Minimum Energy Control with Applications to Spacecraft Rendezvous and Docking

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Abstract

Consider a stabilizable linear system with periodic solutions. For a given periodic orbit, the minimum energy problem is to find the infimum of $\mathcal{L}_2$ norms of controls which steer the state from the orbit to the origin asymptotically. It is shown that the infimum is obtained in terms of the maximal solution of the singular Riccati equation associated with the system. Using this result, a design method of stabilizing feedback controllers which steer the state from the orbit to the origin asymptotically with energy arbitrarily close to the infimum is proposed. As applications, the Halo orbit control problem near the L_2 Lagrangian point of the Earth-Moon-Spacecraft system and the relative orbit transfer problem associated with the Hill-Clohessy-Whiltshire equations are discussed.

Keywords: minimum energy problem, periodic solutions, optimal regulator, singular Riccati equation, null controllability with vanishing energy

1 Introduction

In this lecture, the minimum energy control problem for a stabilizable linear system with periodic solutions given in [4] is introduced. It is a minimization problem in which the infimum of $\mathcal{L}_2$ norms of controls which steer the state from the orbit to the origin asymptotically is sought. As an application, the Halo orbit control problem near the L_2 Lagrangian point of the Earth-Moon-Spacecraft system is discussed. If the system is controllable and the minimum energy is zero, it is referred to as null controllable with vanishing energy (NCVE) [5]. The system which describes the realtive motion of a spacecraft with respect another in a circular orbit is NCVE [6]. The relative orbit transfer problem based on NCVE in [6] is also introduced. It covers rendezvous and docking problems.

To motivate the minimum energy problem, consider the Earth-moon-spacecraft system [4], shown in Fig.1, regarded as the circular restricted three-body problem, where the Earth-moon system, by assumption, rotates with a constant angular velocity ($\omega = 2.661699 \times 10^{-6}$ rad/s) about their composite center of mass, and their orbital motion is not affected by the spacecraft. Then the equations of motion

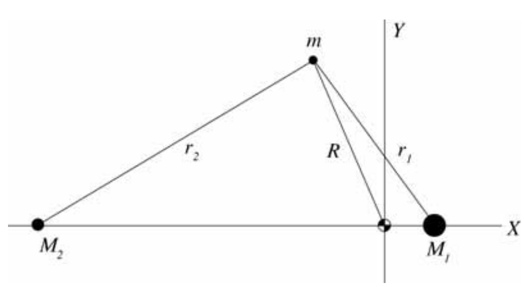

Figure 1: Circular restricted three-body problem.

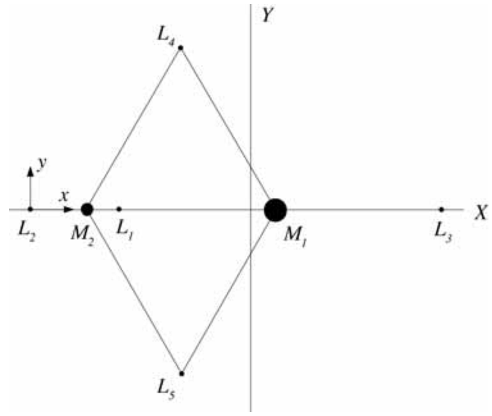

Figure 2: Lagrangian points.

of the spacecraft in nondimensional form are given as follows [7]:

$$
\begin{aligned}
\ddot{X} - 2\dot{Y} - X &= -(1-\rho)(X-\rho)/r_1^3 - \rho(X+1-\rho)/r_2^3 + u_x, \\
\ddot{Y} + 2\dot{X} - Y &= -(1-\rho)Y/r_1^3 - \rho Y/r_2^3 + u_y, \\
\ddot{Z} &= -(1-\rho)Z/r_1^3 - \rho Z r_2^3 + u_z,
\end{aligned}
$$

