A NEW APPROACH TO ELASTICITY THEORY: INTRINSIC METHODS

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OUTLINE

1. BACKGROUND: THE CLASSICAL APPROACH TO LINEAR SHELL THEORY. THE UNCONSTRAINED MINIMIZATION PROBLEM

2. NEW UNKNOWNS AND NEW CONSTRAINED MINIMIZATION PROBLEM. WEAK VERSIONS OF A CLASSICAL THEOREM OF POINCARÉ AND OF ST VENANT'S COMPATIBILITY CONDITIONS.

3. MAIN RESULT: A NECESSARY AND SUFFICIENT CONDITION FOR MATRIX FIELDS TO BE LINEARIZED CHANGE OF METRIC AND CHANGE OF CURVATURE TENSORS.

4. MAIN GOAL: A NEW APPROACH TO EXISTENCE THEORY FOR KOITER'S LINEAR SHELL EQUATIONS.
1. THE CLASSICAL APPROACH TO LINEAR SHELL THEORY.

Conventions: $\boxed{i, j, k, \ldots \in \{1,2,3\}}$ $\boxed{\alpha, \beta, \ldots \in \{1,2\}}$

$$\boxed{a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}\gamma_{\alpha\beta} = \sum_{\alpha,\beta,\sigma,\tau=1,2} a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}\gamma_{\alpha\beta}} \quad ; \quad \boxed{p^i\eta_i = \sum_{i=1,2,3} p^i\eta_i}.$$

Notations:

- $\boxed{E^3}$ = a three-dimensional Euclidean space with vectors $\boxed{\hat{e}^i}$ forming an orthonormal basis; $\boxed{|\mathbf{a}|}$ = the Euclidean norm of $\boxed{\mathbf{a} \in E^3}$; $\boxed{\mathbf{a} \cdot \mathbf{b}}$ = the Euclidean product, $\boxed{\mathbf{a} \wedge \mathbf{b}}$ = the exterior product of $\boxed{\mathbf{a}, \mathbf{b} \in E^3}$. A generic point in $\boxed{E^3}$ will be denoted $\boxed{\hat{x} = (\hat{x}_i)}$, where $\boxed{\hat{x}_i}$ are the Cartesian coordinates, and

$$\boxed{\hat{\partial}_i = \frac{\partial}{\partial \hat{x}_i}}, \boxed{\hat{\partial}_{ij} = \frac{\partial^2}{\partial \hat{x}_i \partial \hat{x}_j}}.$$

- A generic point in $\boxed{\mathbf{R}^2}$ will be denoted $\boxed{y = (y_\alpha)}$; $\boxed{\partial_\alpha = \frac{\partial}{\partial y_\alpha}}$, $\boxed{\partial_{\alpha\beta} = \frac{\partial^2}{\partial y_\alpha \partial y_\beta}}$.

- A generic point in $\boxed{\mathbf{R}^3}$ will be denoted $\boxed{x = (x_i)}$ and $\boxed{\partial_i = \frac{\partial}{\partial x_i}}$, $\boxed{\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}}$.

- If V is a vector space and R a subspace of V, the **quotient space** of V *modulo* R is denoted V/R and the equivalence class of $\eta \in$ V *modulo* R is denoted .

Definitions:

- A **domain** $\omega \subset \mathbf{R}^2$ (or $\Omega \subset \mathbf{R}^3$, or $\hat{\Omega} \subset E^3$) is an **open, bounded connected subset with a Lipschitz continuous boundary**, the set (or respectively ,) being locally on the same side of its boundary.

- Given a domain $\omega \subset \mathbf{R}^2$, a mapping $\boldsymbol{\theta} \in C^1(\overline{\omega}; E^3)$ is an **immersion** if the vectors $\mathbf{a}_\alpha = \partial_\alpha \boldsymbol{\theta}(y)$ are **linearly independent** at all points $y \in \overline{\omega}$.

