Sub-fractional Brownian motion as a model in finance

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1. Fractional and sub-fractional Brownian motions.
   General properties

The fractional Brownian motion (fBm for short) is the best known and most used process with long-dependence property for models in telecommunication, turbulence, finance, etc. This process was first introduced by Kolmogorov (1940) and later studied by Mandelbrot and his coworkers (1968). The fBm is a continuous centered Gaussian process \( \mathcal{B}_k(t) \), starting from zero, with covariance

\[
C_{B_k}(s, t) := E\left( \mathcal{B}_k(t) \mathcal{B}_k(s) \right) = \frac{1}{2} \left( |s|^{2k+1} + |t|^{2k+1} - |t-s|^{2k+1} \right), s, t \in R,
\]

where \( k \in \left( -\frac{1}{2}, \frac{1}{2} \right) \) (\( H = k + \frac{1}{2} \) is called Hurst parameter). The case \( k = 0 \) corresponds to the Brownian motion.

The self-similarity and stationarity of the increments are two main properties for which fBm enjoyed success as a modeling tool. The fBm is the only continuous Gaussian process which is self-similar and has stationary increments.

An extension of Bm which preserves many properties of the fBm, but not the stationarity of the increments, is so called sub-fractional Brownian motion (sfBm for short), i.e., a continuous Gaussian process \( (S^k_t)_{t \geq 0} \), starting from zero, with covariance

\[
C_{S_k}(s, t) := E\left( S^k_t S^k_s \right) = s^{2k+1} + t^{2k+1} - \frac{1}{2} \left[ (s + t)^{2k+1} + |t-s|^{2k+1} \right], s, t \geq 0.
\]

Next we assume \( k \neq 0 \).

The sfBm has properties analogous to those of fBm (see Bojdecki, Gorostiza and Talarczyk, 2004, Dzapotidze-Van Zanten, 2004 and Tudor, 2007):

(i1) Self-similarity: For each \( a > 0 \) the processes \( (S^k_{at})_{t \geq 0} \) has the same distribution as \( (a^{k+\frac{1}{2}} S^k_t)_{t \geq 0} \).

(i2) Covariance: For all \( s, t \geq 0 \),

\[
C_{S_k}(s, t) > 0,
\]

\[
C_{S_k}(s, t) > C_{B_k}(s, t) \text{ if } k \in \left( -\frac{1}{2}, 0 \right),
\]
$C_{Sk}(s, t) < C_{Bk}(s, t)$ if $k \in \left(0, \frac{1}{2}\right)$.

(i3) Non-stationarity of increments: For all $s \leq t$,

$$E \left[|S^k_t - S^k_s|^2\right] = -2^{2k}(t^{2k+1} + s^{2k+1}) + (t + s)^{2k+1} + (t - s)^{2k+1},$$

$$E \left(|S^k_t|^2\right) = (2 - 2^{2k})t^{2k+1},$$

$$(t - s)^{2k+1} \leq E \left[|S^k_t - S^k_s|^2\right] \leq (2 - 2^{2k})(t - s)^{2k+1} \text{ if } k \in \left(-\frac{1}{2}, 0\right),$$

$$(2 - 2^{2k})(t - s)^{2k+1} \leq E \left[|S^k_t - S^k_s|^2\right] \leq (t - s)^{2k+1} \text{ if } k \in \left(0, \frac{1}{2}\right).$$

(i4) Correlation of increments and long-range dependence: For $0 \leq u < v \leq s < t$, define

$$R^k_{u,v,s,t} = E \left[\left(B^k_v - B^k_u\right)\left(B^k_t - B^k_s\right)\right],$$

$$C^k_{u,v,s,t} = E \left[\left(S^k_v - S^k_u\right)\left(S^k_t - S^k_s\right)\right].$$

Then

$$C^k_{u,v,s,t} = \frac{1}{2} \left[(t + u)^{2k+1} + (t - u)^{2k+1} + (s + v)^{2k+1} + (s - v)^{2k+1}
- (t + v)^{2k+1} - (t - v)^{2k+1} - (s + u)^{2k+1} - (s - u)^{2k+1}\right];$$

$$R^k_{u,v,s,t} < C^k_{u,v,s,t} < 0 \text{ if } k \in \left(-\frac{1}{2}, 0\right),$$

$$0 < C^k_{u,v,s,t} < R^k_{u,v,s,t} \text{ if } k \in \left(0, \frac{1}{2}\right).$$

For $u \geq 0, r > 0$ let $\rho_{Bk}(u, r)$ and $\rho_{Sk}(u, r)$ denote the correlation coefficients of $B^k_{u+r} - B^k_u, B^k_{u+2r} - B^k_{u+r}$ and $S^k_{u+r} - S^k_u, S^k_{u+2r} - S^k_{u+r}$. Then

$$|\rho_{Sk}(u, r)| \leq |\rho_{Bk}(u, r)|,$$

and we have the long-range dependence

$$R^k_{u,v,s+t,t} \sim k(2k + 1)(t - s)(v - u)\tau^{2k-1} \text{ as } \tau \to \infty,$$

$$C^k_{u,v,s+t,t} \sim k(2k + 1)(1 - 2k)(v^2 - u^2)\tau^{2(k-1)} \text{ as } \tau \to \infty.$$
Therefore the covariance of increments of sfBm over non-overlapping intervals have the same sign but are smaller in absolute value than those of fBm and the increments on the intervals \([u, u + r], [u + r, u + 2r]\) are more weakly correlated than those of fBm. Moreover the long-range dependence decays at a higher rate for sfBm than for fBm (these properties justifies the name sfBm).

\[ (i_5) \text{ Short memory: For each } a > 0, \]
\[ \sum_{n \geq a + 1} \text{cov} \left( S_{a+1}^k - S_a^k, S_{n+1}^k - S_n^k \right) < \infty. \]

The above mentioned properties make sfBm a possible candidate for models which involve long-dependence, self-similarity and non-stationarity of the increments.

\[ (i_6) \text{ } S^k \text{ is not a Markov process.} \]

\[ (i_7) \text{ Hölder paths: For each } \varepsilon < k + \frac{1}{2} \text{ and each } T > 0 \text{ there exists a random variable } K_{\varepsilon,T} \text{ such that} \]
\[ |S_t^k - S_s^k| \leq K_{\varepsilon,T} |t - s|^{k + \frac{1}{2} - \varepsilon}, \quad s, t \in [0, T], \ a.s. \]

\[ (i_8) \text{ Variance: For each } T > 0, \]
\[ \sum_{i=0}^{n-1} \left| S_{\frac{(i+1)T}{n}}^k - S_{\frac{iT}{n}}^k \right|^p \xrightarrow{L^2} 0 \text{ if } p > \frac{2}{2k + 1}, \]
\[ \sum_{i=0}^{n-1} \left| S_{\frac{(i+1)T}{n}}^k - S_{\frac{iT}{n}}^k \right|^p \xrightarrow{L^2} \rho_{2k+1}^2 T \text{ if } p = \frac{2}{2k + 1} \]
\[ \sum_{i=0}^{n-1} \left| S_{\frac{(i+1)T}{n}}^k - S_{\frac{iT}{n}}^k \right|^p \xrightarrow{n \to \infty} \infty \text{ if } p < \frac{2}{2k + 1}, \]

where \( \rho_p = E \left( |N(0, 1)|^p \right) \).

