THE BING-BORSUK AND THE BUSEMANN CONJECTURES

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Bucharest, June 10, 2009
Definition: Let $Y$ be a metric space. $Y$ is said to be an *absolute neighborhood retract (ANR)* provided for every closed embedding $e : Y \to Z$ of $Y$ into a metric space $Z$, there is an open neighborhood $U$ of the image $e(Y)$ which retracts to $e(Y)$. That is, there is a continuous surjection $r : U \to e(Y)$ with $r(x) = x$ for all $x \in e(Y)$. 
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Fact: Let $Y$ be a finite-dimensional, locally contractible separable metric space. Then $Y$ is an ANR.

Definition: A topological space $X$ is said to be homogeneous if, for any two points $x_1, x_2 \in X$, there is a homeomorphism of $X$ onto itself taking $x_1$ to $x_2$. 
**Definition:** A (closed) *topological n-manifold* \( n \in \mathbb{N} \) is a connected, compact \( n \)-dimensional metric space which is *locally Euclidean* (i.e. homeomorphic to \( \mathbb{R}^n \)).
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**Jakobsche Theorem (1978):** In dimension $n = 3$, the Bing-Borsuk Conjecture implies the Poincaré Conjecture.
**Definition:** An $n$-dimensional compact metric space $X$ is called an $n$-dimensional Cantor manifold if whenever $X$ can be expressed as the union $X = X_1 \cup X_2$ of its proper closed subsets, then $\dim(X_1 \cap X_2) \geq n - 1$. 
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**Brouwer Theorem (1910):** Every topological $n$-manifold has the invariance of domain property.

**Łysko Theorem (1976):** Every finite-dimensional connected homogeneous ANR space is a Cantor manifold and it has the invariance of domain property.
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**Definition:** An $n$-dimensional ($n \in \mathbb{N}$) locally compact Hausdorff space $X$ is called a \(\mathbb{Z}\)-homology $n$-manifold (\(n\)-hm\(\mathbb{Z}\)) if for every point $x \in X$ and all $k \in \mathbb{N}$, $H_k(X, X - \{x\}; \mathbb{Z}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z})$. 
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**Bredon Theorem (1967):** If $X$ is an $n$-dimensional homogeneous ENR ($n \in \mathbb{N}$) and for some (and, hence all) points $x \in X$, the groups $H_k(X, X - \{x\}; \mathbb{Z})$ are finitely generated, then $X$ is a $\mathbb{Z}$-homology $n$-manifold.
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**Remark:** This theorem was reproved by Bryant in 1987 with a more geometric argument.
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The following classical result was proved for $n \leq 2$ by Wilder, for $n = 3$ by Armentrout, for $n = 4$ by Quinn and for $n \geq 5$ by Siebenmann.
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**Cell-like Approximation Theorem:** Every cell-like map between topological manifolds is a near-homeomorphism.
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**Remark:** By the work of Quinn, these nonresolvable generalized manifolds must be *totally singular.*
**Definition:** A metric space $X$ is said to have the *disjoint disks property* (DDP) if for every $\varepsilon > 0$ and every pair of maps $f, g : B^2 \to X$ there exist $\varepsilon$-approximations $f', g' : B^2 \to X$ with disjoint images.
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**Bryant-Ferry-Mio-Weinberger Theorem (2007):** For every $n \geq 7$ there exist non-resolvable generalized $n$-manifolds with the DDP.
Bing-Borsuk Conjecture: Resolvability and the Modified Conjecture

**Bryant-Ferry-Mio-Weinberger Conjecture (2007):** Every generalized $n$-manifold ($n \geq 7$) satisfying the disjoint disks property, is homogeneous.
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**Remark:** If the Bryant-Ferry-Mio-Weinberger Conjecture is true, then the Bing Borsuk conjecture is false for $n \geq 7$. 
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Remark: If the Bryant-Ferry-Mio-Weinberger Conjecture is true, then the Bing Borsuk conjecture is false for $n \geq 7$.