- Given a domain $\boxed{\boldsymbol{\Omega} \subset \mathbf{R}^3}$, a mapping $\boxed{\boldsymbol{\Theta} \in C^1(\overline{\boldsymbol{\Omega}}; E^3)}$ is an **immersion** if the vectors $\boxed{\mathbf{g}_i = \partial_i \boldsymbol{\Theta}(x)}$ are linearly independent at all points $\boxed{x \in \overline{\boldsymbol{\Omega}}}$ (equivalently, the matrix $\boxed{\nabla\boldsymbol{\Theta}(x) = (\partial_j \Theta_i)_{i,j=1,2,3}}$ is invertible at all points $\boxed{x \in \overline{\boldsymbol{\Omega}}}$).

- **Spaces of vector-valued**, or **matrix-valued functions** over a domain D are denoted by boldface letters and the norms of the spaces $\boxed{L^2(D)}$ or $\boxed{\mathbf{L}^2(D)}$, and $\boxed{H^m(D)}$ or $\boxed{\mathbf{H}^m(D)}$ are denoted $\boxed{\|\cdot\|_{0,D}}$ and $\boxed{\|\cdot\|_{m,D}}$.

- Given a domain $\boxed{\omega \subset \mathbf{R}^2}$ and an immersion $\boxed{\boldsymbol{\theta} \in C^3(\overline{\omega}; E^3)}$, define the

SURFACE $\boxed{S := \boldsymbol{\theta}(\overline{\omega})}$.

The vector fields $\boxed{\mathbf{a}_\alpha = \partial_\alpha \boldsymbol{\theta}}$ and $\boxed{\mathbf{a}_3 = \dfrac{\mathbf{a}_1 \wedge \mathbf{a}_2}{|\mathbf{a}_1 \wedge \mathbf{a}_2|}}$ form the **covariant bases** along surface S.

$\boxed{a_{\alpha\beta} = a_{\beta\alpha} \in C^2(\overline{\omega})}$ **covariant components of the first fundamental form of the surface *S*,**

$\boxed{b_{\alpha\beta} = b_{\beta\alpha} \in C^1(\overline{\omega})}$ **covariant components of the second fundamental form of the surface *S***

$\boxed{\sqrt{a}\, dy}$ = area element along S, where $\boxed{a := \det(a_{\alpha\beta}) \in C^2(\overline{\omega})}$.

Two other fundamental tensors play a key role in the two-dimensional theory of linearly elastic shells, each one being associated with a displacement vector field

$$\boxed{\tilde{\boldsymbol{\eta}} := \eta_i \mathbf{a}^i}$$

of the surface S, where $\boxed{\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}(\omega) := H^1(\omega) \times H^1(\omega) \times H^2(\omega)}$ and the vector fields $\boxed{\mathbf{a}^i}$, which form the **contravariant bases** along S, are defined by the relations $\boxed{\mathbf{a}^i \cdot \mathbf{a}_j = \delta^i_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}}$.

The covariant components of the **linearized change of metric tensor** are given by

$$\boxed{\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}\left[a_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta}\right]^{lin} = \frac{1}{2}\left(\partial_\beta \tilde{\boldsymbol{\eta}} \cdot \mathbf{a}_\alpha + \partial_\alpha \tilde{\boldsymbol{\eta}} \cdot \mathbf{a}_\beta\right)}$$

The covariant components of the **linearized change of curvature tensor** are:

$$\boxed{\rho_{\alpha\beta}(\boldsymbol{\eta}) := \left[b_{\alpha\beta}(\boldsymbol{\eta}) - b_{\alpha\beta}\right]^{lin} = \left(\partial_{\alpha\beta} \tilde{\boldsymbol{\eta}} - \boldsymbol{\Gamma}^{\sigma}_{\alpha\beta}\, \partial_\sigma \tilde{\boldsymbol{\eta}}\right) \cdot \mathbf{a}_3}$$

where

$\boxed{a_{\alpha\beta}(\boldsymbol{\eta})}$ = covariant components of the **first fundamental form** of the deformed surface $\boxed{(\boldsymbol{\theta} + \tilde{\boldsymbol{\eta}})(\overline{\omega})}$,