Such a result for fBm is obtained mainly by using self-similarity and stationarity of the increments (in particular ergodic theorem). For sfBm the lack of stationarity of the increments is replaced by linear regression.

\[ (i_9) \text{ If } W \text{ is a Brownian motion independent of } S_k \text{ and } k > \frac{1}{4}, \text{ then the process } S^k + W \text{ is a semimartingale equivalent in law with } W. \]

\[ (i_{10}) \text{ The fBm } S^k \text{ is not a quasimartingale. In particular } S^k \text{ is not a semimartingale.} \]
2. Pathwise integral with respect to sfBm

For functions of one variable the Riemann-Stieltjes integral \( \int_0^T f(t)dg(t) \) was extended to functions with unbounded variation, essentially by using fractional integrals or \( p \)-variation (Dudley and Norvaisa, Feyel and Pradelle, Kondurar, Mikosch and Norvaisa, Young and Zähle).

For a function \( f : [0, T] \to \mathbb{R} \) a partition \( \pi : 0 = t_0 < ... < t_N = T \) and \( p \geq 1 \) we define the \( p \)-variation associated to \( f \) by

\[
v_p(f, \pi) = \sum_i |f(t_i) - f(t_{i-1})|^p,
\]

Denote

\[
v_p^0(f) = \lim_{|\pi_n| \to 0} v_p(f, \pi_n), \quad v_p(f) = \sup_{\pi} v_p(f, \pi),
\]

for all homogeneous partitions \( \pi_n = (iT\delta_n), \delta_n \to 0 \).

We say that \( f \) has finite \( p \)-variation if \( v_p^0(f) < \infty \) and bounded \( p \)-variation if \( v_p(f) < \infty \).

**Remark 2.1.** A function \( f : [0, T] \to \mathbb{R} \) has bounded \( p \)-variation if and only if \( f = h \circ g \), where \( h : [0, T] \to \mathbb{R} \) is bounded nondecreasing nonnegative function and \( g : [h(0), h(T)] \to \mathbb{R} \) is \( \frac{1}{p} \)-Hölder.

The family of functions with bounded \( p \)-variation is denoted by \( \mathcal{W}_p \) and it becomes a Banach space under the norm

\[
\|f\|_{[0,T],p} = \max \left( v_p^0(f)^{\frac{1}{p}}, \|f\|_{\infty} \right).
\]

We denote by \( H_{[0,T],\alpha} \) the class of all \( \alpha \)-Hölder functions \( f : [0, T] \to \mathbb{R} \) with \( f(0) = 0 \) and define

\[
\|f\|_{[0,T],\alpha} = \sup_{u \neq v, 0 \leq u < v \leq T} \frac{|f(u) - f(v)|}{(v - u)^\alpha}.
\]

**Remark 2.2** (Young, Dudley-Norvaisa) If \( f \in \mathcal{W}_p, g \in \mathcal{W}_q, \frac{1}{p} + \frac{1}{q} > 1 \) and \( f, g \) have no common discontinuities, then the Stieltjes integral \( \int_0^T f(t)dg(t) \) exists as limit of the corresponding Riemann-Stieltjes sums.
In particular if $f$ is $\alpha$-Hölder, $g$ is $\beta$-Hölder with $\alpha + \beta > 1$, then the Stieltjes integral $\int_0^T f(s)dg(s)$ exists and is $\beta$-Hölder. Moreover for every $0 < \varepsilon < \alpha + \beta - 1$,

$$\left| \int_0^T f(s)dg(s) \right| \leq C(\alpha, \beta) \|f\|_{[0,T],\alpha} \|g\|_{[0,T],\beta} T^{1+\varepsilon}, \quad (2.1)$$

(Feyel-Pradelle).

Concerning the variation of sfBm we have the following result.

**Proposition 2.3.**

\begin{align*}
v^0_p(S^k) &= 0, \quad v_p(S^k) < \infty \ if \ p > \frac{2}{2k+1}, \quad (2.2) \\
v^0_{2^k+1}(S^k) &= v_{2^k+1}(S^k) = \rho \frac{2}{2^k+1}, \quad (2.3) \\
v^0_p(S^k) &= v_p(S^k) = \infty \ if \ p < \frac{2}{2k+1}. \quad (2.4)
\end{align*}

**Remark 2.4.** From the above proposition it follows that a.s. $S^k \in \mathcal{W}_p$ if and only if $p > \frac{2}{2k+1}$.

Moreover, for every process $(u_t)_{t \in [0,T]}$ with paths a.s. in $\mathcal{W}_q$ with $q < \frac{2}{1-2k}$, the Riemann-Stieltjes integral $\int_0^t u_r dS^k_r$ is well defined a.s. In particular if $u$ has $\alpha$-Hölder paths for some $\alpha > \frac{1-2k}{2}$, then the Riemann-Stieltjes integral $\int_0^t u_r dS^k_r$ is well defined and has $\beta$-Hölder paths, for every $\beta < k + \frac{1}{2}$.

Since every Riemann-Stieltjes integral obeys the change of variable formula, we have the following result.