where (X, Y, Z) is the rotating coordinate system with origin at the barycenter, Z is the rotating axis, X in the direction of the Earth, Y is defined to form a right-handed system, $\rho = 0.01215$ is the ratio of the mass of the moon and the total mass of the Earth and moon, r_1 and r_2 denote the distances of the spacecraft from the Earth and moon respectively, and (u_x, u_y, u_z) are thrust accelerations. A unit of time is $1/\omega$ and 27.3 days, a unit of distance is $D = 384,748$ km, and a unit of thrust is $D\omega^2$. For the free system, there are five equilibrium points $L_1(-0.83692, 0, 0)$, $L_2(-1.15568, 0, 0)$, $L_3(1.00506, 0, 0)$, $L_4(-0.48785, \sqrt{3}/2, 0)$, and $L_5(-0.48785, -\sqrt{3}/2, 0)$ known as Lagrangian points see Fig.2. The L_2 point is called translunar point and the linearized equations of motion about this point are given by

$$
\begin{aligned}
\ddot{x} - 2\dot{y} - (2\sigma+1)x &= u_x, \\
\ddot{y} + 2\dot{x} + (\sigma-1)y &= u_y, \\
\ddot{z} + \sigma z &= u_z,
\end{aligned} \tag{1}
$$

where $X = X_0 + x$, $Y = y$, $Z = z$ with $X_0 = -1.15568$, and

$$\sigma = (1-\rho)/|X_0 - \rho|^3 + \rho/|X_0 + 1 - \rho|^3 = 3.19043.$$

The equations for x and y are independent of z and determine the in-plane motion, and its characteristic equation has two real and two imaginary roots: ± 2.15868 and $\pm 1.86265j$. Thus the L_2 point is unstable, but the system (1) has periodic solutions. The L_1 and L_2 points are of practical importance for future space missions involving the stationing of a communication platform or a lunar space station. For lunar far-side communications, it is desirable to maintain a 3500-km halo orbit (periodic trajectory) about the L_2 point. Motivated by this, the orbit transfer of the in-plane motion will be discussed.

Our control problem is described as follows. Suppose a spacecraft is initially in a periodic orbit, and it is required to bring it to the origin (L_2 point) asymptotically by a feedback control with energy close to the infimum in the $\mathcal{L}_2$ sense. To solve this problem, the quadratic regulator theory and the singular Riccati equation are recalled. It is known [5] that its maximal solution gives the infimum over T of the mimimum $\mathcal{L}_2$-norm of the controls which steer a given state to the origin in time T. Using this the optimal initial position of the spacecraft in the orbit is obtained. The transfer problem from a given orbit to another, which is more practical, is reduced to the problem above.

This lecture is organized as follows. In section 2 the linear quadratic regulator theory is briefly reviewed, and the main result of the minimum energy problem studied in [5] is recalled. Then its generalization to stabilizable systems is given, and the solution to our minimum energy problem is shown. If any initial state can be steered to the origin with arbitrarily small amount of energy, the system is referred to as null controllable with vanishing energy (NCVE) [5]. Necessary and sufficient conditions for this are recalled, and a slightly weaker notion and its necessary and sufficient conditions are given. In Section 3 the minimum energy problem for the Earth-moon-spacecraft system is discussed. In Section 4 the Hill-Clohessy-Wiltshire equation (rendezvous equation), which is NCVE [6], is introduced. For this system, feedback controls are designed in the same way.

2 Minimum energy problem

Consider the linear system

$$\dot{x} = Ax + Bu, \tag{2}$$

where $x \in R^n$ and $u \in R^m$. Let $x(t; x_0, u)$ be the solution of (2) with initial condition $x(0) = x_0$. The Euclidean norm of vectors is denoted by $|\cdot|$ and the set of all eigenvalues of A by $\sigma(A)$.

Let A have pure imaginary eigenvalues $\pm j\omega$. Then (2) with $u = 0$ has periodic solutions. Each periodic solution determines an orbit. For each x_0, let $u \in \mathcal{L}_2(0, \infty; R^m)$ be such that $x(t; x_0, u) \to 0$ as $t \to \infty$. Such a control is said to be admissible, and the set of all admissible controls for x_0 is denoted by $\mathcal{U}(x_0)$. If (A, B) is stabilizable, this set is nonempty. Define

$$E(x_0) = \inf\{\| u \|_2^2 \colon u \in \mathcal{U}(x_0)\},$$

where $\| \cdot \|_2$ denotes the norm in $\mathcal{L}_2$ space. For a given orbit $\mathcal{O}$, our minimum energy problem is to find an $x_0^* \in \mathcal{O}$ which gives the minimum of $E(x_0)$, and to design stabilizing feedback controls whose $\mathcal{L}_2$-norms are close to $E(x_0)$ for any $x_0 \in \mathcal{O}$.