$\boxed{b_{\alpha\beta}(\boldsymbol{\eta})}$ = covariant components of the **second fundamental form** of the deformed surface $\boxed{(\boldsymbol{\theta} + \tilde{\boldsymbol{\eta}})(\overline{\omega})}$, the notation $\boxed{[\ \dots\ ]^{lin}}$ represents the *linear part* with respect to $\boxed{\boldsymbol{\eta}=(\eta_i)}$ in the expression $\boxed{[\ \dots\ ]}$, and

$$\boxed{\Gamma^{\sigma}_{\alpha\beta}=\mathbf{a}^{\sigma}\cdot\partial_{\alpha}\mathbf{a}_{\beta}}$$ = **Christoffel symbols** of surface S.

Note that $\boxed{\gamma_{\alpha\beta}(\boldsymbol{\eta})\in L^2(\omega)}$ and $\boxed{\rho_{\alpha\beta}(\boldsymbol{\eta})\in L^2(\omega)}$, since $\boxed{\boldsymbol{\eta}\in\mathbf{V}(\omega)}$.

A general shell structure is fully represented by a middle surface geometry and the thickness at each point of its middle surface. Consider a linearly elastic thin shell with **middle surface** $\boxed{S:=\boldsymbol{\theta}(\overline{\omega})}$ and **constant thickness** $\boxed{2\varepsilon}$, where $\boxed{\varepsilon>0}$. The **reference configuration** of the shell is the three-dimensional set $\boxed{\boldsymbol{\Theta}(\overline{\Omega})}$, where $\boxed{\Omega=\omega\times]-\varepsilon,\varepsilon[\subset\mathbf{R}^3}$, and the mapping $\boxed{\boldsymbol{\Theta}:\overline{\Omega}\to E^3}$ is defined by

$$\boxed{\boldsymbol{\Theta}(y,x_3):=\boldsymbol{\theta}(y)+x_3\mathbf{a}_3(y)}, \text{ for all } \boxed{(y,x_3)\in\overline{\Omega}=\overline{\omega}\times[-\varepsilon,\varepsilon]}.$$

We assume for simplicity that the shell is made of a **homogeneous isotropic material** and that the **reference configuration** $\boxed{\mathbf{\Theta}(\overline{\Omega})}$ is a **natural state**, i.e., stress-free. Hence, the material is characterized by its two **Lamé constants** $\boxed{\lambda > 0}$ and $\boxed{\mu > 0}$, and the contravariant components $\boxed{a^{\alpha\beta\sigma\tau}}$ of the **two-dimensional shell elasticity tensor** are given by

$$\boxed{a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu \, (a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}) \in C^2(\overline{\omega})}.$$

This tensor is **uniformly positive definite**: there exists a constant

$\boxed{C = C(\omega, \boldsymbol{\theta}, \mu) > 0}$ such that $\boxed{C \sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \leq a^{\alpha\beta\sigma\tau} t_{\sigma\tau} t_{\alpha\beta}}$,

for all $\boxed{y \in \overline{\omega}}$ and all symmetric matrices $\boxed{(t_{\alpha\beta})}$ of order two.

Assume that the shell is subjected to applied forces acting only in its interior and on its upper and lower faces, whose resultant after integration across the thickness of the shell has contravariant components $\boxed{p^i \in L^2(\omega)}$. Assume that the **lateral face of the shell is free**, i.e., the displacement is not subjected to any boundary condition there. We are thus considering a **pure traction problem for a linearly elastic shell.**

Classical unknown: $\boxed{\boldsymbol{\eta}^* = (\eta_i^*)}$, $\boxed{\eta_i^* : \overline{\omega} \to \mathbf{R}}$,

$\boxed{\eta_i^*}$ = covariant components of the unknown displacement field

$\boxed{\eta_i^* \mathbf{a}^i}$ of the points of the middle surface S.

