $f(u) = e^u - 1$; (ii) $f(u) = u^p,\ p > 1$; (iii) $f(u) = u[\ln(u+1)]^p,\ p > 2$.

Problem (H. Brezis). Find a necessary and sufficient condition such that problem (P) has a blow-up boundary solution.

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Set

$$\Omega_0 = \mathrm{int}\,\{x \in \Omega : \; b(x) = 0\}$$

and assume that $\overline{\Omega}_0 \subset \Omega$ and $b > 0$ in $\Omega \setminus \overline{\Omega}_0$.
Denote $\lambda_1(\Omega_0)$ the first eigenvalue of $(-\Delta)$ in Ω_0 and set $\lambda_1(\Omega_0) = \infty$ if $\Omega_0 = \emptyset$.

Theorem

Assume f satisfies (A_1) and (A_2). Then problem (P) has a blow-up boundary solution if and only if $\lambda \in (-\infty, \lambda_1(\Omega_0))$.

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Definition

$R : [D, \infty) \to [0, \infty)$ measurable has regular variation at $+\infty$ of index $q \in \mathbb{R}$ (notation: $R \in RV_q$) provided that for all $\xi > 0$,

$$\lim_{u \to \infty} R(\xi u)/R(u) = \xi^q.$$

$q = 0$: *weak variation*.

$R \in RV_q \implies R(u) = u^q L(u), \quad L \in RV_0.$

Examples: (i) $R(u) = u^q$, $R \in RV_q$.

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Lemma

Assume (A_1). *The following properties are equivalent:*
$a)$ $f' \in RV_\rho$
$b)$ $\lim_{u\to\infty} uf'(u)/f(u) := \vartheta < \infty$
$c)$ $\lim_{u\to\infty} (F/f)'(u) := \gamma > 0.$

Remark

We have:
(i) $\rho \geq 0$;
(ii) $\gamma = 1/(\rho+2) = 1/(\vartheta+1)$;
(iii) If $\rho \neq 0$, *then* $(K-O)$. *Converse* **not** *true:* $f(u) = u\ln^4(u+1)$. *It may happen that* $\rho = 0$ *and* $(K-O)$ *is not fulfilled, so Eq. (P) does not have blow-up boundary solutions. Examples:* $f(u) = u$, $f(u) = u\ln(u+1)$.

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Karamata's class.

Let $\mathcal{K}$ denote the class of functions $k : (0, \nu) \to (0, \infty)$ of class C^1, increasing and such that $\lim_{t\to 0^+} \left(\dfrac{\int_0^t k(s)\,ds}{k(t)}\right)^{(i)} := \ell_i,\ i = \overline{0, 1}$.

Then $\ell_0 = 0$ and $\ell_1 \in [0, 1]$, for all $k \in \mathcal{K}$.

Lemma

Assume $S \in C^1[D, \infty)$ such that $S' \in RV_q$, $q > -1$. Then

a) If $k(t) = \exp\{-S(1/t)\}$ $\forall t \le 1/D$, then $k \in \mathcal{K}$ and $\ell_1 = 0$.

b) If $k(t) = 1/S(1/t)$ $\forall t \le 1/D$, then $k \in \mathcal{K}$ and $\ell_1 = 1/(q + 2) \in (0, 1)$.

c) If $k(t) = 1/\ln S(1/t)$ $\forall t \le 1/D$, then $k \in \mathcal{K}$ and $\ell_1 = 1$.

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Remark

If $S \in C^1[D,\infty)$, *then* $S' \in RV_q$ *with* $q > -1$ *if and only if* $\exists\, m > 0$, $C > 0$ *and* $B > D$ *such that* $S(u) = Cu^m \exp\left\{\int_B^u \frac{y(t)}{t}\,dt\right\}$, $\forall u \geq B$, *where* $y \in C[B,\infty)$ *satisfies* $\lim_{u\to\infty} y(u) = 0$. *In such a case,* $S' \in RV_q$ *with* $q = m - 1$.

