A glimpse of noncommutative curvature

Henri Moscovici

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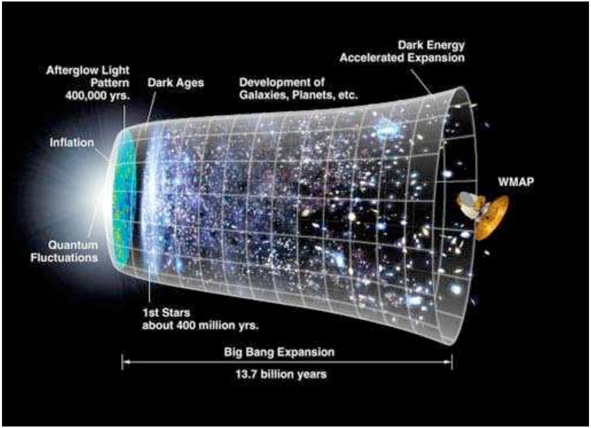
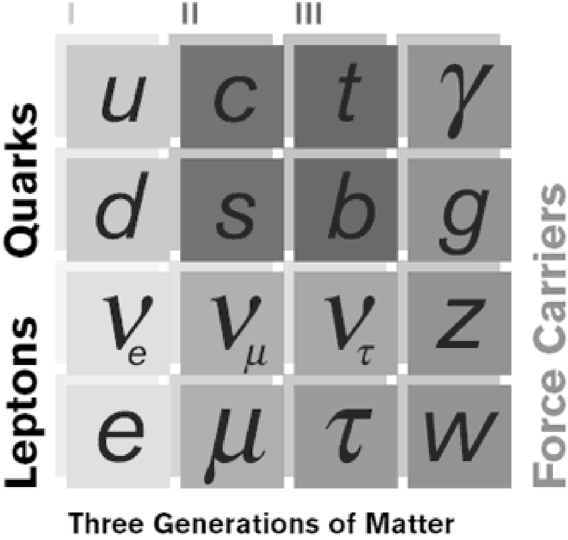
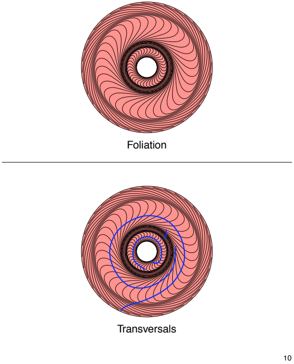
IMAR, Bucharest


Dark Energy
Accelerated Expansion
Afterglow Light
Pattern
400,000 yrs.
Dark Ages
Development of
Galaxies, Planets, etc.
Inflation
WMAP
Quantum
Fluctuations
1st Stars
about 400 million yrs.
Big Bang Expansion
13.7 billion years

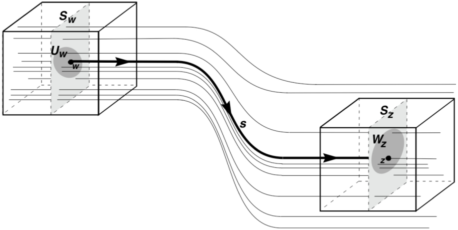

I
II
III
Quarks
Leptons
Force Carriers
u
c
t
γ
d
s
b
g
ν_e
ν_μ
ν_τ
Z
e
μ
τ
W
Three Generations of Matter

Model for noncommutative space: leaf space

Complete transversal



Holonomy



Spectral model: Dirac operator

- **Spin^c manifold**: closed Riemannian manifold (M^{2n}, g) such that the Clifford algebra bundle has "square root":

$$\wedge^\bullet T^*M \otimes \mathbb{C} \cong \mathrm{Cliff}(M) \otimes \mathbb{C} \cong \mathrm{End}(\mathcal{S}) \cong \mathcal{S}^* \otimes \mathcal{S}$$

- **Spin^c-Dirac operator**: "square root" of Laplace operator:

$$\not{D} = \not{D}^*, \qquad \not{D}^2 = \Delta^{\mathcal{S}} + \frac{1}{4}\kappa_g \quad \text{(scalar curvature)}$$