$\boxed{\boldsymbol{\eta}^* = (\eta_i^*) \in}$ $\boxed{\mathbf{V}(\omega) := H^1(\omega) \times H^1(\omega) \times H^2(\omega)}$: space of "**admissible displacements**"

Energy functional $\boxed{j : \mathbf{V}(\omega) \to \mathbf{R}}$ is defined by

$$\boxed{\begin{aligned} j(\boldsymbol{\eta}) &:= \frac{1}{2} \int_\omega \{\varepsilon\, a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\eta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\eta}) \rho_{\alpha\beta}(\boldsymbol{\eta})\} \sqrt{a}\, \mathrm{d}y \\ &\quad - \int_\omega p^i \eta_i \sqrt{a}\, \mathrm{d}y. \end{aligned}}$$

The two-dimensional **Koiter equations** [1970] in the form of a quadratic **minimization problem**:

$$\text{Find } \boxed{\boldsymbol{\eta}^* = (\eta_i^*) \in} \boxed{\mathbf{V}(\omega)} \text{ such that}$$

$$\boxed{j(\boldsymbol{\eta}^*) = \inf_{\boldsymbol{\eta} \in \mathbf{V}(\omega)} j(\boldsymbol{\eta})}.$$

Define the Hilbert **space of infinitesimal rigid displacements of surface** *S*:

$$\boxed{\mathbf{Rig}(\omega) := \{} \boxed{\boldsymbol{\eta} \in} \boxed{\mathbf{V}(\omega)} \boxed{;\, \boldsymbol{\gamma}(\boldsymbol{\eta}) = \boldsymbol{\rho}(\boldsymbol{\eta}) = \mathbf{0}} \text{ in } \boxed{\mathbf{L}^2_{sym}(\omega)\}}$$

$$\boxed{\mathbf{Rig}(\omega) := \{} \boxed{\boldsymbol{\eta} = (\eta_i) \in} \boxed{\mathbf{V}(\omega);\, \eta_i \mathbf{a}^i = \mathbf{a} + \mathbf{b} \wedge \boldsymbol{\theta},\, \mathbf{a} \in \mathbf{R}^3,\, \mathbf{b} \in \mathbf{R}^3\}}.$$

We will assume that the linear form $\boxed{l(\boldsymbol{\eta}) = \int_{\omega} p^i \eta_i \sqrt{a}\, \mathrm{d}y}$ associated with the applied forces satisfies the **compatibility conditions:**

$$\boxed{l(\boldsymbol{\eta}) = 0} \text{ for all } \boxed{\boldsymbol{\eta} \in \mathbf{Rig}(\boldsymbol{\omega})},$$

since these are clearly necessary for the existence of a minimizer of the energy functional j over the space $\boxed{\mathbf{V}(\omega)}$. Then, the above minimization problem becomes:

$$\text{Find } \boxed{\dot{\boldsymbol{\eta}}^*} \boxed{\in \dot{\mathbf{V}}(\boldsymbol{\omega}) := \mathbf{V}(\boldsymbol{\omega}) / \mathbf{Rig}(\boldsymbol{\omega})} \text{ such that } \boxed{j(\dot{\boldsymbol{\eta}}^*) = \inf_{\dot{\boldsymbol{\eta}} \in \dot{\mathbf{V}}(\omega)} j(\dot{\boldsymbol{\eta}})}.$$

In order to establish the **existence and uniqueness of a minimizer of the energy functional** j over the space $\boxed{\dot{\mathbf{V}}(\omega)}$, it suffices, thanks to the positive definiteness of tensor $\boxed{a^{\alpha\beta\sigma\tau}}$, to show that the mapping

$$\boxed{\dot{\boldsymbol{\eta}} \in \dot{\mathbf{V}}(\boldsymbol{\omega}) \rightarrow \| \boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}), \boldsymbol{\rho}(\dot{\boldsymbol{\eta}}) \|_{0,\boldsymbol{\omega}}}$$

is a norm over quotient space $\dot{\mathbf{V}}(\omega)$ equivalent to the quotient norm $\|\cdot\|_{\dot{\mathbf{V}}(\omega)}$.