Theorem

Assume (A_1) *and* $f' \in RV_\rho$, *with* $\rho > 0$. *Suppose* $b \equiv 0$ *on* $\partial\Omega$ *such that*
$(B)\quad b(x) = c\,k^2(d(x)) + o(k^2(d(x)))$ *as* $d(x) \to 0$, *where* $c > 0$ *and* $k \in \mathcal{K}$.

Then, for any $\lambda \in (-\infty, \lambda_{\infty,1})$, *problem (P) has a unique blow-up boundary solution* u_λ. *Moreover,*

$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{h(d(x))} = \xi_0,$$

where $\xi_0 = \left(\frac{2+\ell_1\rho}{c(2+\rho)}\right)^{1/\rho}$ *and* h *is defined by*

$$\int_{h(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = \int_0^t k(s)\,ds, \quad \forall t \in (0, \nu).$$

Examples of admissible functions k:

$k(t) = -1/\ln t$,

$k(t) = t^\alpha$, $k(t) = \exp\{-1/t^\alpha\}$,

$k(t) = \exp\left\{-\ln(1+\frac{1}{t})/t^\alpha\right\}$,

$k(t) = \exp\left\{-\left[\arctan\left(\frac{1}{t}\right)\right]/t^\alpha\right\}$,

$k(t) = t^\alpha/\ln(1+\frac{1}{t})$, where $\alpha > 0$.

Case of non-monotone nonlinearities

Let $f : [0, \infty) \to [0, \infty)$ be a smooth, **increasing**, such that $f(0) = 0$ and $f > 0$ on $(0, \infty)$. According to Keller & Osserman, 1957, problem

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega \end{cases}$$

has a solution if and only if $\int_1^\infty [F(t)]^{-1/2}\, dt < \infty$, where $F(t) = \int_0^t f(s)\, ds$.

Examples: (i) $f(u) = e^u - 1$; (ii) $f(u) = u^p,\ p > 1$; (ii) $f(u) = u^p \ln(u+1)$, $p > 1$; (iv) $f(u) = u^p \arctan u$, $p > 1$; (v) $f(u) = u[\ln(u+1)]^p,\ p > 2$.

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Let $f : [0,+\infty) \to [0,+\infty)$ be such that $f(0) = 0$.

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1}$$

$$\Phi(\alpha) = \frac{1}{\sqrt{2}} \int_\alpha^\infty \frac{ds}{\sqrt{F(s) - F(\alpha)}}\,,\; F(s) = \int_0^s f(t)\,dt$$

We say that f satisfies the Keller-Osserman condition if

$$\exists \alpha > 0 \quad \text{such that} \quad \Phi(\alpha) < \infty. \tag{2}$$

We say that f satisfies the strong Keller-Osserman condition if

$$\liminf_{\alpha\to\infty} \Phi(\alpha) = 0. \tag{3}$$

Example: $f(u) = u^2(1 + \cos u)$ satisfies the strong Keller-Osserman condition and $\limsup_{\alpha\to\infty} \Phi(\alpha) = +\infty$.

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The function f satisfies the Keller-Osserman condition if and only if the BVP (1) *admits at least one positive large solution on* **some** *ball.*

Theorem

The function f satisfies the strong Keller-Osserman condition if and only if the BVP (1) *has at least one positive large solution on* **each** *smooth bounded domain* Ω.

