Law of the Minimal Price

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Law of One Price

"All replicating portfolios of a payoff have the same price!"

Debreu (1959), Sharpe (1964), Lintner (1965),

Merton (1973a, 1973b), Ross (1976), Harrison & Kreps (1979),

Cochrane (2001), . . .

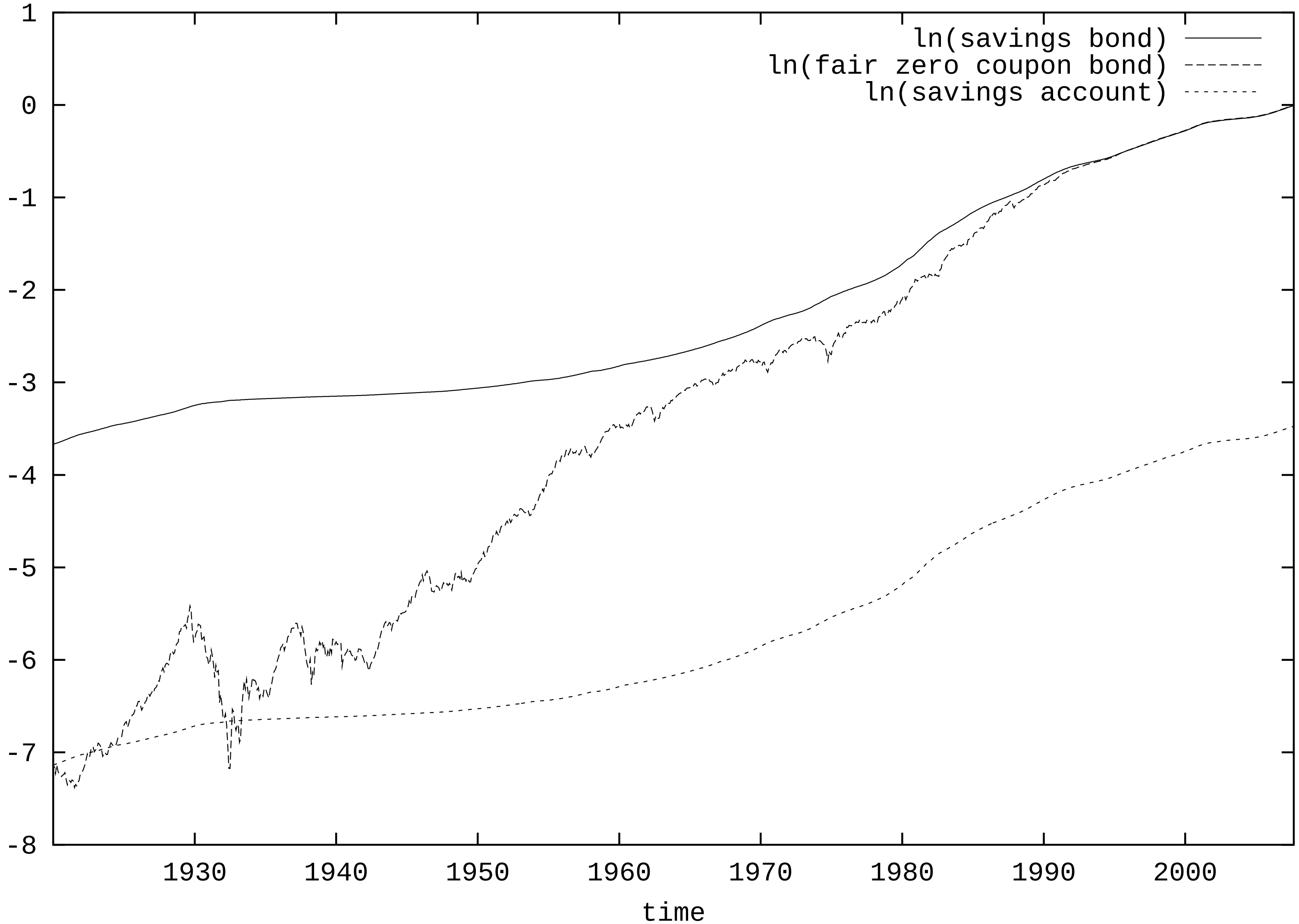

Figure 1: Logarithms of savings bond, fair zero coupon bond and savings account.

Financial Market

- jth primary security account

  S_t^j

  $j \in \{0, 1, \ldots, d\}$

- savings account

  $S_t^0 = B_t$

  $t \geq 0$

- **strategy**

$$\delta = \{\delta_t = (\delta_t^0, \delta_t^1, \ldots, \delta_t^d)^\top,\ t \geq 0\}$$

predictable

- **portfolio**

$$S_t^\delta = \sum_{j=0}^{d} \delta_t^j\, S_t^j$$

- **self-financing**

$$dS_t^\delta = \sum_{j=0}^{d} \delta_t^j\, dS_t^j$$

Numeraire Portfolio

Definition 1 $S^* \in \mathcal{V}_x^+$ numeraire portfolio *if*

$$E_t\left(\frac{\frac{S^\delta_{t+h}}{S^*_{t+h}}}{\frac{S^\delta_t}{S^*_t}} - 1\right) \leq 0$$

for all nonnegative S^δ *and* $t, h \in [0, \infty)$.

- S^* "best" performing portfolio

Long (1990), Becherer (2001), Pl. (2002, 2006),

Bühlmann & Pl. (2003), Goll & Kallsen (2003), Karatzas & Kardaras (2007)

Main Assumption

Assumption 2 *There* ***exists*** *a numeraire portfolio* $S^* \in \mathcal{V}_x^+$.