DEFINE THE NORMS:

$$\|(\mathbf{c},\mathbf{r})\|_{0,\omega} := \{\sum_{\alpha,\beta}\|c_{\alpha\beta}\|_{0,\omega}^2 + \sum_{\alpha,\beta}\|r_{\alpha\beta}\|_{0,\omega}^2\}^{1/2},\quad (\mathbf{c},\mathbf{r}) \in \mathbf{L}^2_{sym}(\omega) \times \mathbf{L}^2_{sym}(\omega),$$

$$\|\boldsymbol{\eta}\|_{\mathbf{V}(\omega)} := \{\sum_{\alpha}\|\eta_\alpha\|_{1,\omega}^2 + \|\eta_3\|_{2,\omega}^2\}^{1/2} \text{ for all } \boldsymbol{\eta} \in \mathbf{V}(\omega),$$

$$\|\dot{\boldsymbol{\eta}}\|_{\dot{\mathbf{V}}(\omega)} := \inf_{\boldsymbol{\xi}\in\mathbf{Rig}(\omega)}\|\boldsymbol{\eta}+\boldsymbol{\xi}\| \text{ for all } \dot{\boldsymbol{\eta}} \in \dot{\mathbf{V}}(\omega) = \mathbf{V}(\omega)/\mathbf{Rig}(\omega).$$

The first stage, due to:

M. Bernadou, P.G. Ciarlet and B. Miara [1994]: ***Existence theorems for two-dimensional linear shell theories, J. Elasticity*** **34, 111-138.**

is to establish a basic **Korn inequality on a surface**, "over the space $\mathbf{V}(\omega)$":

THEOREM. Let there be given a domain $\omega \subset \mathbf{R}^2$ and an immersion $\boldsymbol{\theta} \in C^3(\overline{\omega}; E^3)$. Then there exists a constant $c = c(\omega, \boldsymbol{\theta})$ such that

$$\| \boldsymbol{\eta} \|_{\mathbf{V}(\omega)} \leq c \{ \sum_{\alpha} \| \eta_\alpha \|_{0,\omega}^2 + \| \eta_3 \|_{1,\omega}^2 + \| \boldsymbol{\gamma}(\boldsymbol{\eta}), \boldsymbol{\rho}(\boldsymbol{\eta}) \|_{0,\omega}^2 \}^{1/2}$$

for all $\boldsymbol{\eta} \in \mathbf{V}(\omega) = H^1(\omega) \times H^1(\omega) \times H^2(\omega)$.

PROOF. The essence of this inequality is that the two Hilbert spaces $\mathbf{V}(\omega)$ and

$$\mathbf{W}(\omega) := \{ \boldsymbol{\eta} = (\eta_i) \in L^2(\omega) \times L^2(\omega) \times H^1(\omega); \\ \boldsymbol{\gamma}(\boldsymbol{\eta}) \in \mathbf{L}^2_{sym}(\omega),\ \boldsymbol{\rho}(\boldsymbol{\eta}) \in \mathbf{L}^2_{sym}(\omega) \}$$

coincide. The keystone of the proof is a **fundamental Lemma of J.L.Lions**:

Let Ω be a domain in $\mathbf{R}^n$ and let v be a distribution on Ω. Then

$$\left\{v \in H^{-1}(\boldsymbol{\Omega}) \text{ and } \partial_i v \in H^{-1}(\boldsymbol{\Omega}),\ 1 \leq i \leq n\right\} \Longrightarrow v \in L^2(\boldsymbol{\Omega}).$$

The second stage consists in establishing another basic Korn inequality on a surface, this time "over the quotient space $\dot{\mathbf{V}}(\omega)$":

THEOREM. Let there be given a domain $\omega \subset \mathbf{R}^2$ and an immersion $\boldsymbol{\theta} \in C^3(\overline{\omega}; E^3)$. Then there exists a constant $\dot{c} = \dot{c}(\omega, \boldsymbol{\theta})$ such that

$$\| \dot{\boldsymbol{\eta}} \|_{\dot{\mathbf{V}}(\omega)} \leq \dot{c} \| \boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}), \boldsymbol{\rho}(\dot{\boldsymbol{\eta}}) \|_{0,\omega}$$

for all $\dot{\boldsymbol{\eta}} \in \dot{\mathbf{V}}(\omega) = \mathbf{V}(\omega) / \mathbf{Rig}(\omega)$.