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Assume that the strong Keller-Osserman condition is fulfilled and let u be a positive large solution of (1)*. Then*

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Assume that the strong Keller-Osserman condition is fulfilled and let u be a positive large solution of (1). *Then*

$$\lim_{x\to x_0}\frac{\displaystyle\int_{u(x)}^{\infty}\frac{dt}{\sqrt{2F(t)}}}{\delta(x)}=1,$$

Bifurcation for singular Lane-Emden-Fowler equations

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Implicit function theorem:

$$F(u,\lambda) = \Delta u + \lambda f(u), \quad F : X \times \mathbb{R} \to \mathbb{R},$$

where either

$$X := \{u \in C^{2,\alpha}(\overline{\Omega}) : \ u = 0 \text{ on } \partial\Omega\}, \ Y = C^{0,\alpha}(\overline{\Omega}), \quad (0 < \alpha < 1)$$

or

$$X = \{u \in W^{2,p}(\Omega) : \ u = 0 \text{ on } \partial\Omega\}, \quad Y = L^p(\Omega), \qquad p > N.$$

Then $F(0,0) = 0$ and $F_u(0,0) = \Delta$, hence (IFT) there is $\lambda^* > 0$ such that $\forall\, \lambda \in (0,\lambda^*)$, $\exists\, !\, u(\lambda) \in X$: $F(u(\lambda),\lambda) = 0$.

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Equations with convection terms and singular nonlinearities and potentials

$$\begin{cases} -\Delta u = p(d(x))g(u) + \lambda|\nabla u|^a + \mu f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Notation: $d(x) = \text{dist}(x, \partial\Omega)$, $\lambda \in \mathbb{R}$, $\mu > 0$, and $0 < a \leq 2$.

Assumptions: $g \in C^1(0,\infty)$ verifies

(g1): g is a positive decreasing function such that $\lim_{t \searrow 0} g(t) = +\infty$.

Function $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is Hölder continuous which is nondecreasing with respect to the second variable and such that f is positive in $\Omega \times (0,\infty)$.

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$$m := \lim_{t \to \infty} g(t) \in [0,\infty).$$

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Theorem

Assume that $a = 2$, $\lambda \geq 0$, $\mu > 0$ *and* $p \equiv 1, f \equiv 1$.

(i) The problem (4) *has a solution if and only if* $\lambda(m + \mu) < \lambda_1$;

(ii) Assume $\mu > 0$ *is fixed and let* $\lambda^* = \lambda_1/(m + \mu)$. *Then* (4) *has a unique solution* u_λ *for every* $0 \leq \lambda < \lambda^*$ *and the sequence* $(u_\lambda)_{0 \leq \lambda < \lambda^*}$ *is increasing with respect to* λ. *Moreover, if* $\limsup_{s \searrow 0} s^\alpha g(s) < \infty$, *for some* $\alpha \in (0, 1)$, *then the sequence of solutions* $(u_\lambda)_{0 \leq \lambda < \lambda^*}$ *has the following properties:*

(ii1) there exist two positive constants c_1, c_2 *depending on* λ *such that* $c_1 d(x) \leq u_\lambda \leq c_2 d(x)$ *in* Ω;

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Case (i) for $m = 0$, resp. $m > 0$.

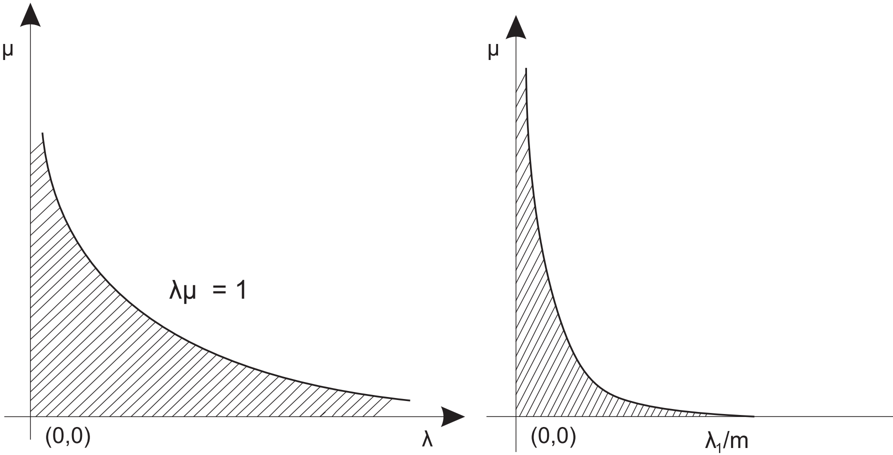

Figure: The bifurcation diagrams in Case (i).