Supermartingale Property

- **benchmarked value**

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^*}$$

Corollary 3 *For nonnegative* S^δ

$$\hat{S}_t^\delta \geq E_t\left(\hat{S}_s^\delta\right)$$

$0 \leq t \leq s < \infty$

nonnegative $\hat{S}^\delta$ *supermartingale*

Corollary 4 *The* ***minimal nonnegative supermartingale*** *that coincides with a given future random variable* ***is the corresponding martingale****.*

see Pl. & Heath (2006)

Law of the Minimal Price

Theorem 5 *If a fair portfolio replicates a nonnegative payoff, then this represents the minimal replicating portfolio.*

- least expensive
- minimal hedge
- economically correct price in a competitive market

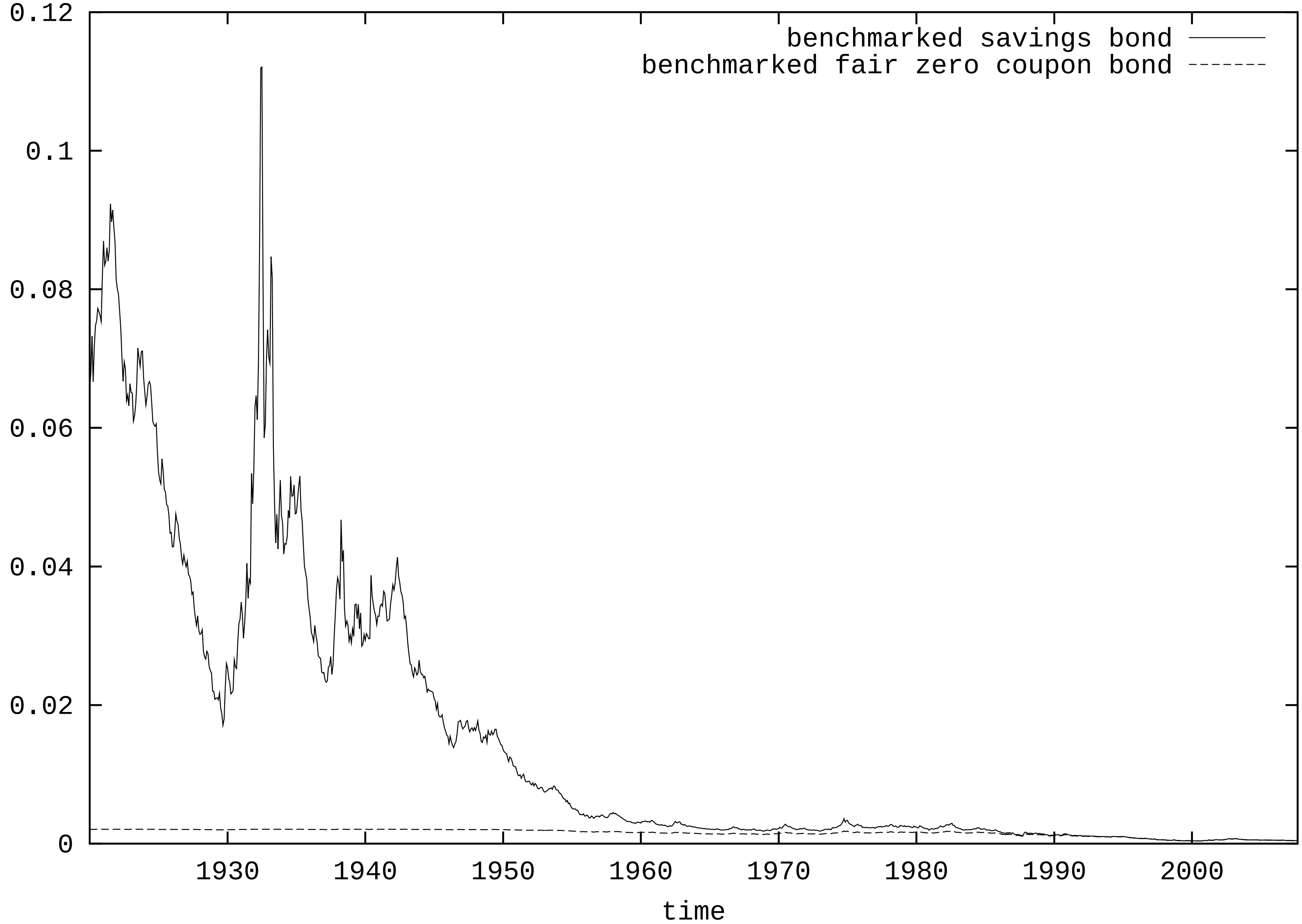

Figure 2: Benchmarked savings bond and benchmarked fair zero coupon bond.

- **claim**

  H_T

$$E_0\left(\frac{H_T}{S_T^*}\right) < \infty$$

Corollary 6

Minimal price for replicable H_T *is given by*

real world pricing formula

$$S_t^{\delta_H} = S_t^* \, E_t\left(\frac{H_T}{S_T^*}\right).$$

- **normalized benchmarked savings account**

$$\Lambda_T = \frac{\hat{S}^0_T}{\hat{B}_0}$$

$$1 = \Lambda_0 \geq E_0(\Lambda_T)$$

- **real world pricing formula** $\Longrightarrow$

$$S_0^{\delta_H} = E_0\left(\Lambda_T \, \frac{B_0}{B_T} \, H_T\right)$$

$\Longrightarrow$

$$S_0^{\delta_H} \leq \frac{E_0\left(\Lambda_T \, \frac{B_0}{B_T} \, H_T\right)}{E_0(\Lambda_T)}$$