**Theorem 2.5.** If $F(t, x) \in C^1$ and the mapping $t \rightarrow \frac{\partial F}{\partial x}(t, S^k_t) \in \mathcal{W}_q$ with $q < \frac{2}{1-2k}$, then for all $s, t \in [0, T]$,\n
$$F(t, S^k_t) - F(s, S^k_s) = \int_s^t \frac{\partial F}{\partial x}(r, S^k_r) dS^k_r + \int_s^t \frac{\partial F}{\partial t}(r, S^k_r) dr. \quad (2.5)$$

**Remark 2.6.** (a) The pathwise integral may not exist: for example for $k \in (-\frac{1}{2}, 0)$ the integral $\int_0^T S^k_t dS^k_t$ does not exists. Indeed if we assume that the integral exists, then using (2.4) we obtain

$$\infty = v^0_2(S^k) = \lim_{n \to \infty} \sum_i \left| S^k_{\frac{i}{n}} - S^k_{\frac{i-1}{n}} \right|^2 = \lim_{n \to \infty} \sum_i S^k_{\frac{i}{n}} \left( S^k_{\frac{i}{n}} - S^k_{\frac{i-1}{n}} \right)$$

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\[ - \lim_{n \to \infty} \sum_i S_{i/n}^k \left( S_{i+1/n}^k - S_{i/n}^k \right) = \int_0^T S_t^k \, dS_t - \int_0^T S_t^k \, dS_t = 0. \]

(b) If the pathwise integral exists, then it may have not zero expectation: for example if \( k \in \left(0, \frac{1}{2}\right) \), by using (2.5) we obtain

\[ E \left( \int_0^T S_t^k \, dS_t \right) = \frac{1}{2} E \left( |S_T^k|^2 \right) = (1 - 2^{2k-1}) T^{2k+1}. \]

3. Wiener integral and the sub-fractional Girsanov theorem

Next we restrict to a finite time interval \([0, T]\). Denote by \( \mathcal{E} \) the family of elementary deterministic functions \( f : [0, T] \to \mathbb{R} \), i.e., \( f = \sum_{j=0}^{n-1} f_j 1_{[t_j, t_{j+1})} \).

For \( f \in \mathcal{E} \) we define the Wiener integral by

\[ I(f) = \sum_{j=0}^{n-1} f_j \left( S_{t_{j+1}}^k - S_{t_j}^k \right), \]

and the bilinear and symmetric form

\[ (f, g)_{S^k} = E \left( I(f) I(g) \right). \]

The closure of the step functions with respect to the above inner product (denoted by \( \Lambda_{S^k,T}^f \)) is called the domain of the Wiener integral.

Let \( f : [0, T] \to \mathbb{R} \) be a measurable application and \( \alpha, \sigma, \eta \in \mathbb{R} \). We define the Erdély-Kober-type fractional integral

\[ \left( I_{T-\sigma,-\eta}^\alpha f \right)(s) = \frac{\alpha^\sigma}{\Gamma(\alpha)} \int_s^T t^{\sigma(1-\alpha)-1} f(t) \left( t^\sigma - s^\sigma \right)^{1-\alpha} \, dt, \quad s \in [0, T], \quad \alpha > 0, \quad \eta > 0, \quad (3.1) \]

\[ \left( I_{T-\sigma,-\eta}^\alpha f \right)(s) = s^{\sigma n} \left( -\frac{d}{\sigma s^{\sigma-1} \, ds} \right)^n s^{\sigma(\alpha-n)} \left( I_{T-\sigma,-\eta}^{\alpha+n} f \right)(s), \quad s \in [0, T], \quad \alpha > -n, \quad (3.2) \]

\[ \left( I_{0+,-\eta}^\alpha f \right)(s) = \frac{\alpha^{\sigma(-\alpha-\eta)}}{\Gamma(\alpha)} \int_0^s t^{\sigma(1-\alpha)-1} f(t) \left( s^\sigma - t^\sigma \right)^{1-\alpha} \, dt, \quad s \in [0, T], \quad \alpha > 0, \quad (3.3) \]
\begin{equation}
(I_{0+,\sigma_n}^\alpha f)(s) = s^{-(\alpha+\eta)} \left( \frac{d}{\sigma s^{\alpha-1}ds} \right)^n s^{\alpha+\eta} (I_{0+,\sigma_n}^{\alpha+n} f)(s), \quad s \in [0,T], \quad \alpha > -n. \tag{3.4}
\end{equation}

We introduce the following kernels

\begin{equation}
n(t, s) = \frac{\sqrt{\pi}}{2k} f_{T-2,k}^{1+k} \left( u_k 1_{[0,t]} \right)(s), \tag{3.5}
\end{equation}

\begin{equation}
\psi(t, s) = \frac{s^k}{\Gamma(1-k)} \left[ t^{k-1} \left( t^2 - s^2 \right)^{-k} - (k-1) \int_s^t \left( u^2 - s^2 \right)^{-k} u^{k-1} du \right] 1_{(0,t)}(s). \tag{3.6}
\end{equation}

**Remark 3.1.** The function $\psi(t, \cdot) \in \Lambda_{k,T}^{sf}$ and satisfies (uniquely) the equality

\begin{equation}
\frac{\sqrt{\pi}}{2k} f_{T-2,k}^{1+k} \left( u_k \psi(t, \cdot) \right)(s) = 1_{(0,t)}(s). \tag{3.7}
\end{equation}

**Theorem 3.2** (Dzhaparidze-Van Zanten, 2004, Tudor, 2007). The process

\begin{equation}
W^k_t = \int_0^t \psi(t, s) dS^k_s,
\end{equation}

is the unique Brownian motion such that

\begin{equation}
S^k_t = c_k \int_0^t n(t, s) dW^k_s, \quad t \in [0,T], \tag{3.8}
\end{equation}

\begin{equation}
c_k^2 = \frac{\Gamma (2k + 2) \sin \pi \left( k + \frac{1}{2} \right)}{\pi}. \tag{3.9}
\end{equation}

Moreover $S^k$ and $W^k$ generate the same filtration.

**Theorem 3.3** (Tudor, 2007). (i) If $-\frac{1}{2} < k < 0$, then the space $\left( \Lambda_{k,T}^{sf}, \langle \cdot, \cdot \rangle_{\Lambda_{k,T}^{sf}} \right)$, where

\begin{equation}
\Lambda_{k,T}^{sf} = \left\{ f : [0,T] \to R : \exists \varphi_f \in L^2([0,T]), I_{T-2,k+\frac{1}{2}} \left( \frac{2k}{\sqrt{\pi}} \varphi_f \right)(t) = t^k f(t) \right\}, \tag{3.10}
\end{equation}

is the unique Brownian motion such that

\begin{equation}
S^k_t = c_k \int_0^t n(t, s) dW^k_s, \quad t \in [0,T], \tag{3.11}
\end{equation}

\begin{equation}
c_k^2 = \frac{\Gamma (2k + 2) \sin \pi \left( k + \frac{1}{2} \right)}{\pi}. \tag{3.12}
\end{equation}

Moreover $S^k$ and $W^k$ generate the same filtration.
\[
\langle f, g \rangle_{\Lambda^s_{k,T}} = c_k^2 \int_0^T \varphi_f(t)\varphi_g(t)dt,
\]

(3.11)
is the domain of the Wiener integral and

\[
\int_0^T f(t)dS^k_t = c_k \int_0^T \varphi_f(t)dW^k_t.
\]

(3.12)

(ii) If \(0 < k < \frac{1}{2}\), then the space \(\left(\Lambda^s_{k,T}, \langle \cdot, \cdot \rangle_{\Lambda^s_{k,T}}\right)\), where

\[
\Lambda^s_{k,T} = \left\{ f \in \mathcal{D}(0,T) : \exists f^* \in \mathcal{S}', \ f^* \text{ odd, supp}(f^*) \subset [-T,T] \right\},
\]

\[
f^* |_{[0,T]} = f, \ \int_R \left| \hat{f}^*(x) \right|^2 |x|^{-2k} dx < \infty, \]

(3.13)
is the domain of the Wiener integral.