Case (ii), $\lambda > 0$ and $\mu =$ fixed.

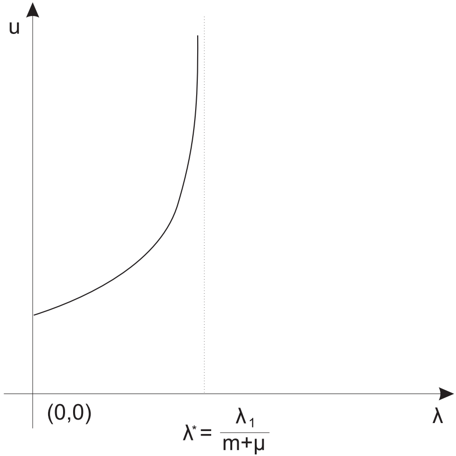

Figure: The bifurcation diagram in Case (ii).

Competition of terms and signs

Consider the problem

$$\begin{cases} -\Delta u \pm p(d(x))g(u) = \lambda f(x,u) + \mu|\nabla u|^a & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P)^{\pm}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary, $d(x) = \operatorname{dist}(x, \partial\Omega)$, $\lambda > 0$, $\mu \in \mathbb{R}$, and $0 < a \leq 2$.

Assumptions:

(i) $g \in C^1(0,\infty)$ is a positive decreasing function and

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$(g1)$ $\qquad \lim_{t \to 0^+} g(t) = +\infty.$

(ii) $f : \overline{\Omega} \times [0, \infty) \to [0, \infty)$ is a Hölder continuous function which is nondecreasing with respect to the second variable and such that f is positive on $\overline{\Omega} \times (0, \infty)$. Furthermore, f is either linear or f is sublinear with respect to the second variable. This last case means that f fulfills the hypotheses

$(f1)$ $\qquad (0, \infty) \ni t \longmapsto \dfrac{f(x,t)}{t}$ is nonincreasing, for all $x \in \overline{\Omega}$;

$(f2)$ $\qquad \lim_{t \to 0^+} \dfrac{f(x,t)}{t} = +\infty$ and $\lim_{t \to +\infty} \dfrac{f(x,t)}{t} =$ 0, uniformly for $x \in \overline{\Omega}$.

(p) $p : (0, +\infty) \to (0, +\infty)$ is nonincreasing and Hölder continuous.

Problem $(P)^+$

Theorem

Assume that $\int_0^1 p(t)g(t)dt = +\infty$. *Let* $\Phi : \overline{\Omega} \times [0, +\infty) \to \mathbb{R}$ *be a Hölder continuous function. Then the inequality boundary value problem*

$$\begin{cases} -\Delta u + p(d(x))g(u) \leq \Phi(x,u) + C\,|\nabla u|^2 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

has no classical solutions.

Corollary

Assume that $\int_0^1 p(t)g(t)dt = +\infty$. *Then problem* $(P)^+$ *has no classical solutions.*

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The next result shows that condition $\int_0^1 p(t)g(t)dt < +\infty$ is sufficient for the existence of a classical solution to $(P)^+$ provided $\mu \leq 0$ and $\lambda > 0$ is sufficiently large.

Theorem

Assume that $\displaystyle\int_0^1 p(t)g(t)dt < +\infty.$

(i) *If* $\mu = -1$, *then there exists* $\lambda^* > 0$ *such that* $(P)^+$ *has at least one classical solution if* $\lambda > \lambda^*$ *and no solution exists if* $0 < \lambda < \lambda^*$.