- **Dirac spectral triple**: $\quad (C^\infty(M^{2n}), L^2(\mathcal{S}^\pm), \not{D}^\pm)$,

$$\not{D} = \not{D}^*, \qquad \not{D} \circ \gamma = -\gamma \circ \not{D}, \qquad \gamma = \text{chirality operator}$$

$$(\not{D}^2 + 1)^{-\frac{1}{2}} \in \mathcal{L}^{2n+\varepsilon},$$

$$[\not{D}, f] := c(df) \in \mathcal{L}(\mathfrak{H}), \quad \forall f \in \mathcal{A} = C^\infty(M).$$

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Spectral presentation of spaces

- **Spectral triple**: $(\mathcal{A}, \mathfrak{H}, D)$, $\mathcal{A} \subset \mathcal{L}(\mathfrak{H})$ involutive subalgebra

$$D = D^* \quad \text{unbounded} \qquad (F = \mathrm{Sign}\, D)$$
$$(D^2 + 1)^{-\frac{1}{2}} \in \mathcal{L}^p(\mathfrak{H}), \quad \text{for some} \quad p \geq 1$$
$$[D, a] := D\,a - a\,D \quad \text{bounded} \quad \forall\, a \in \mathcal{A}\,.$$

- **Grading (= even)**: $\gamma = \gamma^* \in \mathcal{L}(\mathfrak{H})\,, \quad \gamma^2 = 1,$
  $a\,\gamma = \gamma\,a, \quad \forall\, a \in \mathcal{A}, \qquad D\,\gamma = -\gamma\,D\,.$

- **Simplest example: the circle**

$$u^{-1}[D, u] = 1, \quad uu^* = u^* u = 1,$$
$$\rightsquigarrow\Rightarrow \qquad S^1 = \mathrm{Spec}(\mathcal{A}), \quad \mathcal{A} = C^\infty(u), \quad D = -\frac{1}{i}\frac{d}{dx}$$
$$\Rightarrow \qquad \mathrm{d}(x_1, x_2) = \sup_{\|[D,f]\| \leq 1} |x_1(f) - x_2(f)|.$$

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Local-Global Principle in Global Analysis

Theorem (Atiyah-Singer)

M^{2n} = closed spin manifold, $D^{\pm}$ = Dirac operator, E = vector bundle

$$\mathrm{Index}(D_E^+) = \int_M \hat{A}(R_g)\,ch(E,\nabla)$$

- cohomological expression to the index pairing:

$$\begin{array}{ccccc}
K_*(M) & \otimes & K^*(M) & \xrightarrow{\mathrm{Index}} & \mathbb{Z} \\
\Big\downarrow ch_* & & \Big\downarrow ch^* & & \Big\downarrow \\
H_*(M) & \otimes & H^*(M) & \longrightarrow & \mathbb{R}
\end{array}$$

- $\lim_{t\nearrow\infty} \mathrm{Tr}\left(\gamma e^{-tD_E^2}\right) = \lim_{t\searrow 0} \mathrm{Tr}\left(\gamma e^{-tD_E^2}\right)$

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Cyclic cohomology and index pairing [Connes (1981)]

Let A be a C^*-algebra represented in $\mathcal{L}(\mathfrak{H})$, and let

$$F \in \mathcal{L}(\mathfrak{H}), \quad F^2 = I, \quad F\gamma = -\gamma F, \quad [F, a] \in \mathcal{L}^p(\mathfrak{H}), \quad \forall\, a \in \mathcal{A} \subset A.$$

CONNES turned the 'parametrix' index formula

$$\mathrm{Index}\left(e\,F^+\,e\right) \;=\; (-1)^\ell\, \mathrm{Tr}\left(\gamma e\,[F, e]^n\right), \qquad n = 2\ell > p,$$

into a **cyclic cohomological formulation of the index pairing**:

$$K^*(A) \otimes K_*(A) \to HC^*(\mathcal{A}) \otimes HC_*(\mathcal{A}) \to \mathbb{Z}$$

$$\mathrm{Index}\left(e\,F^+\,e\right) \;=:\; \langle [F], [e] \rangle \;=\; \langle ch^*(F), ch_*(e) \rangle$$

where

- $ch^*(F) \;:= [\tau_F^n] \in HC^0(\mathcal{A})$, with $\tau_F^n \in HC_\lambda^n(\mathcal{A})$ given by

  $$\tau_F^n(a^0, a^1, \ldots, a^n) := \; c_n \mathrm{Tr}\left(\gamma\, F[F, a^0] \ldots [F, a^n]\right), \quad a^i \in \mathcal{A};$$