To solve this problem, the linear quadratic regulator theory is now reviewed. Consider the quadratic functional

$$J(u; x_0) = \int_0^\infty [x(t; x_0, u)' Q x(t; x_0, u) + |u(t)|^2] dt, \tag{3}$$

where $Q = C'C$ for some matirix C. The optimal regulator problem is to minimize $J(u; x_0)$ over $u \in \mathcal{L}_2(0, \infty; R^m)$ such that $x(t; x_0, u) \in \mathcal{L}_2(0, \infty; R^n)$ and $x(t; x_0, u) \to 0$ as $t \to \infty$. If (C, A, B) is a stabilizable and detectable triple, there exists a unique stabilizing solution $X \geq 0$ to the algbraic Riccati equation

$$A'X + XA + Q - XBB'X = 0. \tag{4}$$

The feedback control

$$u^* = -B'Xx \tag{5}$$

is optimal and $J(u^*; x_0) = x_0' X x_0$. If the functional is replaced by

$$J_T(u; x_0) = \int_0^T [x(t; x_0, u)' Q x(t; x_0, u) + |u(t)|^2] dt + x(T; x_0, u)' Q_f x(T; x_0, u), \ Q_f \geq 0, \tag{6}$$

then (4) is replaced by the Riccati differential equation

$$-\dot{X} = A'X + XA + Q - XBB'X, \ X(T) = Q_f. \tag{7}$$

In this case the feedback control

$$u^* = -B'X(t)x \tag{8}$$

is optimal, and $J_T(u^*; x_0) = x_0' X(0) x_0$.

Let $\mathcal{U}(T; x_0)$ be the set of $u \in \mathcal{L}_2(0, T; R^m)$ such that $x(T; x_0, u) = 0$. For a controllable system the following theorem is known [5].

Theorem 2.1 *Suppose (A, B) is controllable. Then for each $Q \geq 0$ there exists a maximal solution $X \geq 0$ of (5) such that*

$$x_0' X x_0 = \inf_{T>0} \inf_{\mathcal{U}(T; x_0)} \int_0^T [x(t; x_0, u)' Q x(t; x_0, u) + |u(t)|^2] dt. \tag{9}$$

Recall that X is maximal if it is symmetric and $Y \leq X$ for all solutions of (4).

The controllability assumption is relaxed to stabilizability, and the following results are obtained in [4]. Let X_ϵ be the stabilizing solution of (4) with Q replaced by $Q + \epsilon I$, $\epsilon > 0$.

Lemma 2.1 *Suppose (A, B) is stabilizable. Then $X_\epsilon \geq X$ for any solution X of (4).*

Proof. The difference $X_\epsilon - X$ satisfies the following equation:

$$(A - BB'X_\epsilon)'(X_\epsilon - X) + (X_\epsilon - X)(A - BB'X_\epsilon) + \epsilon I + (X_\epsilon - X)BB'(X_\epsilon - X) = 0.$$

Because $A - BB'X_\epsilon$ is stable, $X_\epsilon - X \geq 0$.

Lemma 2.2 *X_ϵ is monotone decreasing as $\epsilon \to 0$, and its limit X_0 is the maximal solution of (4).*

For our energy problem $Q = 0$, and the Riccati equation becomes singular:

$$A'X + XA - XBB'X = 0. \tag{10}$$

Theorem 2.2 *Suppose that (A, B) is stabilizable. Then*

$$E(x_0) = \inf_{\mathcal{U}(x_0)} \| u \|_2^2 = x_0' X x_0, \tag{11}$$

where X is the maximal solution of (10).