If we define

\[
|\Lambda|_{k,T}^s = \left\{ f : [0,T] \longrightarrow R : I_{T-\frac{1-k}{2}}^{k} \left( u^k |f| \right) \in L^2 ([0,T]) \right\},
\]

(3.15)
then we have the strict inclusion \(|\Lambda|_{k,T}^s \subset \Lambda^s_{k,T}\) and

\[
\int_0^T f(t)dS^k_t = c_k \int_0^T I_{T-\frac{1-k}{2}}^{k} \left( \frac{\sqrt{\pi}}{2^k} u^k f \right) (t)dW^k_t, \ f \in L^2 ([0,T]).
\]

(3.16)

Moreover, if \(k \in \left(0, \frac{1}{2}\right)\), \(f \in |\Lambda|_{k,T}^s\),

\[
\langle f, g \rangle_{\Lambda^s_{k,T}} = c_k^2 \left( I_{T-\frac{1-k}{2}}^{k} \left( \frac{\sqrt{\pi}}{2^k} u^k f \right), I_{T-\frac{1-k}{2}}^{k} \left( \frac{\sqrt{\pi}}{2^k} u^k g \right) \right)_{L^2([0,T])}
\]

\[
= \int_0^T \int_0^T f(u)g(v)\varphi_k(u,v)dudv.
\]

(3.17)
\( \varphi_k(u, v) = k(2k + 1) \left[ |u - v|^{2k-1} - (u + v)^{2k-1} \right] . \)

**Theorem 3.4 (Prediction).** For every \( 0 \leq t \leq T, \)

\[
\hat{S}^k_{T|t} := E \left[ \hat{S}^k_T \mid \mathcal{F}^S_t \right] = S^k_t + \int_0^t \Psi^k_{t,t}(u) dS^k_u
\]

\[
\Psi^k_{t,t}(u) = e^k \int_0^t n(T, u) dW^k_u, \tag{3.18}
\]

\[
\Psi^k_{t,t}(u) = \frac{2 \sin \pi k}{\pi} u(t^2 - u^2)^{-k} \int_t^T \frac{(z^2 - t^2)^k}{z^2 - u^2} z^k dz. \tag{3.19}
\]

In particular (since \( n(T, u) > 0 \) on \( (0, T) \)) we have the equality \( \mathcal{F}^S_t = \mathcal{F}^S_{\hat{S}^k} \).

Denote

\[
d_k = \frac{2^k}{e_k \Gamma(1 - k) \sqrt{\pi}}.
\]

The process

\[
M^k_t = d_k \int_0^t s^{-k} dW^k_s, \tag{3.20}
\]

is called the sub-fractional fundamental martingale.

The following result is straightforward.

**Remark 3.5.** For every \( s < t, \) \( M^k_t - M^k_s \) is independent of \( \mathcal{F}^S_s \) and \( \mathcal{F}^S_s = \mathcal{F}^{M^k}_s = \mathcal{F}^{W^k}_s. \)

For \( f : [0, T] \to R \) with \( \int_0^T f^2(s) s^{-2k} ds < \infty \) define the probability \( Q_f \) by

\[
\frac{dQ_f}{dP} |_{\mathcal{F}^S_t} = \exp \left( \int_0^t f(s) dM^k_s - \frac{1}{2} \int_0^t f^2(s) \langle M^k \rangle_s \right)
\]

\[
= \exp \left( \int_0^t f(s) dM^k_s - \frac{d^2_k}{2} \int_0^t f^2(s) s^{-2k} ds \right), \tag{3.21}
\]

and denote

\[
(\Psi_k f)(s) = \frac{1}{\Gamma(1 - k)} I_{0+2,-k}^k f(s). \tag{3.22}
\]
Theorem 3.6 (Girsanov). For $f$ as above, the process
\[ S^k_t - \int_0^t (\Psi_k f)(s) \, ds, \quad t \in [0, T], \]
is a $Q_f$-sfBm.
In particular if $f \equiv a \in R$ it follows that the process $(S^k_t - at)_{t \in [0,T]}$ is $Q_a$-sfBm.

4. Anticipating stochastic calculus for sfBm
and Clark-Ocone representation formula

Multiple integrals w.r.t. fBm were introduced by Dasgupta-Kallianpur and Duncan-Hu and Pasik-Duncan for the fBm with. The techniques used in these papers involve Wick product and reproducing kernel Hilbert space theory.
We study multiple fractional and subfractional integrals by using this transfer principle from multiple Brownian integrals via a Gamma type operator. Then the chaos form of the sub-fractional anticipating integral is considered.

We assume that $k \in \left(0, \frac{1}{2}\right)$.
For a function $f : [0, T]^n \to R$ we consider the $n$–dimensional form $I^{\alpha,n}_{-\sigma,\eta} f$ of the Erdély-Kober-type fractional integrals (3.1),
\[
\left(I^{\alpha,n}_{-\sigma,\eta} f\right)(s_1, \ldots, s_n) = \left(\frac{\sigma}{\Gamma(\alpha)}\right)^n \prod_{j=1}^n s_j^{\sigma}\]
\[
\times \int_{s_1}^T \cdots \int_{s_n}^T \prod_{j=1}^n \frac{t_j^{\sigma(1-\alpha-\eta)-1}}{(t_j^\sigma - s_j^\sigma)^{1-\alpha}} f(t_1, \ldots, t_n) dt_1 \cdots dt_n, \quad s_j \in [0, T], \quad \alpha > 0. \tag{4.1}
\]
\[
\left(I^{\alpha,n}_{\sigma,\eta} f\right)(s) = \left(\frac{\sigma}{\Gamma(\alpha)}\right)^n \prod_{j=1}^n s_j^{-\sigma(\alpha+\eta)}
\times \int_0^{s_1} \cdots \int_0^{s_n} \prod_{j=1}^n \frac{t_j^{\sigma(1-\alpha-\eta)-1}}{(s_j^\sigma - t_j^\sigma)^{1-\alpha}} f((t_1, \ldots, t_n)) dt_1 \cdots dt_n, \quad s_j \in [0, T], \quad \alpha > 0. \tag{4.2}
\]
We shall denote by $I^{1/2}_{\sigma} (f)$ the multiple Wiener-Itô integral with respect to $W^k$ and we introduce the space
\[
|\Lambda^{n|s,f}_{k,T} = \left\{ f : [0, T]^n \to R : I^{k,n}_{-\sigma,\eta,\frac{1}{2}} \left( \prod_{j=1}^n u_j^{k} f \right) \in L^2 ([0, T]^n) \right\}. \]
**Definition 4.2.** If \( f \in |\Lambda^{n}|^{sf}_{k,T} \), \( f \) symmetric, then we define the multiple sub-fractional integral of \( f \) with respect to \( S^k \) by