USING THIS KORN INEQUALITY AND THE POSITIVE DEFINITENESS OF THE ELASTICITY TENSOR, WE OBTAIN THE EXISTENCE AND THE UNIQUENESS OF A SOLUTION FOR THE KOITER EQUATIONS.

2. OBJECTIVE: CONSIDER THE NEW PRIMARY UNKNOWNS (INSTEAD OF THE DISPLACEMENT)

NEW UNKNOWNS:

$\boxed{\gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)}$: covariant comps. of the **linearized change of metric tensor**

$\boxed{\rho_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)}$: covariant comps. of the **linearized change of curvature tensor**

NEW ENERGY FUNCTIONAL

$$\boxed{\kappa(\boldsymbol{\gamma},\boldsymbol{\rho}) := \frac{1}{2}\int_{\omega}\{\varepsilon\, a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}\gamma_{\alpha\beta} + \frac{\varepsilon^{3}}{3}a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}\rho_{\alpha\beta}\}\sqrt{a}\,\mathrm{d}y - l(\boldsymbol{\gamma},\boldsymbol{\rho})}$$

NEW SPACE FOR "ADMISSIBLE UNKNOWNS"

$$\boxed{\mathbf{T}(\omega) := \{(\boldsymbol{\gamma},\boldsymbol{\rho}) \in \mathbf{L}^{2}_{sym}(\omega)\times\mathbf{L}^{2}_{sym}(\omega)}\,;\ \boxed{\mathbf{R}(\boldsymbol{\gamma},\boldsymbol{\rho}) = \mathbf{0}}\ \text{in}\ \boxed{\mathbf{H}^{-2}(\hat{\Omega})\}}\,.$$

NEW MINIMIZATION PROBLEM

$$\text{Find}\ \boxed{(\boldsymbol{\gamma}^{*},\boldsymbol{\rho}^{*}) \in \mathbf{T}(\omega)}\ \text{such that}\ \boxed{\kappa(\boldsymbol{\gamma}^{*},\boldsymbol{\rho}^{*}) = \inf_{(\boldsymbol{\gamma},\boldsymbol{\rho})\in\mathbf{T}(\omega)}\kappa(\boldsymbol{\gamma},\boldsymbol{\rho})}$$

WHY?

- **MATHEMATICAL NOVELTY**
- **PRACTICAL ADVANTAGE** : As the constitutive equations of linear shell theory are invertible, the new minimization problem can be recast as a minimization problem with the **stress resultants** and **bending moments** as the only unknowns, which are of great interest from the mechanical and computational viewpoints:

$$\boxed{n^{\alpha\beta} = \varepsilon a^{\alpha\beta\sigma\tau}\gamma_{\alpha\beta}(\boldsymbol{\eta})}$$ contrav. comps. of the stress resultant tensor field

$$\boxed{m^{\alpha\beta} = \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau}\rho_{\alpha\beta}(\boldsymbol{\eta})}$$ contrav. comps. of the bending moment tensor field

- **EXTENSION TO FULLY NONLINEAR INTRINSIC SHELL THEORY** (where the "full" change of metric, and change of curvature, tensors appear in the energy, instead of their linearized versions considered here). This approach should provide **existence theorems** that are so far essentially lacking for **nonlinear Koiter shell equations**.

WEAK VERSIONS OF A CLASSICAL THEOREM OF POINCARÉ AND OF ST VENANT'S COMPATIBILITY CONDITIONS.

P.G. Ciarlet and P. Ciarlet Jr. [2005] : Consider the **linearized strain tensor** $\boxed{\mathbf{e} \in \mathbf{L}^2_{sym}(\Omega)}$ as the **primary unknown** instead of the displacement, for the pure traction problem of **linearized three-dimensional elasticity**.

Their objective was to characterize those symmetric 3×3 matrix fields with components $e_{ij} \in L_{sym}(\Omega)$ that can be written as

$$e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j),$$

for some $\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega)$, where Ω is a domain in $\mathbb{R}^3$.