(ii) *If* $\mu = +1$ *and* $0 < a < 1$, *then there exists* $\lambda^* > 0$ *such that* $(P)^+$ *has at least one classical solution for all* $\lambda > \lambda^*$ *and no solution exists if* $0 < \lambda < \lambda^*$.

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Problem $(P)^-$

Theorem

Assume that $\int_0^1 tp(t)dt = +\infty$. *Then the inequality boundary value problem*

$$\begin{cases} -\Delta u + C|\nabla u|^2 \geq p(d(x))g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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Assume that $\int_0^1 tp(t)dt = +\infty$. *Then the problem* $(P)^-$ *has no classical solutions.*

Problem $(P)^-$ in the sublinear case

Theorem

Assume $\lambda = 1$, $\int_0^1 tp(t)dt < +\infty$ *and conditions* $(f1)$, $(f2)$, $(g1)$ *and* $0 < a \leq 2$ *are fulfilled.*

(i) *If* $0 < a < 1$, *then problem* $(P)^-$ *has at least one solution, for all* $\mu \in \mathbb{R}$;

(ii) *If* $1 < a \leq 2$, *then there exists* $\mu^* > 0$ *such that* $(P)^-$ *has at least one classical solution for all* $\mu < \mu^*$ *and no solution exists if* $\mu > \mu^*$.

Corollary

Assume $\mu = \pm 1$, $\int_0^1 tp(t)dt < +\infty$ *and conditions* $(f1)$, $(f2)$, $(g1)$ *and* $0 < a \leq 2$ *are fulfilled.*

(i) *If* $0 < a < 1$, *then problem* $(P)^-$ *has at least one solution, for all* $\lambda > 0$;

(ii) *If* $< 1 < a \leq 2$ *and* $\mu = -1$, *then problem* $(P)^-$ *has at least one solution, for all* $\lambda > 0$;

(iii) *If* $1 < a \leq 2$ *and* $\mu = +1$, *then there exists* $\lambda^* > 0$ *such that* $(P)^-$ *has at least one classical solution for all* $\lambda > \lambda^*$ *and no solution exists if* $\lambda < \lambda^*$.

Problem $(P)^-$ in the linear case

Consider the problem

$$\begin{cases} -\Delta u = p(d(x))g(u) + \lambda u + \mu|\nabla u|^a & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $\lambda > 0$ and p, g are as above.

Theorem

Assume that $\int_0^1 tp(t)dt < +\infty$ and conditions $(g1)$, $0 < a < 1$ are fulfilled. Then for $\mu \geq 0$ the problem (6) has solutions if and only if $\lambda < \lambda_1$.

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Example.

Consider the problem

$$\begin{cases} -\Delta u = d(x)^{-\alpha} u^{-\beta} + f(x,u) + \mu |\nabla u|^a & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Recall that if $\int_0^1 tp(t)dt < +\infty$ and μ belongs to a certain range, then this problem has at least one classical solution u_μ.

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Recall that if $\int_0^1 tp(t)dt < +\infty$ and μ belongs to a certain range, then this problem has at least one classical solution u_μ.

Theorem

(i) *If $\alpha \geq 2$, then the problem* (7) *has no classical solutions.*

(ii) *If $\alpha < 2$, then $\exists\, \mu^* \in (0, +\infty]$ (with $\mu^* = +\infty$ if $0 < a < 1$) such that* (7) *has at least one classical solution u_μ, $\forall -\infty < \mu < \mu^*$. Moreover, $\forall\, 0 < \mu < \mu^*$, $\exists\, 0 < \delta < 1$ and $\exists\, C_1, C_2 > 0$ such that*

(ii1) *If $\alpha + \beta > 1$, then*

$$C_1 d(x)^{\frac{2-\alpha}{1+\beta}} \leq u_\mu(x) \leq C_2 d(x)^{\frac{2-\alpha}{1+\beta}}, \quad \text{for all } x \in \Omega;$$