Lemma 2.3 *Suppose that (A, B) is controllable. Then*

$$E(x_0) = x_0' X x_0,$$

Proof. Since $\mathcal{U}(T; x_0) \subset \mathcal{U}(x_0)$, $E(x_0) \leq x_0' X x_0$. To prove the converse inequality, consider the Riccati equation

$$-\dot{X} = A'X + XA - XBB'X, \;\; X(T) = X.$$

Then the solution is $X(t) = X$, and the equality below holds:

$$\int_0^T |u(t)|^2 dt + x'(T)Xx(T) = x_0' X x_0 + \int_0^T |u + B'Xx|^2 dt$$

for any $u \in \mathcal{L}_2(0, T; R^m)$. Choose, in particular, $u \in \mathcal{U}_(x_0)$ to obtain

$$\int_0^\infty |u(t)|^2 dt \geq x_0' X x_0.$$

Hence $E(x_0) \geq x_0' X x_0$.

Proof of Theorem 2.2. Let X_ϵ be the stabilizing solution of (5) with $Q = \epsilon I$. Then

$$\begin{aligned} x_0' X_\epsilon x_0 &= \int_0^\infty [\epsilon |x_\epsilon(t)|^2 + |u_\epsilon(t)|^2] dt, \\ &\geq \int_0^\infty |u_\epsilon(t)|^2 dt, \end{aligned}$$

where $u_\epsilon = -B'X_\epsilon x$ and x_ϵ is its response. Note that $u_\epsilon \in \mathcal{U}_(x_0)$. Hence

$$\begin{aligned} x_0' X x_0 &= \inf_\epsilon \; x_0' X_\epsilon x_0, \\ &\geq \inf_\epsilon \int_0^\infty |u_\epsilon(t)|^2 dt, \\ &\geq \inf_{\mathcal{U}_(x_0)} \| u \|_2^2, \\ &= E(x_0). \end{aligned}$$

The converse inequality is proved in Lemma 2.3.

Since $E(x_0) = x_0' X x_0$, $E(x_0)$ is a continuous function of x_0, the minimum of $E(x_0)$ over any periodic orbit $\mathcal{O}$ (a compact set) exists, and its search is a constrained minimization. Now let x_0^* be the minimizing point on the orbit. Then apply the feedback law

$$u = -B' X_\epsilon x$$

with sufficiently small $\epsilon > 0$, when the state reaches x_0^*. Then it steers x_0^* asymptotically to the origin, and $E(x_0) \leq \| u \|_2^2 \leq x_0' X_\epsilon x_0$. Hence our problem has been solved. In the next section the minimization of $E(x_0)$ for the Earth-moon-spacecraft system will be shown in detail.

When the maximal solution of (10) is $X = 0$, then a new definition is introduced in [5].

Definition 2.1 *The system* (2) *is said to be null controllable with vanishing energy* (*NCVE for short*) *if for each initial* $x(0) = x_0$ *there exists a sequence of pairs* (T_N, u_N), $0 < T_N \uparrow \infty$, $u_N \in L_2(0, T_N; R^m)$ *such that* $x(T_N; x_0, u_N) = 0$ *and*

$$\lim_{N\to\infty} \int_0^{T_N} |u_N(t)|^2 dt = 0. \tag{12}$$

(A, B) is said to be NCVE if the system (2) is NCVE. Necessary and sufficient conditions for NCVE are given as follows [5].

Theorem 2.3 (A, B) *is NCVE if and only if*
(*a*) *it is controllable, and*
(*b*) $X = 0$ *is the maximal solution of the algebraic Riccati equation* (10).

Theorem 2.4 (A, B) *is NCVE if and only if*
(*a*) (A, B) *is controllable, and*
(*b*) $Re(\lambda) \leq 0$ *for any* $\lambda \in \sigma(A)$.

In view of Theorem 2.2, the notion of NCVE can be relaxed [4]:

Definition 2.2 *The system* (2) *is said to be stabilizable with vanishing energy* (*SVE*) *if* $E(x_0) = 0$ *for all* x_0.

Theorem 2.5 (A, B) *is SVE if and only if*
(*a*) (A, B) *is stabilizable, and*
(*b*) $Re(\lambda) \leq 0$ *for any* $\lambda \in \sigma(A)$.