- $e^2 = e \in M_\infty(\mathcal{A})$, $ch_\bullet(e) \in HP_0(\mathcal{A})$ given by

  $$:= \mathrm{tr}_0(e) + \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{k!} \mathrm{tr}_{2k}\Big(\big(e - \frac{1}{2}\big) \otimes e^{\otimes(2k)}\Big).$$

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Locality and zeta-function regularization

- Spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ with discrete dimension spectrum: $\exists$ discrete subset $\Sigma \subset \mathbb{C}$, such that $\zeta_b(z) = \mathrm{Tr}(b\,|D|^{-z})$, $\Re z > p$, $b \in \mathcal{B}$, admit holomorphic extensions to $\mathbb{C} \setminus \Sigma$, and $\Gamma(z)\,\zeta_b(z)$ decays rapidly on finite vertical strips.
  Simple dimension spectrum: Σ consists of simple poles.
- Then the residue functional

$$⨍ P := \mathrm{Res}_{z=0}\,\zeta_P(2z)\,, \qquad P \in \Psi^\bullet(\mathcal{A}, \mathfrak{H}, D)$$

  is an algebraic trace (assuming s. d. s.).
- $⨍ P$ vanishes on $\Psi^\bullet(\mathcal{A}, \mathfrak{H}, D) \cap \mathcal{L}^1(\mathfrak{H})$, and thus descends to a trace on the algebra of symbols $\mathcal{CS}^\bullet(\mathcal{A}, \mathfrak{H}, D)$.

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Local Index Formula in NcG [Connes - M, 1995]

For an (odd) spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ such that $\exists$ residues

$$⨍ T := \mathrm{Res}_{s=0} \mathrm{Tr}(T|D|^{-2s}), \quad T \in \{\mathcal{A}, [D, \mathcal{A}], |D|^{-z};\, z \in \mathbb{C}\}$$

1. $[(\varphi_n)_{n=1,3,\ldots}]$ is a cocycle in the (b, B)-bicomplex of $\mathcal{A}$,

$$\varphi_n(a^0, \ldots, a^n) = \sum_{\mathbf{k}} c_{n,\mathbf{k}} ⨍ a^0 [D, a^1]^{(k_1)} \ldots [D, a^n]^{(k_n)} |D|^{-n-2|k|}$$

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Example: the Dirac spectral triple

1. The dimension spectrum of the Dirac spectral triple $(C^\infty(M^m), L^2(\mathcal{S}), D\!\!\!/)$ is simple, and $= \{1, \ldots, m\}$
2. $⨍ P =$ Wodzicki-Guillemin residue $(P), \quad \forall P \in \Psi DO(M)$
3. $⨍ f^0 [D\!\!\!/, f^1]^{(k_1)} \ldots [D\!\!\!/, f^n]^{(k_n)} \, |D\!\!\!/|^{-(n+2|k|)} = 0 \,, \quad \text{if} \quad |k| > 0$
4. $⨍ f^0 [D\!\!\!/, f^1] \ldots [D\!\!\!/, f^n] \, |D\!\!\!/|^{-n} \; =$
$$c_n \int_M \det \left(\frac{\nabla^2 / 4\pi i}{\sinh \nabla^2 / 4\pi i} \right)^{\frac{1}{2}} \wedge f^0 \, df^1 \wedge \ldots \wedge df^n$$
5. Connes' H-K-R iso: $\quad HP^*(C^\infty(M^m)) \cong H_*^{\mathrm{dR}}(M, \mathbb{C})$