Bryant Modified Bing-Borsuk Conjecture (2002): Every homogeneous ($n \geq 3$)-dimensional ENR is a generalized $n$-manifold.
**Definition:** A space $X$ is *homologically arc-homogeneous* provided that for every path $\alpha : [0, 1] \to X$, the inclusion induced map

$$H_*(X \times 0, X \times 0 - (\alpha(0), 0); \mathbb{Z}) \to H_*(X \times I, X \times I - \Gamma(\alpha); \mathbb{Z})$$

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The following result answers in affirmative a question asked by Quinn at the 2003 Oberwolfach workshop on exotic homology manifolds.

**Bryant Theorem (2006):** Every $n$-dimensional homologically arc-homogeneous ENR is a generalized $n$-manifold.
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Bing-Borsuk Conjecture: History of Results

In 1991 Repovš, Skopenkov and Ščepin proved the *smooth version* of the Bing-Borsuk Conjecture.

**Definition:** A subset $K \subset \mathbb{R}^n$ is said to be $C^1$–*homogeneous* if for every pair of points $x, y \in K$ there exist neighborhoods $O_x, O_y \subset \mathbb{R}^n$ of $x$ and $y$, respectively, and a $C^1$–*diffeomorphism*

$$h : (O_x, O_x \cap K, x) \rightarrow (O_y, O_y \cap K, y),$$

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i.e. $h$ and $h^{-1}$ have continuous first derivatives.

**Repovš-Skopenkov-Ščepin Theorem (1991):** Let $K$ be a locally compact (possibly nonclosed) subset of $\mathbb{R}^n$. Then $K$ is $C^1$–*homogeneous* if and only if $K$ is a $C^1$–*submanifold* of $\mathbb{R}^n$. 
**Remark:** This theorem clearly does not work for all *homeomorphisms*, a counterexample is the *Antoine Necklace* – a wild Cantor set in $\mathbb{R}^3$ which is clearly *homogeneously* (but not $C^1$–*homogeneously*) embedded in $\mathbb{R}^3$. In fact, it does not even work for *Lipschitz* homeomorphisms, i.e. the maps for which $d(f(x), f(y)) < \lambda d(x, y)$, for all $x, y \in X$. 
Remark: This theorem clearly does not work for all homeomorphisms, a counterexample is the Antoine Necklace – a wild Cantor set in $\mathbb{R}^3$ which is clearly homogeneously (but not $C^1$–homogeneously) embedded in $\mathbb{R}^3$. In fact, it does not even work for Lipschitz homeomorphisms, i.e. the maps for which $d(f(x), f(y)) < \lambda \ d(x, y)$, for all $x, y \in X$.

Malešič-Repovš Theorem (1999): There exists a Lipschitz homogeneous wild Cantor set in $\mathbb{R}^3$. 
**Remark:** This theorem clearly does not work for all homeomorphisms, a counterexample is the Antoine Necklace – a wild Cantor set in \( \mathbb{R}^3 \) which is clearly homogeneously (but not \( C^1 \)-homogeneously) embedded in \( \mathbb{R}^3 \). In fact, it does not even work for Lipschitz homeomorphisms, i.e. the maps for which 

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**Malešič-Repovš Theorem (1999):** There exists a Lipschitz homogeneous wild Cantor set in \( \mathbb{R}^3 \).

**Garity-Repovš-Željko Theorem (2005):** There exist uncountably many rigid Lipschitz homogeneous wild Cantor sets in \( \mathbb{R}^3 \).
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**Busemann Conjecture (1955):** Every $n$-dimensional $G$-space $(n \in \mathbb{N})$ is a topological $n$-manifold.
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**Menger Convexity:** Given distinct points \(x, y \in X\), there is a point \(z \in X - \{x, y\}\) such that \(d(x, z) + d(z, y) = d(x, y)\).
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**Local Extendibility:** For every point \(w \in X\), there is a radius \(\rho_w > 0\), such that for any pair of distinct points \(x, y \in B(w, \rho_w)\), there is a point \(z \in \text{int } B(w, \rho_w) - \{x, y\}\) such that \(d(x, y) + d(y, z) = d(x, z)\).
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**Uniqueness of the Extension:** Given distinct points $x, y \in X$, if there are points $z_1, z_2 \in X$ for which both

$$d(x, y) + d(y, z_i) = d(x, z_i) \quad \text{for } i = 1, 2,$$

and

$$d(y, z_1) = d(y, z_2)$$

hold, then $z_1 = z_2$. 
Facts: From these basic properties, a rich structure on a $G$-space can be derived. Let $(X, d)$ be a $G$-space and let $x \in X$. Then $(X, d)$ satisfies the following properties:
Busemann Conjecture: History of Results