Proof. To show necessity, note that the controllable part of the system is NCVE. Hence $Re(\lambda) \leq 0$ for any controllable λ, and (b) holds. To show sufficiency, let $X \neq 0$. Then there exists an eigenvector p of A corresponding to a λ such that $p^* X p \neq 0$. Then the Riccati equation (10) yields

$$2Re(\lambda) p^* X p - p^* X B B' X p = 0.$$

If $Re(\lambda) < 0$, it is a contradiction. Thus $Re(\lambda) = 0$. Then $B' X p = 0$. Again by the Riccati equation

$$A' X p + X A p - X B B' X p = 0,$$

which yields $A' X p = -j\omega X p$, where $\lambda = j\omega$. By stabilizability of (A, B), $Xp = 0$, which is a contradiction.

The notion of NCVE is defined for discrete-time systems in [1], [3], and is extended to periodic systems [2], where the relative orbit transfer along an elliptical orbit is considered.

3 Halo orbit transfer near L_2 point

Consider the in-plane motion of (1)

$$\ddot{x} - 2\dot{y} - (2\sigma + 1)x = u_x, \tag{13}$$
$$\ddot{y} + 2\dot{x} + (\sigma - 1)y = u_y. \tag{14}$$

The state space form of these equations is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2\sigma+1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 1-\sigma & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}, \tag{15}$$

where $\mathbf{x} = [\, x \quad \dot{x} \quad y \quad \dot{y} \,]'$ and $\mathbf{u} = [\, u_x \quad u_y \,]'$. The periodic solution of this equation is given by

$$\begin{aligned} x(t) &= x_0 \cos \omega t + \dot{x}_0/\omega \sin \omega t, \\ y(t) &= y_0 \cos \omega t + \dot{y}_0/\omega \sin \omega t, \end{aligned}$$

where $\omega = 1.86265$. By (13), y_0 and $\dot{y}_0$ are given by

$$y_0 = (\omega^2 + 2\sigma + 1)/2\omega^2 \ \dot{x}_0, \tag{16}$$

$$\dot{y}_0 = -(\omega^2 + 2\sigma + 1)/2 \ x_0. \tag{17}$$

These are the conditions which should be satisfied by the periodic solution. Now it is parametrized as follows:

$$x(t) = a \sin(\omega t + \alpha_0), \tag{18}$$

$$y(t) = \gamma a \cos(\omega t + \alpha_0), \tag{19}$$

where $a = (x_0^2 + \dot{x}_0^2/\omega^2)^{1/2}$, $\gamma = (\omega^2 + 2\sigma + 1)/2\omega = 2.91260$, and α_0 is determined by $\cos \alpha_0 = \dot{x}_0/a\omega$ and $\sin \alpha_0 = x_0/a$. In view of (18), (19), the periodic orbit $\mathcal{O}$ is an ellipse

$$(x/a)^2 + (y/\gamma a)^2 = 1.$$

The period of this solution is $\tau = 2\pi/\omega = 3.37154$, which corresponds to the actual period 14.6607 (days). In view of (16) and (17), the initial condition for (15) is given by

$$\mathbf{x}_0 = [\, x_0 \quad \dot{x}_0 \quad \gamma/\omega \ \dot{x}_0 \quad -\gamma\omega \ x_0 \,]'.$$

The maximal solution of the singular Riccati equation approximated by X_ϵ with $\epsilon = 10^{-14}$ is given by

$$\begin{bmatrix} 36.12404 & 10.56518 & -6.75656 & 6.65862 \\ 10.56518 & 3.08999 & -1.97609 & 1.94744 \\ -6.75656 & -1.97609 & 1.26373 & -1.24541 \\ 6.6586 & 1.94744 & -1.24541 & 1.22736 \end{bmatrix},$$