similar for any numeraire

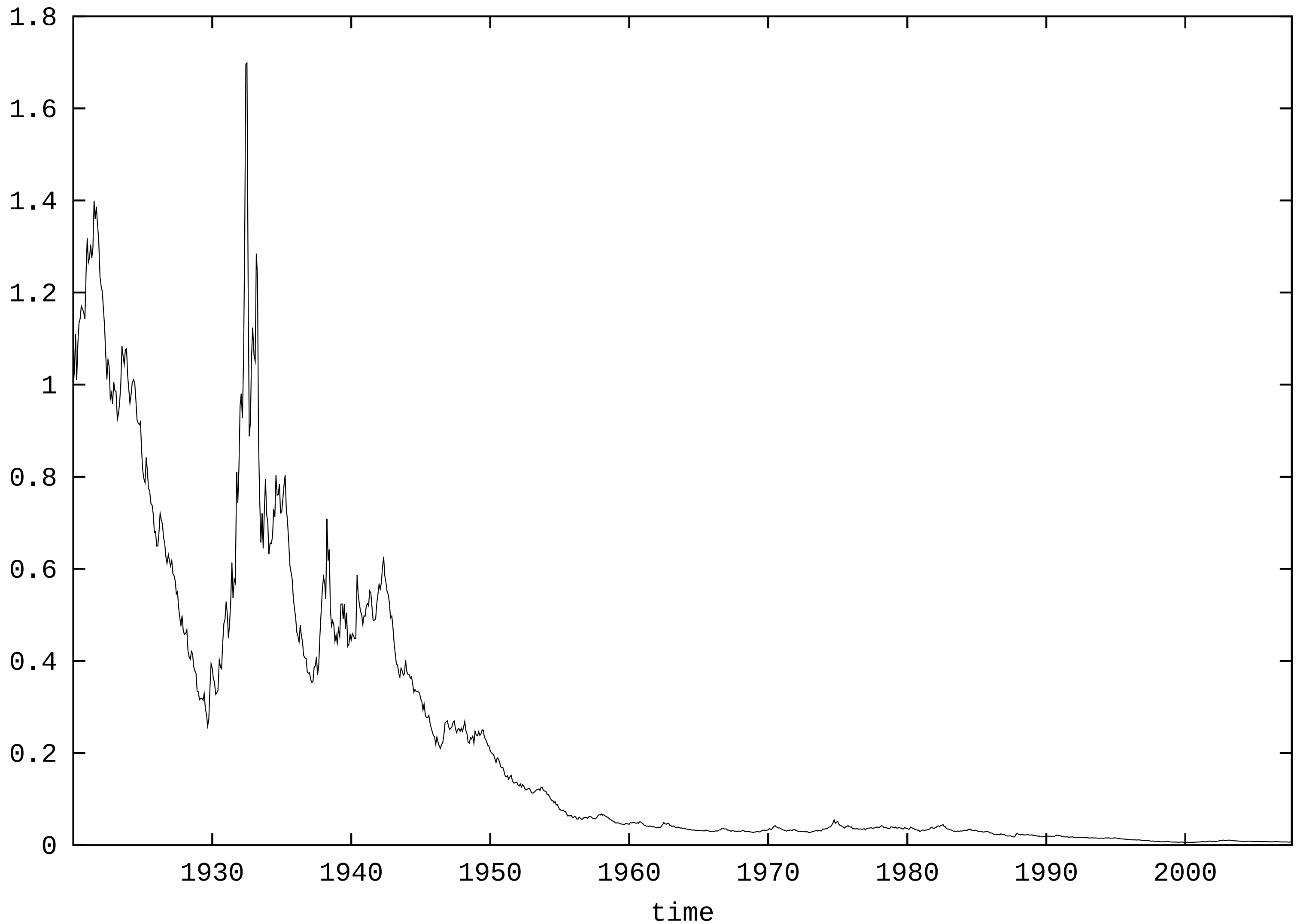

Figure 3: Candidate Radon-Nikodym derivative of hypothetical risk neutral measure for any complete model of real market.

- **special case when savings account is fair:**

$$\Longrightarrow \qquad \Lambda_T = \frac{dQ}{dP} \text{ forms martingale;} \quad E_0(\Lambda_T) = 1;$$

equivalent risk neutral probability measure Q exists;

Bayes' formula $\Longrightarrow$
risk neutral pricing formula

$$S_0^{\delta_H} = E_0^Q\left(\frac{B_0}{B_T}\,H_T\right)$$

Harrison & Kreps (1979), Ingersoll (1987),

Constatinides (1992), Duffie (2001), Cochrane (2001), ...