(ii2) *If $\alpha + \beta = 1$, then for all $x \in \Omega$ with $d(x) < \delta$*

$$C_1 d(x)(-\ln d(x))^{\frac{1}{2-\alpha}} \leq u_\mu(x) \leq C_2 d(x)(-\ln d(x))^{\frac{1}{2-\alpha}};$$

(ii3) *If $\alpha + \beta < 1$, then*

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(ii3) *If* $\alpha + \beta < 1$*, then*

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Theorem

(i) *If* $\alpha \geq 2$, *then the problem* (7) *has no classical solutions.*

(ii) *If* $\alpha < 2$, *then* $\exists\, \mu^* \in (0, +\infty]$ *(with* $\mu^* = +\infty$ *if* $0 < a < 1$*) such that* (7) *has at least one classical solution* u_μ, $\forall -\infty < \mu < \mu^*$. *Moreover,* $\forall\, 0 < \mu < \mu^*$, $\exists\, 0 < \delta < 1$ *and* $\exists\, C_1, C_2 > 0$ *such that*

(ii1) *If* $\alpha + \beta > 1$, *then*

$$C_1 d(x)^{\frac{2-\alpha}{1+\beta}} \leq u_\mu(x) \leq C_2 d(x)^{\frac{2-\alpha}{1+\beta}}, \quad \text{for all } x \in \Omega;$$

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Bifurcation for nonhomogeneous operators

The problem

$$\begin{cases} -\Delta u = \lambda u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega\,. \end{cases}$$

has an unbounded sequence of eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots\,.$$

In 1894, Poincaré established the existence of an infinite sequence of eigenvalues and corresponding eigenfunctions for the Laplace operator under the Dirichlet boundary condition. (For the first eigenvalue this was done by H. A. Schwarz in 1885 and for the second eigenvalue by E. Picard in 1893.) This key result is the beginning of spectral theory which has been one the major themes of functional analysis and its role in theoretical physics and differential geometry during the 20th century.

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The anisotropic case

$$\begin{cases} -\Delta u = \lambda a(x) u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

was considered by Bocher (1914), Minakshisundaram and Pleijel (1949), Hess and Kato (1980). Minakshisundaram and Pleijel proved that the above eigenvalue problem has an unbounded sequence of positive eigenvalues if $a \in L^\infty(\Omega)$, $a \geq 0$ in Ω, and $a > 0$ in $\Omega_0 \subset \Omega$, where $|\Omega_0| > 0$.

Case of an indefinite weight $a(\,\cdot\,)$: Szulkin & Willem (1999).

Quasilinear eigenvalue problems: Anane (1987), Lindqvist (1990).

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$$w(x) = \begin{cases} |x|^{-p} & \text{if } 1 < p < N \\ \left(|x| \log \dfrac{1}{|x|}\right)^{-N} & \text{if } p = N. \end{cases}$$

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First example

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{q(x)-2}u, & x \in \Omega \\ u = 0, & x \in \partial\Omega\,, \end{cases} \tag{8}$$

where p, q are continuous on $\overline{\Omega}$ and $\Delta_{p(x)}u := \mathrm{div}\,(|\nabla u|^{p(x)-2}\nabla u)$.

Abstract framework. Assume $p \in C(\overline{\Omega})$ and $p > 1$, on $\in \overline{\Omega}$. Set

$$C_+(\overline{\Omega}) = \{h;\; h \in C(\overline{\Omega}),\; h(x) > 1 \text{ for } x \in \overline{\Omega}\}.$$

For $h \in C_+(\overline{\Omega})$, define

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For any $p \in C_+(\overline{\Omega})$, define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u;\ \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}.$$

Luxemburg norm:

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)}\,dx \le 1\right\}.$$

Then $L^{p(x)}(\Omega)$ is separable and reflexive Banach space.

$L^{p'(x)}(\Omega)$: the dual space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$.