\[
I^k_n(f) = c_k \frac{\sqrt{\pi}}{2^k} n I^k \left( I^{k,n}_{T-2,\frac{1}{2}} \left( \prod_{j=1}^{n} u_j^k f \right) \right), \tag{4.3}
\]

We have the equalities

\[
\|f\|_{|\Lambda^{n}|^{sf}_{k,T}}^2 := \|I^k_n(f)\|_{L^2(\Omega, F, P)}^2 = c_k \frac{\sqrt{\pi}}{2^k} n \|I^{k,n}_{T-2,\frac{1}{2}} \left( \prod_{j=1}^{n} u_j^k f \right)\|_{L^2([0,T]^n)}^2
\]

\[
= \int_{[0,T]^n} f(u_1, \ldots, u_n) f(v_1, \ldots, v_n) \prod_{j=1}^{n} \varphi_k(u_j, v_j) du_j dv_j. \tag{4.4}
\]

We can now define the space

\[
|L^2_k| = \left\{ F \in L^2(\Omega, F, P) : F = \sum_{n=0}^{\infty} I^k_n(f_n), \ f_n \in |\Lambda^{n}|^{sf}_{k,T}, \ f_n \text{ symmetric} \right\}.
\]

**Remark 4.3.** Since \( |\Lambda^{n}|^{sf}_{k,T} \) is not complete, \( |L^2_k| \) is a strict subspace of \( L^2(\Omega, F, P) \).

Like in the fBm case (Bender, Bender-Elliot) the multiplication by an indicator function can increase the norm in \( \Lambda^{sf}_{k,T} \).

Following the ideas of Hu-Oksendal we introduce the following

**Definition 4.4.** For \( F \in |L^2_k| \) and \( t \in [0, T] \), we define the sub-fractional quasi-conditional expectation of \( F \) with respect to \( F_t^{S^k} \) by

\[
\tilde{E} \left[ F \mid F_t^{S^k} \right] = \sum_{n=0}^{\infty} I^k_n(1_{(0,t)^n} f_n), \tag{4.5}
\]

provided the series converges in \( L^2(\Omega, F, P) \), i.e.,

\[
\sum_{n=1}^{\infty} n! \|1_{(0,t)^n} f_n\|_{|\Lambda^{n}|^{sf}_{k,T}}^2 < \infty. \tag{4.6}
\]

**Remark 4.5.** To see the difference between conditional expectation and quasi-conditional expectation note that for \( t \in (0, T) \),

\[
\tilde{E} \left[ S^k_T \mid F_t^{S^k} \right] = S^k_t,
\]
\[ E \left[ S^k_T \mid \mathcal{F}^S_t \right] = S^k_t + \int_0^t \Psi_{t,T}(u) dS^k_u. \]

The sub-fractional quasi-conditional expectation need not exist.

**Definition 4.6.** The random variable \( F = \sum_{n=1}^{\infty} I^k_n(f_n) \in L^2_k \) is sub-fractional Malliavin differentiable if

\[ D^k_t F = \sum_{n=1}^{\infty} n I^k_{n-1}(f_n(\cdot,t)), \quad (4.7) \]

converges in \( L^2_k \) for a.e. \( t \in [0,T] \), i.e.,

\[ \sum_{n=1}^{\infty} nn! \| f_n(\cdot,t) \|^2_{|L^2_{n-1}|^f_{k,T}} < \infty. \quad (4.8) \]

The sub-fractional Clark-Ocone derivative at time \( t \) of \( F \in L^2_k \) is defined by

\[ \nabla^k_t F = \hat{E} \left[ D^k_t F \mid \mathcal{F}^S_t \right], \quad (4.9) \]

provided \( F \) is sub-fractional Malliavin differentiable and the quasi-conditional expectation exists, i.e.,

\[ \sum_{n=1}^{\infty} nn! \| 1_{(0,t)^n-1} f_n(\cdot,t) \|^2_{|L^2_{n-1}|^f_{k,T}} < \infty. \quad (4.10) \]

**Remark 4.7.** The sub-fractional Clark-Ocone derivative need not exist.

Consider the set \( \mathcal{S}^{|f|}_{k,T} \) of all measurable processes \( (u_t)_{t \in [0,T]} \) such that:

(a) \( u_t \in L^2_k \) for a.e. \( t \in [0,T] \) and

\[ u_t = \sum_{n=0}^{\infty} I^k_n(f_n(\cdot,t)), \quad (4.11) \]

with \( f_n(\cdot,t) \in |L^{n+1}|^f_{k,T}, \ f_n \in |L^{n+1}|^f_{k,T} \).

(b) The following series is convergent

\[ \sum_{n=1}^{\infty} (n+1)! \| \text{sym} (f_n) \|^2_{|L^{n+1}|^f_{k,T}} < \infty. \]
**Definition 4.8.** For a process $u \in \mathcal{S}^k_{[k,T]}$ define the chaos sub-fractional Skorohod integral $\delta_{k,T}^h(u)$ of $u$ with respect to $S^k$ by

$$\delta_{k,T}^h(u) := \sum_{n=0}^{\infty} I^k_{n+1} \left( \text{sym} \left( f_n \right) \right).$$

From (b) it follows that the previous converges in $|L^2_k|$.

**Remark 4.9.** Note that the chaos sub-fractional Skorohod integral has zero expectation (in contrast with the pathwise integral).