- otherwise "risk neutral price" $\geq$ real world price

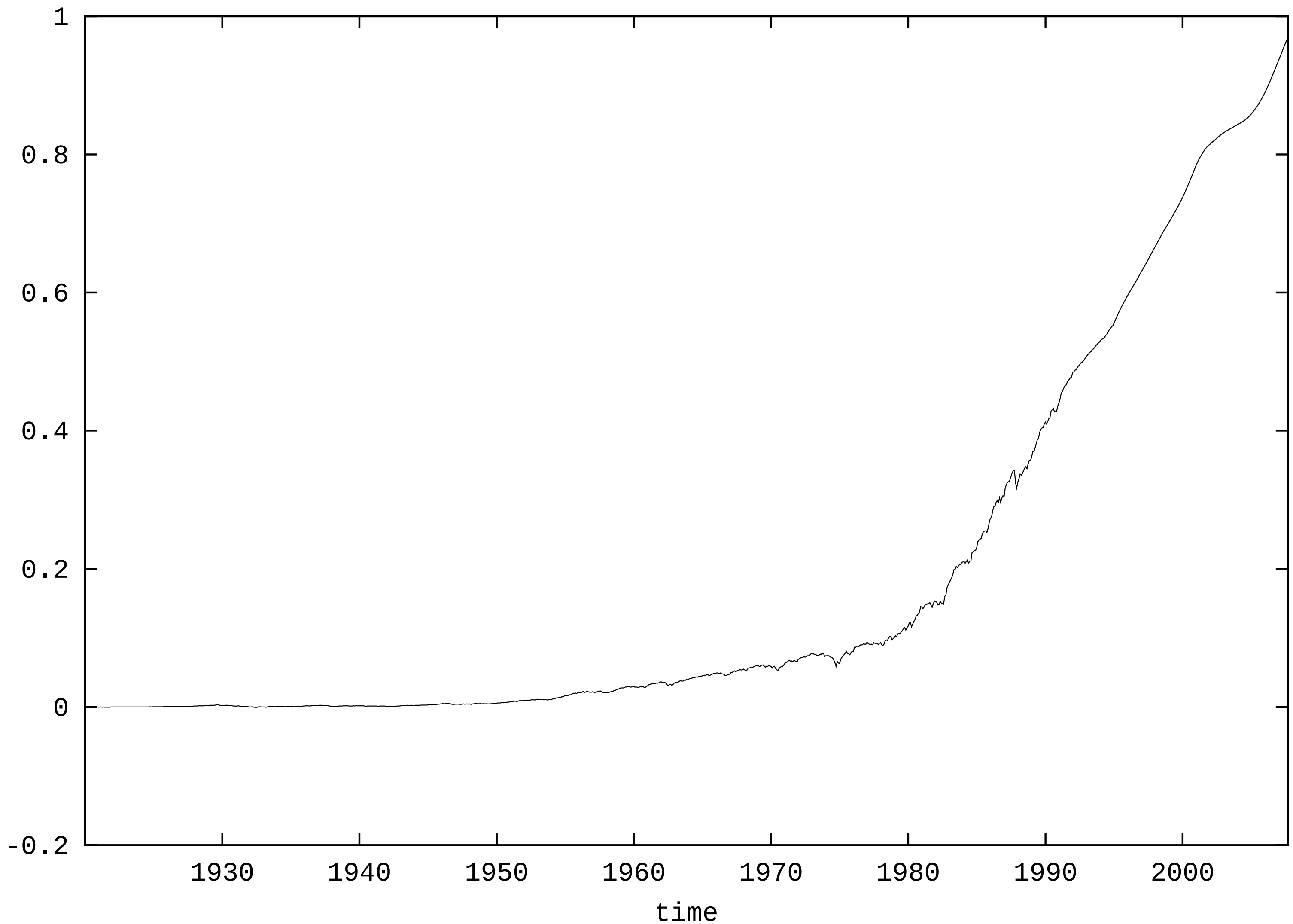

Figure 4: $P(t,T)$ minus savings account.

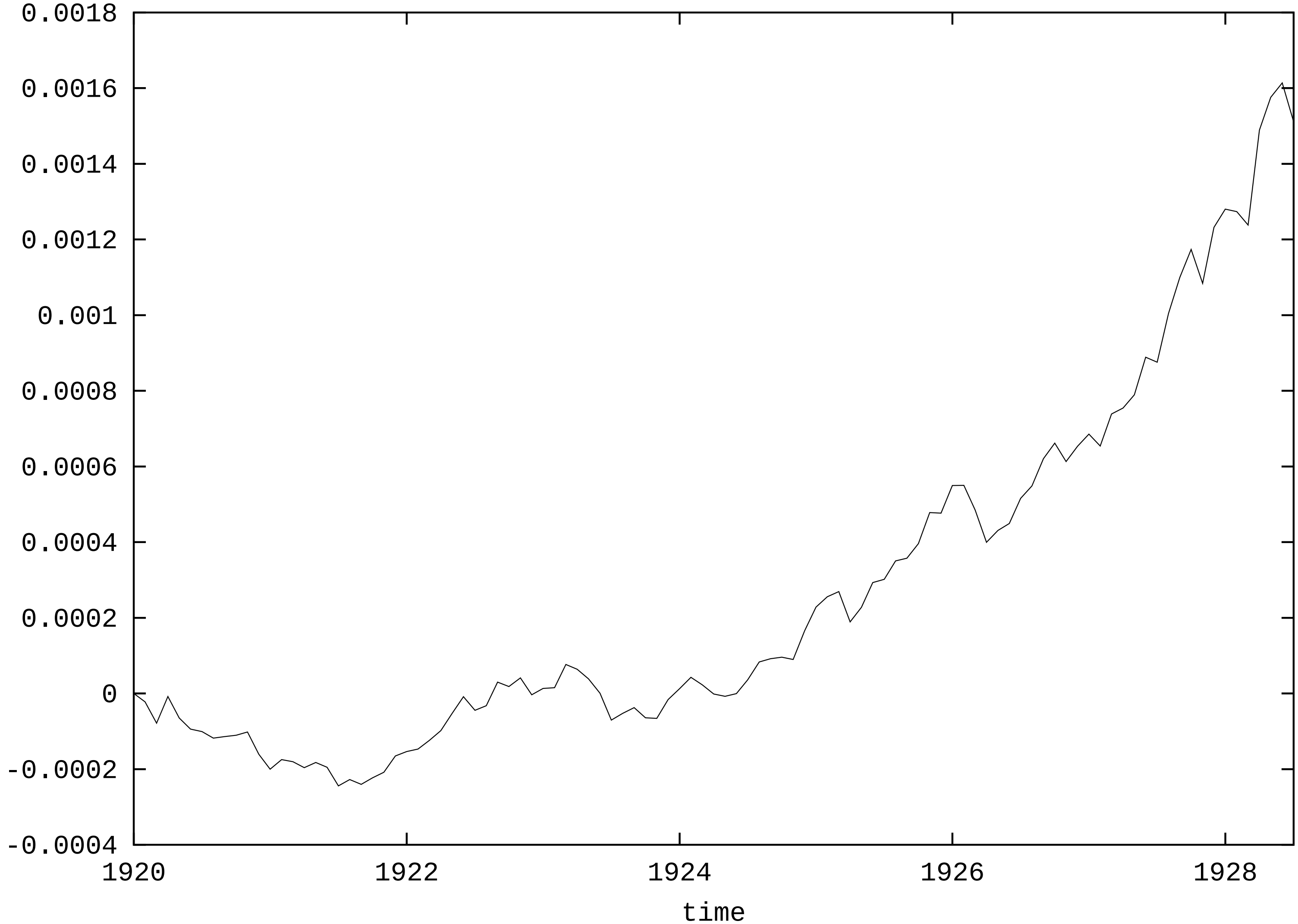

Figure 5: $P(t, T)$ minus savings account.