For $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder type inequality

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(x)} |v|_{p'(x)} \tag{9}$$

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Modular mapping. $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\, dx.$$

If (u_n), $u \in L^{p(x)}(\Omega)$ then

$$|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}$$

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}$$

$$|u_n - u|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho_{p(x)}(u_n - u) \to 0.$$

Define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm $\|u\| = |\nabla u|_{p(x)}$. The space $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is separable and reflexive. If $s \in C_+(\overline{\Omega})$ and $s(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ is compact and continuous, where $p^\star(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^\star(x) = +\infty$ if $p(x) \geq N$.

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Theorem

Assume that $q(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ and

$$1 < \min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x) < \max_{x\in\overline{\Omega}} q(x).$$

Then there exists $\lambda^\star > 0$ such that any $\lambda \in (0, \lambda^\star)$ is an eigenvalue for problem (8).

Remark. If $\max_{x\in\overline{\Omega}} p(x) < \min_{x\in\overline{\Omega}} q(x)$ and $q(x) < p^\star(x)$ then a mountain pass argument shows that **any** $\lambda > 0$ is an eigenvalue of problem (8).

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Second example

Consider the problem

$$\begin{cases} -\mathrm{div}((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u) = \lambda |u|^{q(x)-2}u, & x \in \Omega \\ u = 0, & x \in \partial\Omega\,, \end{cases} \tag{10}$$

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$$\lambda_1 := \inf_{u \in W_0^{1,p_1(x)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \frac{1}{p_1(x)} |\nabla u|^{p_1(x)}\,dx + \int_\Omega \frac{1}{p_2(x)} |\nabla u|^{p_2(x)}\,dx}{\displaystyle\int_\Omega \frac{1}{q(x)} |u|^{q(x)}\,dx}.$$

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Any $\lambda \in [\lambda_1, \infty)$ is an eigenvalue of problem (10). Moreover, there exists a positive constant λ_0 such that $\lambda_0 \leq \lambda_1$ and any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem (10).

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Sketch of the proof.

Step 1: $\lambda_1 > 0$.

Step 2: λ_1 is an eigenvalue of problem (10).

STEP 3: any $\lambda \in (\lambda_1, \infty)$ is an eigenvalue of problem (10).

Define

$$J_1(u) = \int_\Omega |\nabla u|^{p_1(x)}\,dx + \int_\Omega |\nabla u|^{p_2(x)}\,dx,$$

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$$\lambda_0 = \inf_{v \in W_0^{1,p_1(x)}(\Omega)\setminus\{0\}} \frac{J_1(v)}{I_1(v)} > 0.$$

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Third example

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u + \\ |u|^{q(x)-2}u = \lambda g(x)|u|^{r(x)-2}u \quad \text{for } x \in \Omega \\ u = 0 \qquad \text{for } x \in \partial\Omega\,, \end{cases} \tag{11}$$

where $p, q, r : \overline{\Omega} \to [2, \infty)$ are Lipschitz; $g : \overline{\Omega} \to [0, \infty)$ is a measurable function for which there exists a nonempty set $\Omega_0 \subset \Omega$ with $|\Omega_0| > 0$ such that $g(x) > 0$ for any $x \in \Omega_0$.

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Assumptions:

$$2 \le p^- \le p^+ < N\,,$$

$$p^+ < r^- \le r^+ < q^- \le q^+ < \frac{Np^-}{N-p^-}\,.$$

$$g \in L^\infty(\Omega) \cap L^{p_0(x)}(\Omega)\,,$$

where $p_0(x) = p^\star(x)/(p^\star(x) - r^-)\ \forall\, x \in \overline{\Omega}$.

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Theorem

The following properties hold true:
(i) $0 < \lambda_0 \le \lambda_1$;
(ii) any $\lambda \in [\lambda_1, \infty)$ *is an eigenvalue of problem* (11) *while any* $\lambda \in (0, \lambda_0)$ *is not an eigenvalue of problem* (11).