Classical Theorem of Poincaré. If functions $h_k \in C^1(\Omega)$ satisfy $\partial_l h_k = \partial_k h_l$ in a simply-connected open subset Ω of $\mathbb{R}^3$, then there exists a function $p \in C^2(\Omega)$ such that $h_k = \partial_k p$ in $L^2(\Omega)$.

Theorem of Poincaré (weak form). Let Ω be a simply-connected domain in $\mathbb{R}^3$. Let $h_k \in H^{-1}(\Omega)$ be distributions that satisfy $\partial_l h_k = \partial_k h_l$ in $H^{-2}(\Omega)$. Then there exists a function $p \in L^2(\Omega)$, unique up to an additive constant, such that $h_k = \partial_k p$ in $H^{-1}(\Omega)$.

1864: the classical compatibility relations of St Venant

Theorem (weak form of St Venant's compatibility conditions). Let Ω be a simply-connected domain in $\mathbb{R}^3$. If $\mathbf{e} = (e_{ij}) \in \mathbf{L}^2_{sym}(\Omega)$ satisfy the **weak St Venant compatibility conditions**:

$$R_{ijkl}(\mathbf{e}) := \partial_{lj} e_{ik} + \partial_{ki} e_{jl} - \partial_{li} e_{jk} + \partial_{kj} e_{il} = 0 \text{ in } H^{-2}(\Omega),$$

then there exists $\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega)$ such that $\mathbf{e} = \frac{1}{2}(\nabla \mathbf{v}^T + \nabla \mathbf{v})$ or

$$e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j),$$

and any other solution differs by an infinitesimal rigid displacement.

3. MAIN RESULT: A NECESSARY AND SUFFICIENT CONDITION FOR MATRIX FIELDS TO BE LINEARIZED CHANGE OF METRIC AND CHANGE OF CURVATURE TENSORS.

(P.G. Ciarlet, L. Gratie, A new approach to linear shell theory, *Math. Models Methods Appl. Sci.*, Vol.15, 2005, pp. 1181-1202).

THEOREM. Let $\boxed{\omega \subset \mathbf{R}^2}$ be a simply-connected domain. There exists $\boxed{\varepsilon_0 > 0}$ and a continuous linear mapping $\boxed{\mathbf{R} : \mathbf{L}^2_{sym}(\omega) \times \mathbf{L}^2_{sym}(\omega) \to \mathbf{H}^{-2}(\hat{\Omega})}$ where $\boxed{\hat{\Omega} = \omega \times]-\varepsilon_0, \varepsilon_0[}$ is an ad hoc tubular neighborhood of the surface $\boxed{\boldsymbol{\theta}(\omega)}$, and $\boxed{\mathbf{H}^{-2}(\hat{\Omega}) := (H^{-2}(\hat{\Omega}))^6}$, s.t. a pair $\boxed{(\mathbf{c}, \mathbf{r}) = ((c_{\alpha\beta}), (r_{\alpha\beta})) \in \mathbf{L}^2_{sym}(\omega) \times \mathbf{L}^2_{sym}(\omega)}$ of symmetric matrix fields satisfies $\boxed{\mathbf{R}(\mathbf{c}, \mathbf{r}) = \mathbf{0}}$ in $\boxed{\mathbf{H}^{-2}(\hat{\Omega})}$, if and only if there exists a vector field $\boxed{\boldsymbol{\eta} \in \mathbf{V}(\omega) = H^1(\omega) \times H^1(\omega) \times H^2(\omega)}$ s.t.

$$\boxed{c_{\alpha\beta} = \gamma_{\alpha\beta}(\boldsymbol{\eta})} \text{ and } \boxed{r_{\alpha\beta} = \rho_{\alpha\beta}(\boldsymbol{\eta})} \text{ in } \boxed{L^2_{sym}(\omega)}$$

and any other solution differs by an infinitesimal rigid displacement.