[4] and its eigenvalues are $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (41.70513, 0.00000, 0.00000, 0.00000)$ with corresponding eigenvectors

$$(p_1, p_2, p_3, p_4) = \begin{pmatrix} 0.930687 & -0.355604 & -0.000000 & 0.085840 \\ 0.272197 \ , & 0.767633 \ , & 0.533184 \ , & 0.228829 \\ -0.174074 & -0.224116 & 0.000000 & 0.958890 \\ 0.171550 & 0.483794 & -0.845999 & 0.144217 \end{pmatrix}.$$

Now

$$E(\mathbf{x}_0) = \mathbf{x}_0' X \mathbf{x}_0 = q^2 p_1' X p_1 = \lambda_1 q^2,$$

where $q = \mathbf{x}_0' p_1$. Hence the minimization of q^2 subject to $x_0^2 + \dot{x}_0^2/\omega^2 = a^2$ yields the optimal initial position. To minimize q^2 consider the function

$$\begin{aligned} h &= q^2 - \mu(x_0^2 + \dot{x}_0^2/\omega^2 - a^2), \\ &= [(q_1 - \gamma\omega q_4)x_0 + (q_2 + \gamma/\omega \ q_3)\dot{x}_0)]^2 - \mu(x_0^2 + \dot{x}_0^2/\omega^2 - a^2), \end{aligned}$$

where $p_1 = [\, q_1 \quad q_2 \quad q_3 \quad q_4 \,]'$. Setting

$$\begin{aligned} \partial h/\partial x_0 &= 2(q_1 - \gamma\omega q_4)q - 2\mu x_0 = 0, \\ \partial h/\partial \dot{x}_0 &= 2(q_2 + \gamma/\omega \ q_3)q - 2\mu \dot{x}_0/\omega^2 = 0, \end{aligned}$$

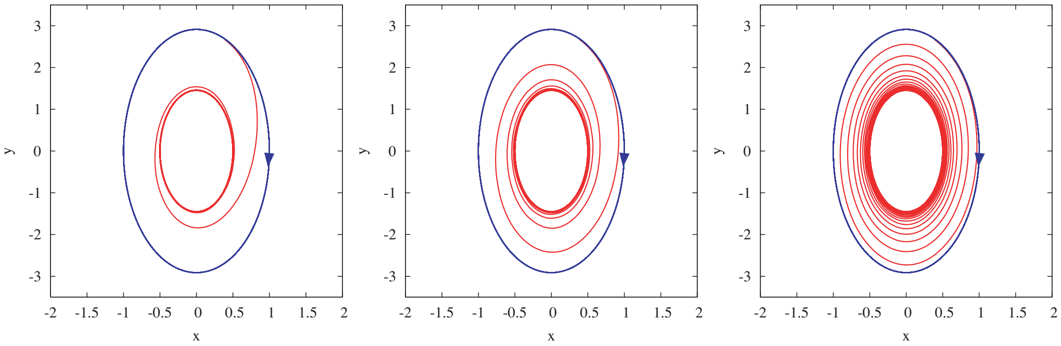

Figure 3: Asymptotic orbit transfer.

the following relation is obtained:

$$\dot{x}_0 = \beta\omega^2/\alpha \; x_0,$$

where $\alpha = q_1 - \gamma\omega q_4$ and $\beta = q_2 + \gamma/\omega \; q_3$. Thus the optimal initial condition is given by

$$\begin{aligned} x_0^* &= \pm\alpha/(\alpha^2+\beta^2\omega^2)^{1/2} \; a, \\ \dot{x}_0^* &= \pm\beta\omega^2/(\alpha^2+\beta^2\omega^2)^{1/2} \; a. \end{aligned}$$

The minimum of q^2 is given by $q^{*2} = (\alpha^2+\beta^2\omega^2)a^2$ and the minimum of $E(\mathbf{x}_0)$ by

$$E(\mathbf{x}_0^*) = \mathbf{x}_0^{*\prime} X \mathbf{x}_0{}^* = \lambda_1 q^{*2}.$$

Since $\alpha = 1.12251 \times 10^{-6}$ and $\beta = 1.09315 \times 10^{-6}$, the optimal initial values are $(x_0^*, \dot{x}_0^*) = (0.482784a, 1.60120a)$. Moreover, $q^{*2} = 5.40594 \times 10^{-12}a^2$, and $E(\mathbf{x}_0^*) = 2.25453 \times 10^{-10}a^2$. In Fig. 3 simulation results of asymptotic transfer of orbit are given, where the initial orbit is the larger ellipse and the final orbit is the smaller one. If the unit is taken as 1/100, the size of the larger ellipse is 3847 km. The feedback gain $-B'X_\epsilon$ with three different values of ϵ are considered.