Strong Arbitrage

- market participants can only exploit arbitrage
- **limited liability**

$\Longrightarrow$ nonnegative total wealth of each market participant

Definition 7 *A nonnegative* S^δ *is a* strong arbitrage *if* $S_0^\delta = 0$ *and*

$$P\left(S_t^\delta > 0\right) > 0.$$

Pl. (2002)-mathematical arguments

Loewenstein & Willard (2000)-economic arguments

Theorem 8 *There is no strong arbitrage.*

$\Longrightarrow$ there is no pricing based on excluding strong arbitrage

- Delbaen & Schachermayer (1998)

  free lunches with vanishing risk (FLVR) may exist

- Loewenstein & Willard (2000)

  free snacks & cheap thrills may exist

- Under BA candidate risk neutral Q may ***not*** be equivalent to P since its Radon-Nikodym derivative may be a strict supermartingale.

- Existence of equivalent risk neutral probability measure is **mathematically convenient** but not an economic necessity.

Two Asset Market   Example

- **risky asset** S_t^*

  (S&P500 accumulation index)

- **savings account** B_t

  (US savings account)

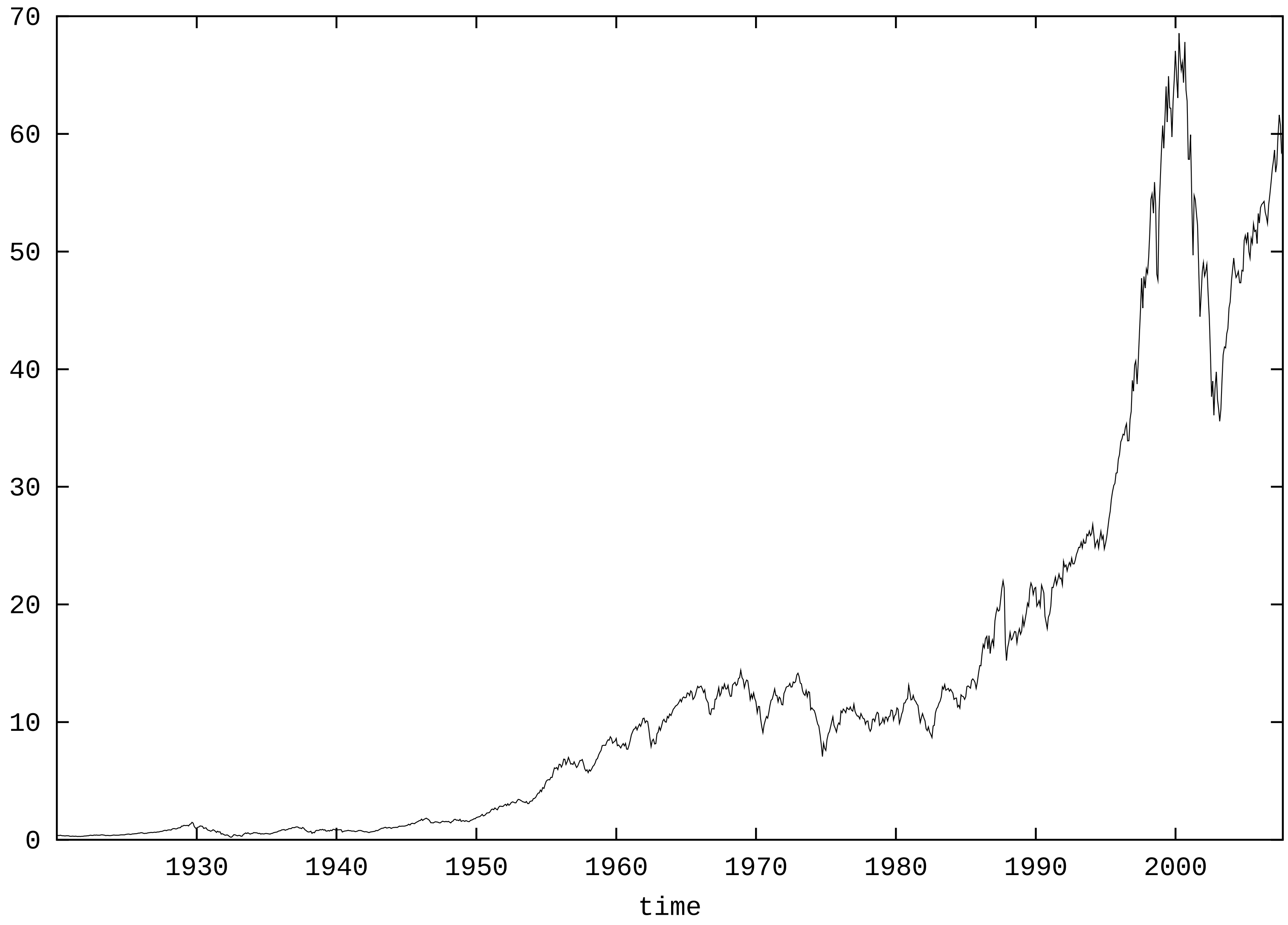

Figure 6: Discounted S&P500.