Define $|D_{k,2}^{1.2}|$ as the family of all $F = \sum_{n=1}^{\infty} I_n^k(f_n) \in |L^2_k|$ such that

$$\sum_{n=1}^{\infty} \frac{n!}{|f_n|^2_{\mathcal{A}^{n+1}} k^{k+1}} < \infty. \quad (4.12)$$

**Theorem 4.10 (Clark-Ocone representation formula).** If $F \in |D_{k,2}^{1.2}|$ then the Clark-Ocone derivative exists and satisfies

$$\nabla^k_t F = \sum_{n=1}^{\infty} n^{k-1} f_{n-1}(, t), \quad (4.13)$$

and

$$\int_{[0,T]^2} E \left( \left| \nabla^k_t F \right| \right) \varphi_k(s, t) ds dt < \infty. \quad (4.14)$$

Moreover, $\nabla^k F$ is sub-fractional Skorohod integrable and the following Clark-Ocone representation formula holds

$$F = E(F) + \delta_{k,T}^h \left( \nabla^k F \right). \quad (4.15)$$

**Remark 4.11.** A representation of a random variable $F \in |L^2_k|$ in the form

$$F = E(F) + \delta_{k,T}^h (u), \quad (4.16)$$

need not be unique.

For example: By using the product formula it is easily see that qwe have the relations

$$\left( S^k_T \right)^2 = \delta_{k,T}^h \left( S^k_T \right) + (2 - 2^{2k}) T^{2k+1},$$

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Proposition 4.12. A representation of the form (4.16) with u adapted is unique.

Remark 4.13. In (4.15) the integrand is adapted.

Proposition 4.14. For $0 < \tau < \sigma$ and $F \in L^2(R, N(0, \sigma^2))$ define
\[
G_{\sigma, \tau}^F(x) = \frac{1}{\sqrt{2\pi (\sigma^2 - \tau^2)}} \int_R F(y) e^{-\frac{(x-y)^2}{2(\sigma^2 - \tau^2)}} dy.
\] (4.17)

If $f \in |\Lambda|_{k,T}^{sf}$, $(a,b) \subset [0, T]$ and $\|1_{(a,b)}f\|_{\Lambda_k^{sf}} < \|f\|_{\Lambda_k^{sf}}$, then for every $F \in L^2(R, N(0, \|f\|_{\Lambda_k^{sf}}^2))$ we have the equality
\[
E \left[ F \left( I_1^k (f) \right) | S_t^k : a \leq t \leq b \right] = G_{\|f\|_{\Lambda_k^{sf}}, 1_{(a,b)}f \|_{\Lambda_k^{sf}}, 1_{(a,b)}f}^F \left( I_1^k (1_{(a,b)}f) \right).
\] (4.18)

Moreover, if $F \in C^1(R)$, $F' \in L^2(R, N(0, \|f\|_{\Lambda_k^{sf}}^2))$, then the following relation holds
\[
F \left( I_1^k (f) \right) = E \left[ I_1^k (f) \right] + \delta_{k,T}^{ch} \left( G_{\|f\|_{\Lambda_k^{sf}}, 1_{(a,b)}f \|_{\Lambda_k^{sf}}, 1_{(a,b)}f}^{F'} \left( I_1^k (1_{(a,b)}f) \right) f \right).
\] (4.19)

5. Sub-fractional Black-Scholes model

Next we consider the sub-fractional Black-Scholes model.

In this model the bank account has the dynamics
\[
\frac{dB_t}{B_t} = rB_t dt, \ 0 \leq t \leq T, \ B_0 = 1,
\] (5.1)

so that $B_t = \exp(rt)$ and the price of the risky asset has sub-fractional log normal dynamics
\[
\frac{dS_t}{S_t} = \mu S_t dt + \sigma S_t dS_t^k, \ 0 \leq t \leq T, \ S_0 = s_0 > 0,
\] (5.2)

where $\mu$ is the mean rate of return and $\sigma > 0$ is the volatility.
If we interpret the stochastic integral in (5.2) as Riemann-Stieltjes, then by the change of variable formula (2.5) the solution of (5.2) is

\[ S_t = s_0 \exp \left( \mu t + \sigma S_t^k \right). \]  

(5.3)

A couple \( \pi = \{(u_t)_{t \in [0,T]} : (v_t)_{t \in [0,T]}\} \) of \( \mathcal{F}_t^S \)-adapted processes is called portfolio. The wealth or the value of the portfolio \( \pi \) is the process

\[ V_t^\pi = u_t \exp (rt) + v_t S_t, \quad 0 \leq t \leq T. \]  

(5.4)

We say that the portfolio \( \pi \) is self-financing if

\[ V_t^\pi = V_0^\pi + r \int_0^t u_s \exp (rs) \, ds + \int_0^t v_s dS_s, \quad 0 \leq t \leq T. \]  

(5.5)

An arbitrage is an self-financing portfolio \( \pi \) such that \( V_0^\pi = 0 \), \( V_T^\pi \geq 0 \) and \( P(V_T^\pi > 0) > 0 \).

As soon as we have the dynamics of the risky asset given by Riemann-Stieltjes integral, always this leads to the existence of arbitrage opportunities. As example we recall the Shiryaev construction or an arbitrage for \( \mu = r \), \( \sigma = 1 \) :

\[ \pi = (u, v), \quad u_t = 1 - \exp(2S^k_t), \quad v_t = 2 \left[ \exp(S^k_t) - 1 \right]. \]

From the change of variables formula (2.5) it follows that \( \pi \) is self-financing and moreover \( \pi \) is arbitrage, since

\[ V_T^\pi = \left[ \exp(S_T^k) - 1 \right]^2 \exp(rT) > 0. \]

An alternative which works in the case of pathwise cost is to restrict the class of admissible portfolios, but to remain stil big enough to cover hedges for relevant options and also to consider mixed cost models.