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Twisted spectral triples

- **Twisted commutators** : $(\mathcal{A}, \mathfrak{H}, D, \sigma)$, where $\sigma \in \mathrm{Aut}(\mathcal{A})$,

$$[D, a]_\sigma := D\,a \,-\, \sigma(a)\,D$$

- **Example:** **codimension 1 foliation**:
$\mathcal{A}_G = C^\infty(S^1) \rtimes G, \qquad f\,U_\phi^{-1} \cdot g\,U_\psi^{-1} := f\,\phi^*(g)\,U_{\phi\psi}^{-1},$
acting on $\mathfrak{H} = L^2(S^1) := \{\xi = \xi(x)\,dx^{\frac{1}{2}} \,|\, \xi \in L^2\}$, via

$$U_\phi^{-1}\,\xi = \phi^*(\xi), \qquad \xi \in L^2(S^1);$$
$$\sigma(f\,U_\phi^{-1}) = \phi'\,f\,U_\phi^{-1}, \quad f \in C^\infty(S^1), \quad \phi \in G,$$
$$D \equiv \not{D} = \frac{1}{i}\,\frac{d}{dx}, \qquad [\not{D}, U_\phi^{-1}]_\sigma = \frac{1}{2i}\,\frac{d}{dx}\left(\log \phi'\right)\,U_\phi^{-1}.$$

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$$U_\phi^{-1}\, \xi \;=\; \phi^*(\xi), \qquad \xi \in L^2(S^1);$$
$$\sigma(f\, U_\phi^{-1}) = \phi'\, f\, U_\phi^{-1}, \quad f \in C^\infty(S^1), \quad \phi \in G,$$
$$D \equiv \not{D} = \frac{1}{i}\,\frac{d}{dx}, \qquad [\not{D}, U_\phi^{-1}]_\sigma = \frac{1}{2i}\,\frac{d}{dx}\left(\log\, \phi'\right)\, U_\phi^{-1}.$$

Conformal twisting

- **Dirac spectral triple** Let $g' = e^{-4h} g$. Natural identifications of spinorial frames [*Hitchin, Bourguignon-Goduchon*] give to the transported Dirac operator the expression

$$ {}^{g}\not{D}^{g'} = e^{(n+1)h} \circ \not{D}^{g} \circ e^{-(n-1)h} ; $$

when taking into account the change $\mathrm{vol}_{g'} = e^{-2nh} \, \mathrm{vol}_g$,

$$ \Longrightarrow \qquad {}^{g}\not{D}^{g'} = e^{h} \, \not{D}^{g} \, e^{h} . $$

- **General spectral triple**. Given $(\mathcal{A}, \mathfrak{H}, D)$ and $h = h^* \in \mathcal{A}$ define $D_h = e^h \, D \, e^h$. $[D_h, a]$ is unbounded if h is not central. However, with the twisting automorphism

$$ \sigma(a) = e^{2h} \, a \, e^{-2h} \Longrightarrow [D_h, a]_\sigma := D_h \, a - \sigma(a) \, D_h \in \mathcal{L}(\mathfrak{H}). $$

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Noncommutative 2-torus

- $A_\theta := C^*\{U, V\}$, U, V unitaries, $VU = e^{2\pi i\theta} UV$, $\quad \theta \in \mathbb{R} \setminus \mathbb{Q}$.
- Action of $s \in \mathbb{R}^2$, $\qquad \alpha_s(U^m V^n) = e^{i<s\,,\,(m,n)>} U^m V^n$.
- Smooth subalgebra:

$$A_\theta^\infty = \Big\{ \sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n : |m|^k |n|^q |a_{m,n}| \le C_{k,q}, \qquad \forall k, q > 0 \Big\}.$$

- Infinitesimal generators : $\qquad$ derivations $\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty$,

$$\delta_1(U) = U,\ \delta_1(V) = 0, \quad \delta_2(U) = 0,\ \delta_2(V) = V.$$

- Unique normalized trace: $\qquad \tau(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n) = a_{0,0}$.
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- **Conformal state**: $\varphi(a) = \tau(ae^{-h})$,

$$\varphi(ab) = \varphi(b\sigma_i(a)), \quad a, b \in A_\theta ,$$

i.e. φ is KMS with 1-parameter group

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Asssociated spectral triples over A_θ

- $\partial_z = \delta_1 + z\delta_2, \qquad \partial_z^* = \delta_1 + \bar{z}\delta_2, \qquad z = x + iy, \quad y > 0$,

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Gauss-Bonnet

For a Riemannian surface Σ, $\zeta(s) = \mathrm{Tr}(\Delta^{-s})$ satisfies

$$\zeta(0) + \dim \mathrm{Ker}\Delta = \frac{1}{12\pi}\int_\Sigma K = \frac{1}{6}\chi(\Sigma),$$

in particular, $\zeta(0)$ is a topological invariant.