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**Local Cones:** There is a radius $\epsilon_x > 0$ for which the closed metric ball $B_{\epsilon_x}(x)$ is homeomorphic to the cone over its boundary.

**Homogeneity:** $X$ is homogeneous and the homogeneity homeomorphisms can be chosen so that it is isotopic to the identity.
Busemann Conjecture: History of Results

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Busemann predicted: “Although this conjecture is probably true for any $G$-space, the proof seems quite inaccessible in the present state of topology.” His prediction was correct – the proof of the case $n = 4$ required the theory of 4-manifolds, developed almost three decades later.
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**Remarks:** The fact that every finite-dimensional $G$-space is an ANR follows from local contractibility and local compactness.
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A Busemann $G$-space $(X, d)$ has Alexandrov curvature $\leq K$ if geodesic triangles in $X$ are at most as ”fat” as corresponding triangles in a surface $S_K$ of constant curvature $K$, i.e., the length of a bisector of the triangle in $X$ is at most the length of the corresponding bisector of the corresponding triangle in $S_K$. 
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Example: The boundary of a convex region in $\mathbb{R}^n$ has nonnegative Alexandrov curvature.
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Remark: The general case of Busemann’s Conjecture of $n \geq 5$ remains unsolved.
**Definition:** A compact finite-dimensional metric space $X$ is called an *absolute suspension (AS)* if it is a suspension with respect to any pair of distinct points and is called an *absolute cone* if it is a cone with respect to any point.
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In 1974 Szymański proved both de Groot conjectures for dimensions $n \leq 3$. 
Remark: Small metric balls in Busemann $G$-spaces are absolute open cones. Absolute open cones have one point compactifications that are absolute suspensions. Therefore de Groot absolute suspension conjecture in dimension $n$ implies the Busemann conjecture in dimension $n$. 
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**Guilbault Theorem (2007):** The de Groot absolute cone conjecture is true for $n \leq 4$ and false for $n \geq 5$. 
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Remark: Guilbault provides counterexamples to the absolute cone conjecture. Unfortunately, the double of these non-ball counterexamples are spheres. Hence these counterexamples provide no solution to the de Groot absolute suspension conjecture in high dimensions $n \geq 5$. Unfortunately this also leaves the Busemann conjecture in dimensions $n \geq 5$ unsolved.
Mitchell Theorem (1978):

1. An \( n \)-dimensional absolute suspension \( X \) is a regular generalized \( n \)-manifold homotopy equivalent to \( S^n \); all its links are generalized \( (n - 1) \)-manifolds homotopy equivalent to \( S^{n-1} \).
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**Question:** If in Mitchell’s theorem ”homotopy equivalent” could be replaced with ”fine homotopy equivalent”, Mitchell’s theorem would imply resolvability. Is this possible?
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**Remark:** Every Busemann $G$-space is a manifold if and only if small metric spheres are codimension one manifold factors. Equivalently in dimensions $n \geq 5$, every Busemann $G$-space $X$ is a manifold if and only if $X$ is resolvable and small metric spheres $S$ in $X$ satisfy the property that $S \times \mathbb{R}$ has DDP.
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- The disjoint concordances property (Daverman and Halverson)
Summary and Questions

Facts:

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- Moore Conjecture and Resolution Conjecture $\Rightarrow$ Busemann Conjecture (recall that the Resolution Conjecture was shown to be wrong for all $n \geq 6$)
Questions:

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▶ Are all resolvable generalized manifolds codimension one manifold factors?

▶ Are all finite-dimensional homogeneous connected compact metric spaces resolvable?