In this respect we consider the mixed model with the stock price given by

\[ S_t^{k,a} = s_0 \exp \left\{ \sigma S_t^k + aW_t + \mu t - (1 - 2^{2k-1})\sigma^2 t^{2k+1} - \frac{1}{2}a^2 t \right\}, \]  

(5.6)

where \( \sigma, s_0 > 0, a \in \mathbb{R}, W \) is a Bm and the mixed process \( \left( \sigma S_t^k + aW_t \right)_t \) is assumed to be Gaussian (this heapens, for example, if \( S^k \) and \( W \) are independent).
Remark 5.1. The mixed process \( \left( \sigma S^k_t + aW_t \right)_t \) is a semimartingale equivalent with \((aW_t)_t\) if \( k \in \left( \frac{1}{4}, \frac{1}{2} \right) \) and is not a semimartingale if \( 0 < k \leq \frac{1}{4} \).
The following class of restringed class of self-financing portfolios is considered: a self-financing portfolio \( \pi = (u, v) \) is \textit{nds-admissible} (no-doubling strategy) if there exists \( a \geq 0 \) such that \( V_t^\pi \geq a \) for all \( 0 \leq t \leq T \) P-a.s.
A nds-admissible portfolio \( \pi = (u, v) \) is regular if there exists a differentiable function \( \varphi : [0, T] \times R^4_+ \to R \) such that
\[
v_t = \varphi \left( t, S^k_t, \max_{0 \leq s \leq t} S^{k,a}_s, \min_{0 \leq s \leq t} S^{k,a}_s, \int_0^t S^{k,a}_s ds \right).
\]
As a consequence of a more general result due to Bender-Sottined-Valkeila (2006) the following result holds:

**Theorem 5.2.** The mixed model is arbitrage free in the class of regular portfolios. Moreover, European, Asian, lookback options can be hedged with regular portfolios, with the same functionals and hedging prices as in the classical Black-Scholes model.

A different alternative to the pathwise approach of the stock price is to consider the chaos form of the stochastic integral in (5.2).
We shall use the notation \( \int_0^t f(s)\delta_{k,T}^{ch}B^H_s \) for the sub-fractional Skorohod integral \( \delta_{k,T}^{ch} \left( 1_{(0,t)} f \right) \).
Therefore the price of the risky asset has the dynamics
\[
dS_t = \mu S_t dt + \sigma S_t \delta_{k,T}^{ch} S^k_t, \quad 0 \leq t \leq T, \quad S_0 = s_0 > 0.
\]
In the present situation the portofolio \( \pi \) is \textit{self-financing} if \( v,S \in L^1 (\left[ 0, T \right] ) \), \( 1_{(0,t)} v.S \in \left| S^{\text{ch},sf}_{k,T} \right| \) for a.a. \( t \), and
\[
V^\pi_t = V^\pi_0 + \int_0^t \left( r u_s \exp (rs) + \mu v_s S_s \right) ds + \sigma \int_0^t v_s S_s \delta_{k,T}^{ch} S^k_s, \quad 0 \leq t \leq T.
\]
A self-financing portofolio \( \pi \) is \textit{admissible} if \( V^\pi_t \) is bounded below for all \( 0 \leq t \leq T \). In order to find the explicit form of the \( S_t \) in (5.7) we consider the following sub-fractional affine equation
\[
X_t = \eta + \int_0^t \left[ a_0(s) + a(s) X_s \right] ds + \int_0^t \left[ b_0(s) + b(s) X_s \right] \delta_{k,T}^{ch} B^H_s, \quad t \in [0, T],
\]
\( \eta \in L^2(\Omega, \mathcal{F}, P) \) and \( a_0, b_0, a, b : [0, T] \to \mathbb{R} \) are measurable and bounded functions.

**Definition 5.3.** A process \((X_t)_{t \in [0,T]}\) is a strong solution of (5.9) if

(i) \( X \in L^1([0,T]) \) and \( 1_{[0,t]}(.)b(.)X \) is sub-fractional Skorohod integrable for a.a. \( t \in [0,T] \).

(ii) For a.a. \( t \in [0,T] \), the equation (5.9) is satisfied \( P-a.s. \).

We introduce the following notation: Given \( t_1, \ldots, t_p \) and a symmetric function of \( p \) \( j \)-variables \((j < p)\), we denote by \( f_{p,j}(^\top t_i) \) the function \( f_{p,j} \) evaluated in \( t \)'s other than \( t_{i_1}, \ldots, t_{i_j} \).

**Proposition 5.4.** Assume that

\[
\eta = \sum_{p=0}^{\infty} I_p^{H,T}(\eta_p), \quad \sum_{p=1}^{\infty} p! \|\eta_p\|_{[A^n]_{k,T}}^2 < \infty. \tag{5.10}
\]

Define

\[
U(t,s) = \exp\left\{ \int_s^t a(r)dr \right\}
\]

\[
\Phi(t,s) = \exp\left\{ \int_s^t a(u)du + \int_s^t b(u)dB_u^H - \frac{1}{2} \|b_{1[s,t]}\|_{[A^n]_{k,T}}^2 \right\}.
\]

Then (5.9) has a unique strong solution \( X \) with the chaos decomposition

\[
X_t = \sum_{p=0}^{\infty} I_p^k(f^k_p), \tag{5.11}
\]

\[
f^0_0 = U(t,t_0)\eta_0 + \int_{t_0}^t U(t,s)a_0(s)ds, \tag{5.12}
\]

\[
f^1_1(t_1) = U(t,t_0)\eta_1(t_1) + 1_{[t_0,t]}(t_1)U(t,t_1)b_0(t_1) +
\]

\[
1_{[t_0,t]}(t_1)U(t,t_0)b(t_1)\eta_0 + 1_{[t_0,t]}(t_1)b(t_1)\int_{t_0}^{t_1} U(t,s)a_0(s)ds, \tag{5.13}
\]

and for \( p \geq 2 \),

\[
f^p_{p}(t_1, ..., t_p) = U(t,t_0)\eta_p(t_1, ..., t_p) + \frac{1}{p} \sum_{j=1}^{p} 1_{[t_0,t]}(t_j)U(t,t_j)b(t_j)f^j_{p-1}(\hat{t}_j), \tag{5.14}
\]

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or in an explicit form
\[
F_p^t(t_1, ..., t_p) = U(t, t_0)\eta_p(t_1, ..., t_p) + \\
U(t, t_0) \sum_{j=1}^{p} \frac{1}{j!} \text{sym} \left\{ 1_{[t_0, t]}(t_1, ..., t_j) b(t_1) ... b(t_j) \eta_{p-j}(\hat{t}_1, ..., \hat{t}_j) \right\} + \\
\text{sym} \left\{ 1_{t_0 < t_1 < ... < t_p < t} U(t, t_1) b_0(t_1) b(t_2) ... b(t_p) \right\} + \\
\frac{1}{p!} 1_{[t_0, t]}(t_1, ..., t_p) b(t_1) ... b(t_p) \int_{t_0}^{t_1 \wedge ... \wedge t_p} U(t, s) a_0(s) ds.
\] (5.15)

In particular if \( \eta = \eta_0 \in R \) the solution is
\[
X_t = \Phi(t, t_0) \eta_0 + \int_{t_0}^{t} \Phi(t, s) a_0(s) ds + \int_{t_0}^{t} \Phi(t, s) b_0(s) dB^H_s.
\] (5.16)

**Corollary 5.5.** The stock price \( S_t \) in (5.7) is given by
\[
S_t = s_0 \exp \left\{ \mu t - \left( 1 - 2^{2k-1} \right) \sigma^2 t^{2k+1} + \sigma S_t^k \right\}.
\] (5.17)

**Remark 5.6.** The standard Black-Scholes model is markovian and the log-returns \( R_{t,t+s} = \log \frac{S_{t+s}}{S_t} \) are stationary independent Gaussian random variables.