Theorem (Connes-Tretkoff)

For $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $z = 0$, let $h \in A_\theta^\infty$, and $\triangle_\varphi = \partial_\varphi^* \partial_\varphi \sim e^{\frac{h}{2}} \triangle e^{\frac{h}{2}}$. Then $\zeta_{\triangle_\varphi}(0)$ of the operator is independent of h.

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Pseudodifferential calculus [Connes,1980]

Symbols of order n: $\mathcal{S}_n$ formed of smooth maps $\rho : \mathbb{R}^2 \to A_\theta^\infty$ such that $\forall\, i_1, i_2, j_1, j_2 \in \mathbb{Z}^+$, $\exists$ constant $C_{i,j}$,

$$||\delta_1^{i_1}\delta_2^{i_2}\partial_1^{j_1}\partial_2^{j_2}\rho(\xi)|| \leq C_{i,j}(1+|\xi|)^{n-j_1-j_2},$$

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$$\lim_{\lambda\to\infty} \lambda^{-n}\rho(\lambda\xi_1, \lambda\xi_2) = \sigma(\xi_1, \xi_2).$$

Associated pseudodifferential operator on A_θ^∞ :

$$P_\rho(a) = (2\pi)^{-2}\iint e^{-is\cdot\xi}\rho(\xi)\alpha_s(a)\, ds\, d\xi.$$

Elliptic: $\quad \rho(\xi)$ is invertible for $\xi \neq 0$, and $\exists$ constant c such that

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Pseudodifferential calculus [Connes,1980]

Symbols of order n: $\mathcal{S}_n$ formed of smooth maps $\rho : \mathbb{R}^2 \to A_\theta^\infty$ such that $\forall\, i_1, i_2, j_1, j_2 \in \mathbb{Z}^+$, $\exists$ constant $C_{i,j}$,

$$||\delta_1^{i_1}\delta_2^{i_2}\partial_1^{j_1}\partial_2^{j_2}\rho(\xi)|| \leq C_{i,j}(1+|\xi|)^{n-j_1-j_2},$$

and $\quad \exists \quad \sigma \in C^\infty(\mathbb{R}^2 \setminus \{0\}, A_\theta^\infty)$ such that

$$\lim_{\lambda\to\infty} \lambda^{-n}\rho(\lambda\xi_1, \lambda\xi_2) = \sigma(\xi_1, \xi_2).$$

Associated pseudodifferential operator on A_θ^∞ :

$$P_\rho(a) = (2\pi)^{-2} \iint e^{-is\cdot\xi}\rho(\xi)\alpha_s(a)\, ds\, d\xi.$$

Elliptic: $\quad \rho(\xi)$ is invertible for $\xi \neq 0$, and $\exists$ constant c such that

$$||\rho(\xi)^{-1}|| \leq c(1+|\xi|)^{-n}, \qquad \xi >> 0.$$

Symbolic calculations

$$\sigma(k\triangle k+1) \;=\; a_2(\xi)+a_1(\xi)+a_0(\xi), \qquad k=e^{\frac{h}{2}}$$

where

$$a_2(\xi)=\xi_1^2k^2+|z|^2\xi_2^2k^2+2x\xi_1\xi_2k^2,$$

$$a_1(\xi)=2\xi_1k\delta_1(k)+2|z|^2\xi_2k\delta_2(k)+2x\xi_1k\delta_2(k)+2x\xi_2k\delta_1(k),$$

$$a_0(\xi)=k\delta_1^2(k)+|z|^2k\delta_2^2(k)+2xk\delta_1\delta_2(k).$$

Inverse symbol $= b_0+b_1+b_2+\cdots,$ with

$$b_n=-\sum_{\substack{2+j+\ell_1+\ell_2-i=n,\\ 0\le j<n,\,0\le i\le 2}}\frac{1}{\ell_1!\ell_2!}\partial_1^{\ell_1}\partial_2^{\ell_2}(b_j)\delta_1^{\ell_1}\delta_2^{\ell_2}(a_i)b_0,$$

of order $-2-n$.

