THE BING-BORSUK AND THE BUSEMANN CONJECTURES

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Definition: Let Y be a metric space. Y is said to be an *absolute neighborhood retract (ANR)* provided for every closed embedding $e : Y \to Z$ of Y into a metric space Z, there is an open neighborhood U of the image $e(Y)$ which retracts to $e(Y)$. That is, there is a continuous surjection $r : U \to e(Y)$ with $r(x) = x$ for all $x \in e(Y)$.

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Definition: A topological space X is said to be *homogeneous* if, for any two points $x_1, x_2 \in X$, there is a homeomorphism of X onto itself taking x_1 to x_2.

Definition: A (closed) *topological n-manifold* ($n \in \mathbb{N}$) is a connected, compact n-dimensional metric space which is *locally Euclidean* (i.e. homeomorphic to $\mathbb{R}^n$).

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Jakobsche Theorem (1978): In dimension $n = 3$, the Bing-Borsuk Conjecture implies the Poincaré Conjecture.

Definition: An n-dimensional compact metric space X is called an *n-dimensional Cantor manifold* if whenever X can be expressed as the union $X = X_1 \cup X_2$ of its proper closed subsets, then $\dim(X_1 \cap X_2) \geq n - 1$.

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Brouwer Theorem (1910): Every topological n-manifold has the invariance of domain property.

Łysko Theorem (1976): Every finite-dimensional connected homogeneous ANR space is a Cantor manifold and it has the invariance of domain property.

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Definition: An n-dimensional ($n \in \mathbb{N}$) locally compact Hausdorff space X is called a *$\mathbb{Z}$-homology n-manifold* (n-hm$_{\mathbb{Z}}$) if for every point $x \in X$ and all $k \in \mathbb{N}$,
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Bredon Theorem (1967): If X is an n-dimensional homogeneous ENR ($n \in \mathbb{N}$) and for some (and, hence all) points $x \in X$, the groups $H_k(X, X - \{x\}; \mathbb{Z})$ are finitely generated, then X is a $\mathbb{Z}$-homology n-manifold.

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Remark: This theorem was reproved by Bryant in 1987 with a more geometric argument.

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Cell-like Approximation Theorem: Every cell-like map between topological manifolds is a near-homeomorphism.

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Remark: By the work of Quinn, these nonresolvable generalized manifolds must be *totally singular*.

Definition: A metric space X is said to have the *disjoint disks property* (DDP) if for every $\varepsilon > 0$ and every pair of maps $f, g : B^2 \to X$ there exist ε-approximations $f', g' : B^2 \to X$ with disjoint images.

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Bryant-Ferry-Mio-Weinberger Theorem (2007): For every $n \geq 7$ there exist non-resolvable generalized n-manifolds with the DDP.

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Bryant Modified Bing-Borsuk Conjecture (2002): Every homogeneous ($n \geq 3$)-dimensional ENR is a generalized n-manifold.

Definition: A space X is *homologically arc-homogeneous* provided that for every path $\alpha : [0,1] \to X$, the inclusion induced map

$$H_*(X \times 0, X \times 0 - (\alpha(0),0); \mathbb{Z}) \to H_*(X \times I, X \times I - \Gamma(\alpha); \mathbb{Z})$$

is an isomorphism, where $\Gamma(\alpha)$ denotes the graph of α.

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Bryant Theorem (2006): Every n-dimensional homologically arc-homogeneous ENR is a generalized n-manifold.

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Definition: A subset $K \subset \mathbb{R}^n$ is said to be C^1*–homogeneous* if for every pair of points $x, y \in K$ there exist neighborhoods $O_x, O_y \subset \mathbb{R}^n$ of x and y, respectively, and a C^1*–diffeomorphism*

$$h : (O_x, O_x \cap K, x) \to (O_y, O_y \cap K, y),$$

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Repovš-Skopenkov-Ščepin Theorem (1991): Let K be a locally compact (possibly nonclosed) subset of $\mathbb{R}^n$. Then K is C^1–homogeneous if and only if K is a C^1–submanifold of $\mathbb{R}^n$.

Remark: This theorem clearly does not work for all *homeomorphisms*, a counterexample is the *Antoine Necklace* – a wild Cantor set in $\mathbb{R}^3$ which is clearly *homogeneously* (but not C^1–*homogeneously* embedded in $\mathbb{R}^3$. In fact, it does not even work for *Lipschitz* homeomorphisms, i.e. the maps for which $d(f(x), f(y)) < \lambda\ d(x, y)$, for all $x, y \in X$.

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Malešič-Repovš Theorem (1999): There exists a Lipschitz homogeneous wild Cantor set in $\mathbb{R}^3$.

Garity-Repovš-Željko Theorem (2005): There exist uncountably many rigid Lipschitz homogeneous wild Cantor sets in $\mathbb{R}^3$.

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Local Extendibility: For every point $w \in X$, there is a radius $\rho_w > 0$, such that for any pair of distinct points $x, y \in B(w, \rho_w)$, there is a point $z \in \text{int } B(w, \rho_w) - \{x, y\}$ such that $d(x, y) + d(y, z) = d(x, z)$.

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Uniqueness of the Extension: Given distinct points $x, y \in X$, if there are points $z_1, z_2 \in X$ for which both

$$d(x, y) + d(y, z_i) = d(x, z_i) \quad \text{for } i = 1, 2,$$

and

$$d(y, z_1) = d(y, z_2)$$

hold, then $z_1 = z_2$

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Homogeneity: X is homogeneous and the homogeneity homeomorphisms can be chosen so that it is isotopic to the identity.

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Busemann predicted: "Although this conjecture is probably true for any G-space, the proof seems quite inaccessible in the present state of topology." His prediction was correct – the proof of the case $n = 4$ required the theory of 4-manifolds, developed almost three decades later.

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Remark: The general case of Busemann's Conjecture of $n \geq 5$ remains unsolved.

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In 1974 Szymański proved both de Groot conjectures for dimensions $n \leq 3$.

Remark: Small metric balls in Busemann G-spaces are absolute open cones. Absolute open cones have one point compactifications that are absolute suspensions. Therefore de Groot absolute suspension conjecture in dimension n implies the Busemann conjecture in dimension n.

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Remark: Guilbaut provides counterexamples to the absolute cone conjecture. Unfortunately, the double of these non-ball counterexamples are spheres. Hence these counterexamples provide no solution to the de Groot absolute suspension conjecture in high dimensions $n \geq 5$. Unfortunately this also leaves the Busemann conjecture in dimensions $n \geq 5$ unsolved.

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Question: If in Mitchell's theorem "homotopy equivalent" could be replaced with "fine homotopy equivalent", Mitchell's theorem would imply resolvability. Is this possible?

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Remark: Every Busemann G-space is a manifold if and only if small metric spheres are codimension one manifold factors. Equivalently in dimensions $n \geq 5$, every Busemann G-space X is a manifold if and only if X is resolvable and small metric spheres S in X satisfy the property that $S \times \mathbb{R}$ has DDP.

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- The 0-stitched disks property (Halverson)
- The disjoint concordances property (Daverman and Halverson)

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- The failure of Busemann Conjecture $\Rightarrow$ The failure of de Groot Absolute Suspension Conjecture
- Moore Conjecture and Resolution Conjecture $\Rightarrow$ Busemann Conjecture (recall that the Resolution Conjecture was shown to be wrong for all $n \geq 6$)

Questions:

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- Are all finite-dimensional homogeneous connected compact metric spaces resolvable?