The fractional Black-Scholes model is nonmarkovian and the log-returns are stationary non-independent Gaussian random variables.

The sub-fractional Black-Scholes model differs from fractional Black-Scholes model by the \textit{non stationarity} of the log-returns
\[
R_{t,t+s} = \mu s - \left( 1 - 2^{2k-1} \right) \sigma^2 \left[ (t+s)^{2k+1} - t^{2k+1} \right] + \sigma \left( S_{t+s}^k - S_t^k \right).
\]

**Definition 5.7.** A probability measure \( Q \) on \( \mathcal{F}_T^S \) which is equivalent with \( P \) (\( Q \sim P \)) is called a \textit{quasi-martingale measure} (or \textit{average risk neutral measure}) if:

(i) There exists a Gaussian process \( (Z_t)_{0 \leq t \leq T} \) with respect to \( Q \) such that
\[
S_tB_t^{-1} = \exp(Z_t), 0 \leq t \leq T.
\] (5.18)

(ii) For every \( 0 \leq t \leq T \),
\[
E_Q(S_tB_t^{-1}) = s_0.
\] (5.19)
Remark 5.8. It is clear that if $Q$ is a quasi-martingale measure then $Q$ is uniquely determined on $\mathcal{F}^{s^k}_T = \mathcal{F}^S_T$.

Define the probability measure $Q$ by

$$\frac{dQ}{dP} \big|_{\mathcal{F}^{s^k}_t} = \exp \left\{ -\frac{\mu - r}{\sigma} M^k_t - \left( \frac{\mu - r}{\sigma} \right)^2 \frac{d^2}{2(1 - 2k)} t^{1 - 2k} \right\}. \quad (5.20)$$

Remark 5.9. (a) By Girsanov’s theorem (Theorem 3.6) the process

$$\left( Z^k_t := S^k_t + \frac{\mu - r}{\sigma} t \right)_t$$

is a sfBm under $Q$.

(b) The relation (5.17) becomes in terms of $Z^k$

$$S_t = \exp \left\{ Z^k_t - (1 - 2^{2k-1}) t^{2k+1} \right\}, \quad (5.21)$$

and it is clear that $S_t$ has under $Q$ the dynamics

$$dS_t = S_t \delta_{k,T}^c Z^k_t, \quad 0 \leq t \leq T; \quad S_0 = s_0. \quad (5.22)$$

The main result is the following

**Theorem 5.10.** (i) The sub-fractional Black-Scholes market is arbitrage free and for every bounded contingent claim $F \in \left| D^{1,2}_k(Q) \right|$ there exist $v_0 \in \mathbb{R}$ and an admissible portfolio $\pi$ such that

$$V^\pi_0 = v_0, \quad V^\pi_T = F \quad \text{P - a.s.}$$

(ii) The following relation holds:

$$E_Q \left[ S_T B^{-1}_T \mid \mathcal{F}^{s^k}_t \right] = S_t B^{-1}_t \exp(K(T,t)), \quad \forall 0 \leq t \leq T, \quad (5.23)$$

where

$$K(T,t) = \exp \left\{ d_k(r - \mu) \int_0^t k^{T_k - T_{k-1}} \left( \mathbf{1}_{(0,t]} \psi_{t,T} \right)(s) s^{-k} ds \right\}$$

$$-\sigma^2 c^2_k \int_0^t n^2(T,s) ds + \sigma \int_0^t \psi_{t,T}(s) dS^k_s + \sigma^2(1 - 2^{2k-1}) t^{2k+1}. \quad (5.24)$$

In particular $Q$ is the unique quasi-martingale measure and $Q$ is not a martingale measure.
Remark 5.11. The price $C_T(F)$ of the contingent claim $F$ is given by

$$C_T(F) = E_Q \left( B_T^{-1} F \right).$$

(5.25)

The corresponding replicating portfolio $\pi = (u, v)$ also can be described. The price of the contingent claim $f(S_T)$ is given by the formula

$$C_T(f(S_T)) = \frac{\exp(-rT)}{\sqrt{2\pi}} \times \int_R f s_0 \exp \left\{ \left( \sigma T^{k+\frac{1}{2}} y + rT - \sigma^2 \left( 1 - 2^{2k-1} \right) \right) T^{2k+1} \right\} \exp(-\frac{y^2}{2}) dy$$

(5.26)

In particular the price of of an European call is

$$C_T((S_T - K)^+) = s_0 \Phi(y_1) - K \exp(-rT) \Phi(y_2),$$

(5.27)

$$y_1 = \frac{\log \frac{s_0}{K} + rT + \sigma^2 \left( 1 - 2^{2k-1} \right) T^{2k+1}}{\sigma \sqrt{2 - 2^{2k} T^{k+\frac{1}{2}}}},$$

$$y_1 = \frac{\log \frac{s_0}{K} + rT - \sigma^2 \left( 1 - 2^{2k-1} \right) T^{2k+1}}{\sigma \sqrt{2 - 2^{2k} T^{k+\frac{1}{2}}}}.$$

Comments. (a) In the mixed sub-fractional model the class of regular portfolios is a arbitrage-free class that is sufficiently large to cover hedges for most known relevant options. A recent result by Bender, Sottinen and Valkeila (2006) extends no-arbitrage property and robust hedges to a class of non-semimartingale models larger than the mixed processes and a larger class of portfolios. It should be noted that the quadratic variation is the main property which is necessary for pricing in non-semimartingale models.

(b) The use of chaos form of the Skorohod integral in the sub-fractional Black-Scholes model does not have a nice economic interpretation (Bjork-Hult, 2005 for fBm case) and also this is problematic from the mathematical point of view.
(Nualart-Taqu, 2006). It happens that different Gaussian processes with the same variation as \( S^k \) give the same price for European call options. Therefore in above mentioned both cases it is not the distribution of the process which determines uniquely the prices, but the variation of the process. 

(c) An alternative to rescue the sub-fractional Black-Scholes model is to use so called market observers according to an idea by Øksendal (Bender, 2003)

References