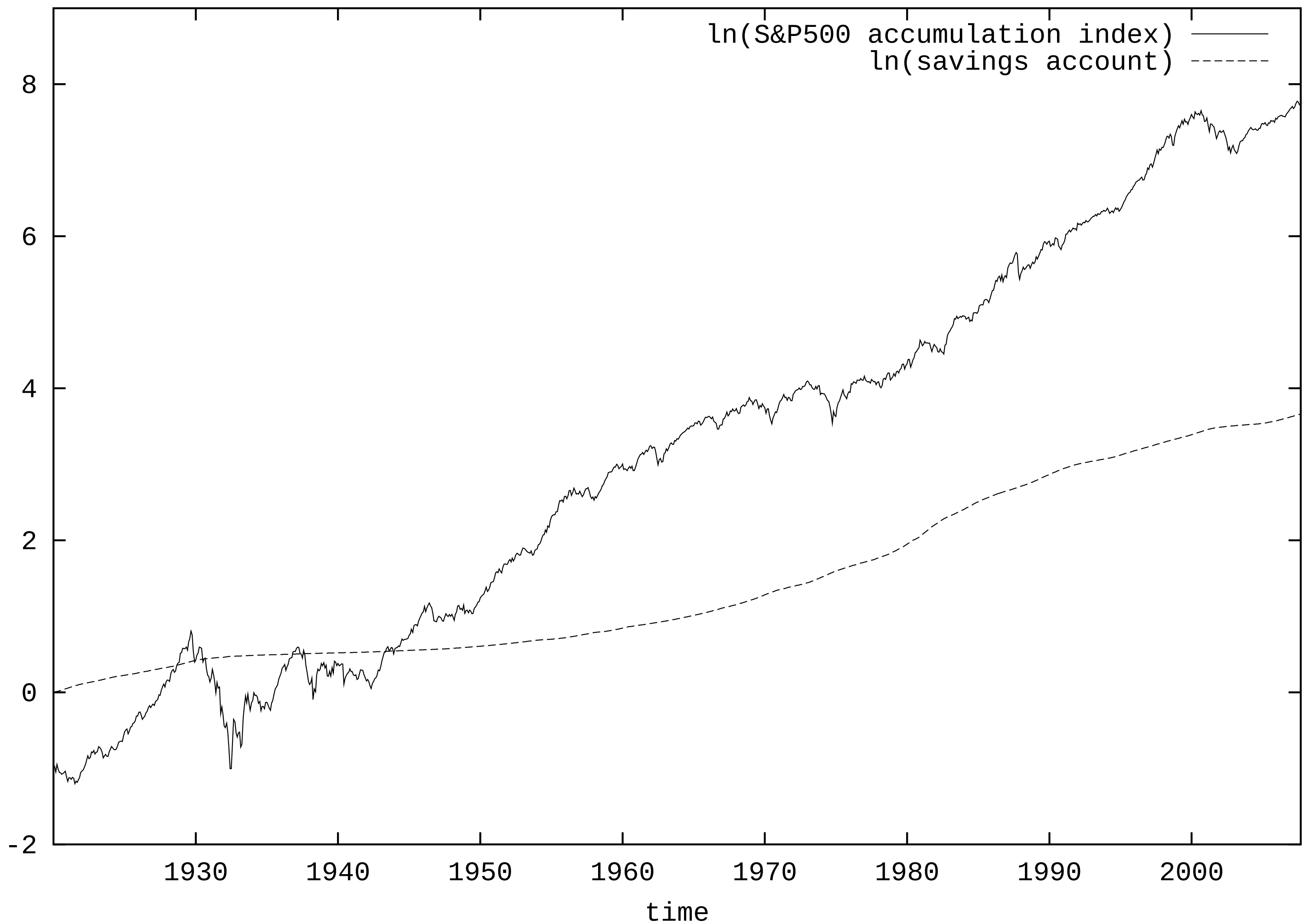

Figure 7: ln(S&P500 accumulation index) and ln(savings account).