<u>MAIN SURPRISE</u>: **A NEW PROOF OF KORN'S INEQUALITY ON A SURFACE.** Our new approach provides "as by-products" different proofs of the classical Korn's inequalities on a surface.

4. <u>MAIN GOAL</u>: A NEW "INTRINSIC" APPROACH TO KOITER'S LINEAR SHELL EQUATIONS.

We are now in a position to answer the main question addressed here, at least for the so-called PURE TRACTION PROBLEM for a linearly elastic shell modeled by Koiter's equations. Recall that, in this case, the quadratic functional j of the **classical approach** is to be minimized over the space

$$\boxed{\mathbf{V}(\omega) = H^1(\omega) \times H^1(\omega) \times H^2(\omega)}.$$

In the "intrinsic approach", we now minimize a quadratic functional defined over $\boxed{L^2_{sym}(\omega)}$-spaces:

THEOREM. Given a simply-connected domain $\boxed{\omega \subset \mathbf{R}^2}$ and an immersion $\boxed{\boldsymbol{\theta} \in C^3(\overline{\omega}; E^3)}$, define the quadratic functional $\boxed{\boldsymbol{\kappa}: \mathbf{L}^2_{sym}(\omega) \times \mathbf{L}^2_{sym}(\omega) \to \mathbf{R}}$ by

$$\boxed{\kappa(\mathbf{c}, \mathbf{r}) := \int_\omega \{\varepsilon\, a^{\alpha\beta\sigma\tau} c_{\sigma\tau} c_{\alpha\beta} + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau} r_{\sigma\tau} c_{\alpha\beta}\} \sqrt{a}\, \mathrm{d}y - l \circ \mathbf{H}}$$

Then the minimization problem (with the linearized change of metric, and change of curvature, tensors as the new unknowns):

Find $\boxed{(\mathbf{c}^*,\mathbf{r}^*)\in\mathbf{T}(\omega)}$ such that $\boxed{\kappa(\mathbf{c}^*,\mathbf{r}^*)=\inf_{(\mathbf{c},\mathbf{r})\in\mathbf{T}(\omega)}\kappa(\mathbf{c},\mathbf{r})}$

has one and only one solution. Furthermore,

$$\boxed{(\mathbf{c}^*,\mathbf{r}^*)=(\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}^*),\boldsymbol{\rho}(\dot{\boldsymbol{\eta}}^*))},$$

where $\boxed{\dot{\eta}^*}$ is the unique solution of the "classical" minimization problem:

Find $\boxed{\dot{\boldsymbol{\eta}}^*\in\mathbf{V}(\omega)/\mathbf{Rig}(\omega)}$ such that $\boxed{j(\dot{\boldsymbol{\eta}}^*)=\inf_{\dot{\boldsymbol{\eta}}\in\dot{\mathbf{V}}(\omega)}j(\dot{\boldsymbol{\eta}})}$.

CONCLUDING REMARKS

(a) We obtained a new constrained minimization problem over the space $\boxed{\mathbf{L}^2_{sym}(\omega)\times\mathbf{L}^2_{sym}(\omega)}$ with 6 unknowns. The **constraints** (in the sense of optimization theory) are the **compatibility relations** $\boxed{\mathbf{R}(\mathbf{c},\mathbf{r})=\mathbf{0}}$ in $\boxed{\mathbf{H}^{-2}(\hat{\Omega})}$.

(b) There remains the task of devising efficient **numerical schemes** for approaching such a constrained minimization problem.

(c) A highly challenging task consists in extending the present approach to **nonlinear elastic shells**, where the full differences

$$\boxed{\lfloor a_{\alpha\beta}(\boldsymbol{\eta})-a_{\alpha\beta}\rfloor} \text{ and } \boxed{\lfloor b_{\alpha\beta}(\boldsymbol{\eta})-b_{\alpha\beta}\rfloor}$$

are considered as the NEW UNKNOWNS, instead of their linearizations $\boxed{\gamma_{\alpha\beta}(\boldsymbol{\eta})}$ and $\boxed{\rho_{\alpha\beta}(\boldsymbol{\eta})}$ as here.