The Lagrangian points L_4 and L_5 are stable in that linearized equations of motion around them have only pure imaginary characteristic roots. Hence these systems with control accelerations become NCVE.

4 Relative orbit transfer of Hill's equations

The relative motion of a spacecraft (chaser) with respect to another (target) in a circular orbit around the Earth is described by autonomous nonlinear differential equations. The linearized equations are known as Hill-Clohessy-Wiltshire equations [7]. The motion in the orbit plane (the in-plane motion) is independent of the out-of-plane motion, and a controlled version has the form

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}, \tag{20}$$

where $\mathbf{x} = [x \quad \dot{x} \quad y \quad \dot{y}]'$, x axis is along the radial direction, y axis along the flight direction of the target, and ω is the orbit rate (angular velocity) of the target. The system (20) is controllable, and the eigenvalues of the state matrix are $(0, 0, \pm j\omega)$. Thus it is NCVE, and $E(x_0) = 0$ for all x_0. It has periodic solutions,

$$\begin{aligned} x(t) &= 2c + a\cos[\omega(t-t_0)+\alpha], \\ y(t) &= d - 3\omega c(t-t_0) - 2a\sin[\omega(t-t_0)+\alpha], \end{aligned}$$

where

$$a = [(3x_0 + 2\dot{y}_0/\omega)^2 + (\dot{x}_0/\omega)^2]^{1/2}, \;\; d = y_0 - 2\dot{x}_0/\omega, \;\; c = 2x_0 + \dot{y}_0/\omega,$$
$$\cos\alpha = -(1/a)\,(3x_0 + 2\dot{y}_0/\omega)\,, \;\; \sin\alpha = -\dot{x}_0/(\omega a),$$

which constitute relative orbits of the chaser, and are useful for rendezvous and flyaround operations. They form ellipses

$$\frac{[x(t) - 2c]^2}{a^2} + \frac{[y(t) - d + 3\omega c(t - t_0)]^2}{(2a)^2} = 1. \tag{21}$$

A small relative orbit which encircles the origin is useful for inspection of the target spacecraft. Using the feedback law

$$u = -B' X_\epsilon x$$

the relative orbit transfer can be fulfilled with arbitrarily little energy in $\mathcal{L}_2$ sense. Actual energy is evaluated by the $\mathcal{L}_1$ norm, but it decreases in general as $\mathcal{L}_2$ norm. This problem is studied in detail in [6]. If the final orbit collapses to the origin, it corresponds to the rendezvous and docking.

References

[1] A. Ichikawa, Null controllability with vanishing energy for discrete-time systems, Systems & Control Letters, to appear.

[2] A. Ichikawa, Null controllability with vanishing energy for infinite dimensional periodic systems, Proc. 17th International Symposium on Mathematical Theory of Networks and Systems, July, 2006, pp. 638-642.

[3] A. Ichikawa, Null controllability with vanishing energy for discrete-time systems in Hilbert space, SIAM J. Control Optim., Vol. 46, No. 2,(2007), pp. 683-693.

[4] A. Ichikawa, Minimum energy control and its applications, Fourth International Conference of Applied Mathematics and Computing, Plovdiv, Bulgaria, August 12-18, 2007.

[5] E. Priola and J. Zabczyk, Null controllability with vanishing energy, SIAM J. Control Optim., Vol. 42, No. 3, (2003), pp. 1013-1032.

[6] M. Shibata and A. Ichikawa, Orbital rendezvous and flyaround based on null controllability with vanishing energy, Journal of Guidance, Control, and Dynamics, Vol. 30, No. 4, 2007, pp.934-945.

[7] B. Wie, Space Vehicle Dynamics and Control, American Institute of Aeronautics and Astronautics, 1998.