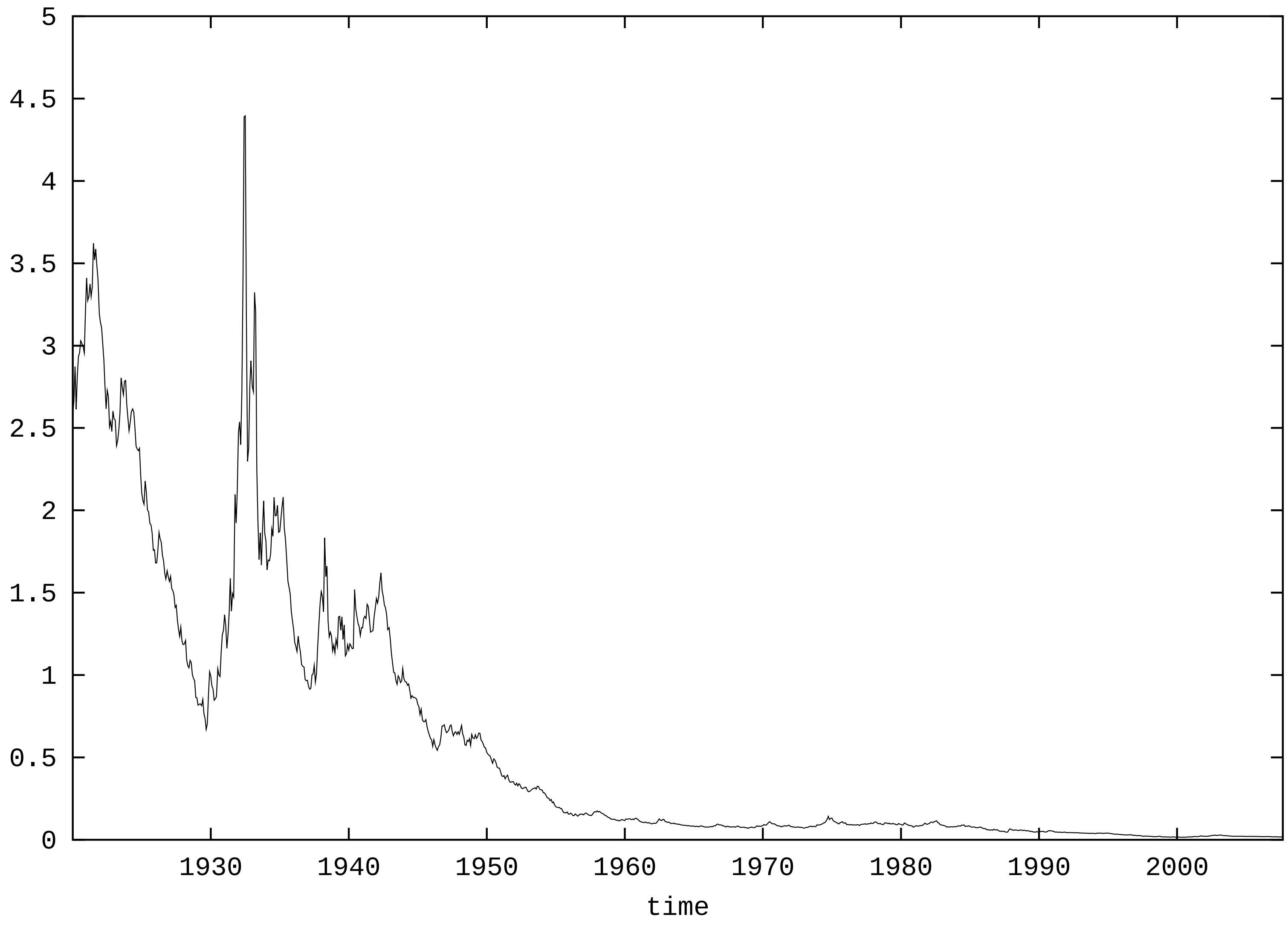

Figure 8: US-benchmarked savings account.

- assume short rate deterministic

- **savings bond**

$$P^*(t,T) = \frac{B_t}{B_T}$$

- select index S^* as **numeraire portfolio**

  nonnegative benchmarked securities supermartingales

  best performing portfolio

- fair zero coupon bond

$$P(t,T) = S_t^* \, E_t\left(\frac{1}{S_T^*}\right)$$

- **discounted numeraire portfolio**

$$\bar{S}^*_t = \frac{S^*_t}{B_t}$$

for continuous market generally satisfies SDE

$$d\bar{S}^*_t = \alpha_t \, dt + \sqrt{\bar{S}^*_t \, \alpha_t} \, dW_t$$

is time transformed squared Bessel process, dimension 4

- need model with downward trending $\frac{B_t}{S^*_t}$

  to reflect reality

- model drift of discounted index as

$$\alpha_t = \alpha \exp\{\eta\, t\}$$

$\Longrightarrow$ **Minimal Market Model**

MMM, see Pl. & Heath (2006)

- **net growth rate**

$\eta \approx \mathbf{0.0511}$ with R^2 of $\mathbf{0.88}$

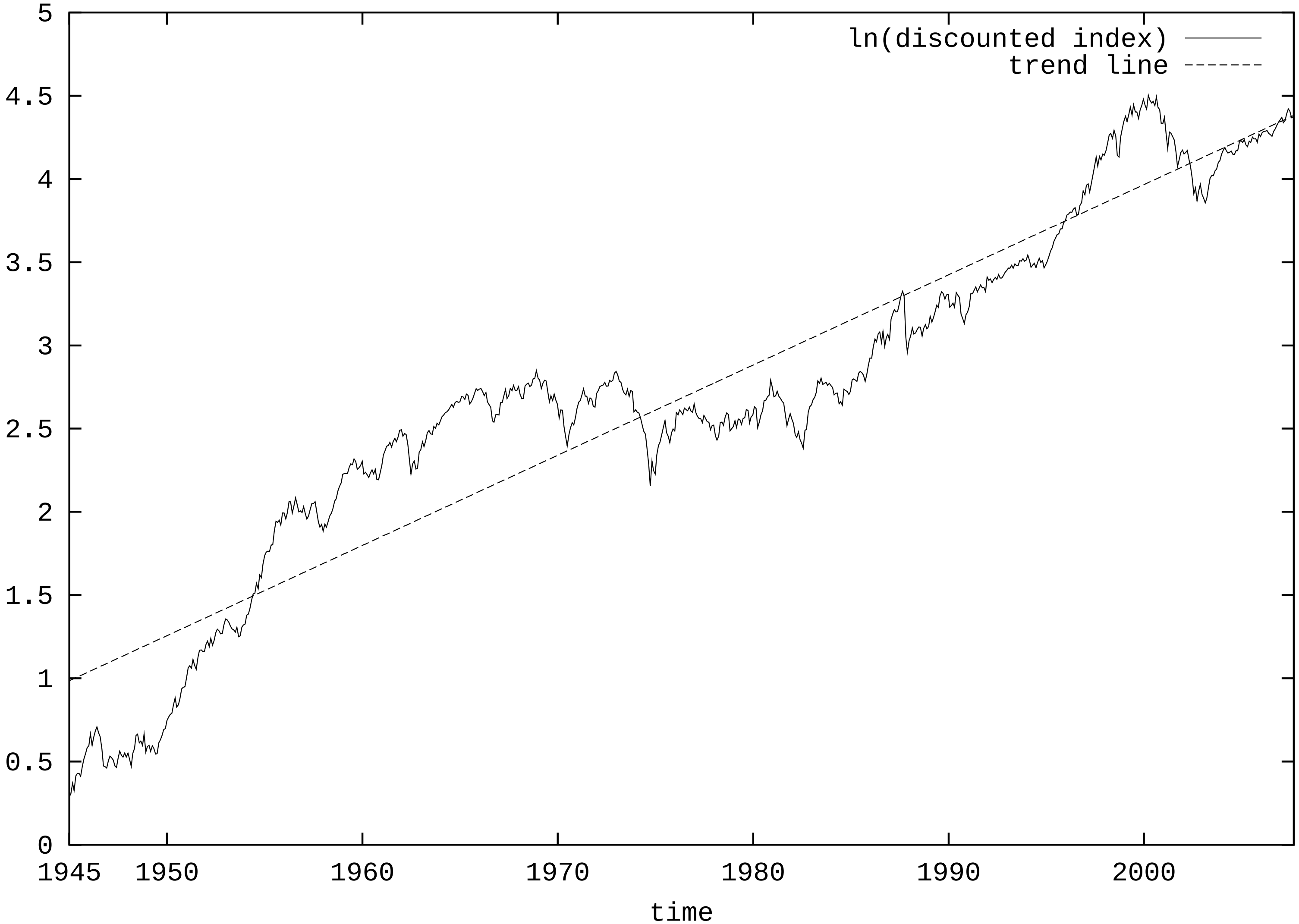

Figure 9: Logarithm of discounted index.