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$$a_1(\xi) = 2\xi_1 k\delta_1(k) + 2|z|^2\xi_2 k\delta_2(k) + 2x\xi_1 k\delta_2(k) + 2x\xi_2 k\delta_1(k),$$

$$a_0(\xi) = k\delta_1^2(k) + |z|^2 k\delta_2^2(k) + 2xk\delta_1\delta_2(k).$$

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$$b_n = -\sum_{\substack{2+j+\ell_1+\ell_2-i=n,\\ 0\le j<n,\, 0\le i\le 2}} \frac{1}{\ell_1!\ell_2!}\partial_1^{\ell_1}\partial_2^{\ell_2}(b_j)\delta_1^{\ell_1}\delta_2^{\ell_2}(a_i)b_0,$$

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$$b_0 = (\xi_1^2 k^2 + |z|^2 \xi_2^2 k^2 + 2x\xi_1\xi_2 k^2 + 1)^{-1},$$
$$b_1 = -(b_0 a_1 b_0 + b_1 a_1 b_0 + \partial_1(b_0)\delta_1(a_2)b_0 + \partial_2(b_0)\delta_2(a_2)b_0),$$

$$\begin{gathered}
b_2 = -b_0 k \delta_1^2(k) b_0 - 2x b_0 k \delta_1\delta_2(k) b_0 - |z|^2 b_0 k \delta_2^2(k) b_0 + \\
6\xi_1^2 b_0^2 k^2 \delta_1(k)^2 b_0 + \xi_1^2 b_0^2 k^2 \delta_1^2(k) b_0 k + 5\xi_1^2 b_0^2 k^3 \delta_1^2(k) b_0 + \\
2\xi_1^2 b_0 k \delta_1(k) b_0 \delta_1(k) b_0 k + 6\xi_1^2 b_0 k \delta_1(k) b_0 k \delta_1(k) b_0 + \\
6x\xi_1^2 b_0^2 k^2 \delta_1(k)\delta_2(k) b_0 + 6x\xi_1^2 b_0^2 k^2 \delta_2(k)\delta_1(k) b_0 + \\
2x\xi_1^2 b_0^2 k^2 \delta_1\delta_2(k) b_0 k + 10x\xi_1^2 b_0^2 k^3 \delta_1\delta_2(k) b_0 + \\
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12x\xi_1\xi_2 b_0^2 k^2 \delta_1(k)^2 b_0 + 2x\xi_1\xi_2 b_0^2 k^2 \delta_1^2(k) b_0 k + \\
2x\xi_1\xi_2 b_0^2 k^2 \delta_1^2(k) b_0 k 10x\xi_1\xi_2 b_0^2 k^3 \delta_1^2(k) b_0 + \ldots
\end{gathered}$$

$(\ldots + 100\text{s more such terms})$

Scalar curvature

Theorem (Connes)

Let θ be an irrational number and k an invertible positive element of A_θ^∞. Then the value at the origin of the zeta function of the twisted spectral triple $(A_\theta^\infty, \mathcal{H}, D)$ is given for any $a \in A_\theta^\infty$ by

$$\mathrm{Tr}(a\,|D|^{-s})|_{s=0} = \tau\left(a\left(K(\log\Delta)(\triangle(h)) + \sum_{j=1}^{2} K(\nabla_{(1)}, \nabla_{(2)})\delta_j(h)\delta_j(h)\right)\right).$$

Thus, the scalar curvature of the metric D_φ with $\triangle_\varphi \cong e^{\frac{h}{2}}\triangle e^{\frac{h}{2}}$ is

$$\kappa_h = K(\log\Delta)(\triangle(h)) + \sum_{j=1}^{2} K(\nabla_{(1)}, \nabla_{(2)})\delta_j(h)\delta_j(h)\,,$$

where $\nabla_{(i)}$ is the action of $\log\Delta$ on the i-th factor of the product.

Scalar curvature functions

$$K(u) = \frac{e^u - e^{u/2}u - 1}{\left(-1 + e^{u/2}\right)^3}, \qquad K(s,t) = \left(1 + \mathrm{ch}\left(\frac{s+t}{2}\right)\right) K_0(s,t)$$

$$K_0(s,t) = \frac{-t(s+t)\mathrm{ch}(s) + s(s+t)\mathrm{ch}(t) - (s-t)(s+t+\mathrm{sh}(s)+\mathrm{sh}(t)-\mathrm{sh}(s+t))}{st(s+t)\mathrm{sh}\left(\frac{s}{2}\right)\mathrm{sh}\left(\frac{t}{2}\right)\mathrm{sh}\left(\frac{s+t}{2}\right)^2}$$

$$K_0(t,s) = -K_0(s,t); \quad K_0(-s,-t) = -K_0(s,t).$$

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Conformal index of a spectral triple

$(\mathcal{A}, \mathfrak{H}, D) =$ p-summable spectral triple with good pseudodifferential calculus, *i.e.* $\exists$ asymptotic expansions

$$\mathrm{Tr}\left(A\, e^{-tD_s^2}\right) =_{t \searrow 0} \sum_{j=0}^{\infty} a_j(A, s)\, t^{\frac{j-N-p}{2}} + 0(1),$$

for any $A \in \mathcal{D}^N(\mathcal{A}, \mathfrak{H}, D)$; $\qquad h = h^* \in \mathcal{A}, \quad D_s = e^{sh}\, D\, e^{sh}$.
Moreover, assume good resolvent approximation, implying that these expansions can be differentiated in the term-by-term in $s \in [-1, 1]$.

Theorem (Conformal index à la Branson-Ørsted)

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Proof.

$$D_s = e^{sh}\, D\, e^{sh} \Longrightarrow \frac{de^{-D_s^2}}{ds} = -\int_0^1 e^{-uD_s^2}\, \frac{dD_s^2}{ds}\, e^{-(1-u)D_s^2}\, du$$

$$\frac{dD_s}{ds} \equiv \frac{d}{ds}\left(e^{sh}\, D\, e^{sh}\right) = hD_s + D_s h,$$

$$\begin{aligned} \frac{d}{ds}\mathrm{Tr}\left(e^{-tD_s^2}\right) &= -t\,\mathrm{Tr}\left((D_s^2 h + 2D_s h D_s + hD_s^2)e^{-tD_s^2}\right) \\ &= -4t\,\mathrm{Tr}\left(h\, D_s^2\, e^{-tD_s^2}\right) = 4t\,\frac{d}{dt}\mathrm{Tr}\left(h\, e^{-tD_s^2}\right). \end{aligned}$$

Equating the coefficients of the asymptotic expansions,

$$\frac{d}{ds}a_j(s) = 2(j-p)\, a_j(h,s), \quad \text{in particular} \quad \frac{d}{ds}a_p(s) = 0.$$

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Corollary (1) (Connes-Tretkoff, Khalkhali-Fathi Zadeh)

Gauss-Bonnet for the noncommutative 2-torus with any translation invariant metric.

Proof.

Same argument as above, but for the Laplacian $\triangle_s = e^{sh} \triangle e^{sh}$:

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Corollary (2) (Connes-Chamseddine)

The constant term in the SM spectral action $\mathrm{Tr}\, F \left(\frac{D^2}{m^2} \right) + \langle \Psi | D | \Psi \rangle$ *is independent of dilaton field rescaling* $D^2 \mapsto e^h D^2 e^h$.

$$\mathrm{Tr}\, F \left(\frac{D^2}{m^2} \right) \simeq \frac{m^4}{2} f_0 ⨍ D^{-4} + \frac{m^2}{2} f_2 ⨍ D^{-2} + f_4 \, \zeta_{D^2}(0) + \ldots$$

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