- **normalized index**

$$Y_t = \frac{\bar{S}^*_t}{\alpha_t}$$

$$V_{t,h} = \sum_{\ell=1}^{i_t} \left(\sqrt{Y_{t_\ell}} - \sqrt{Y_{t_{\ell-1}}} \right)^2 \approx \left[\sqrt{Y} \right]_t = \frac{t}{4}$$

- **scaling parameter** $\alpha \approx 0.01429$ with R^2 of 0.995

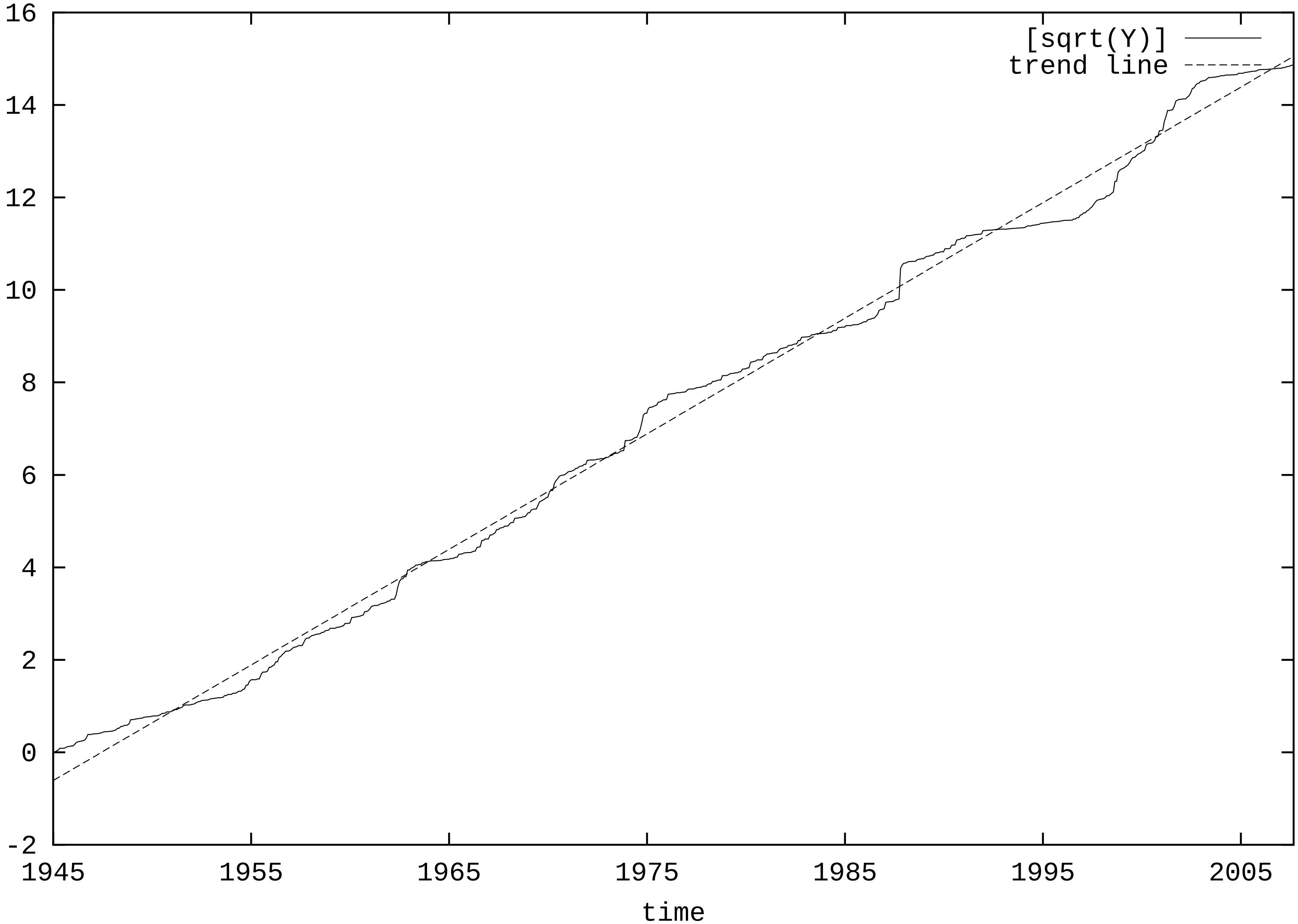

Figure 10: Quadratic Variation of $\sqrt{Y_t}$.

- **fair zero coupon bond (MMM)**

$$P(t,T) = P^*(t,T)\left(1 - \exp\left\{-\frac{2\,\eta\,\bar{S}^*_t}{\alpha\,(\exp\{\eta\,T\} - \exp\{\eta\,t\})}\right\}\right)$$

- **initial prices** in 1920:

$$P^*(0,T) = 0.025496$$

$$P(0,T) = 0.000795$$

$$\frac{P(0,T)}{P^*(0,T)} < 0.0312 \qquad \Longrightarrow \qquad 3.12\%$$

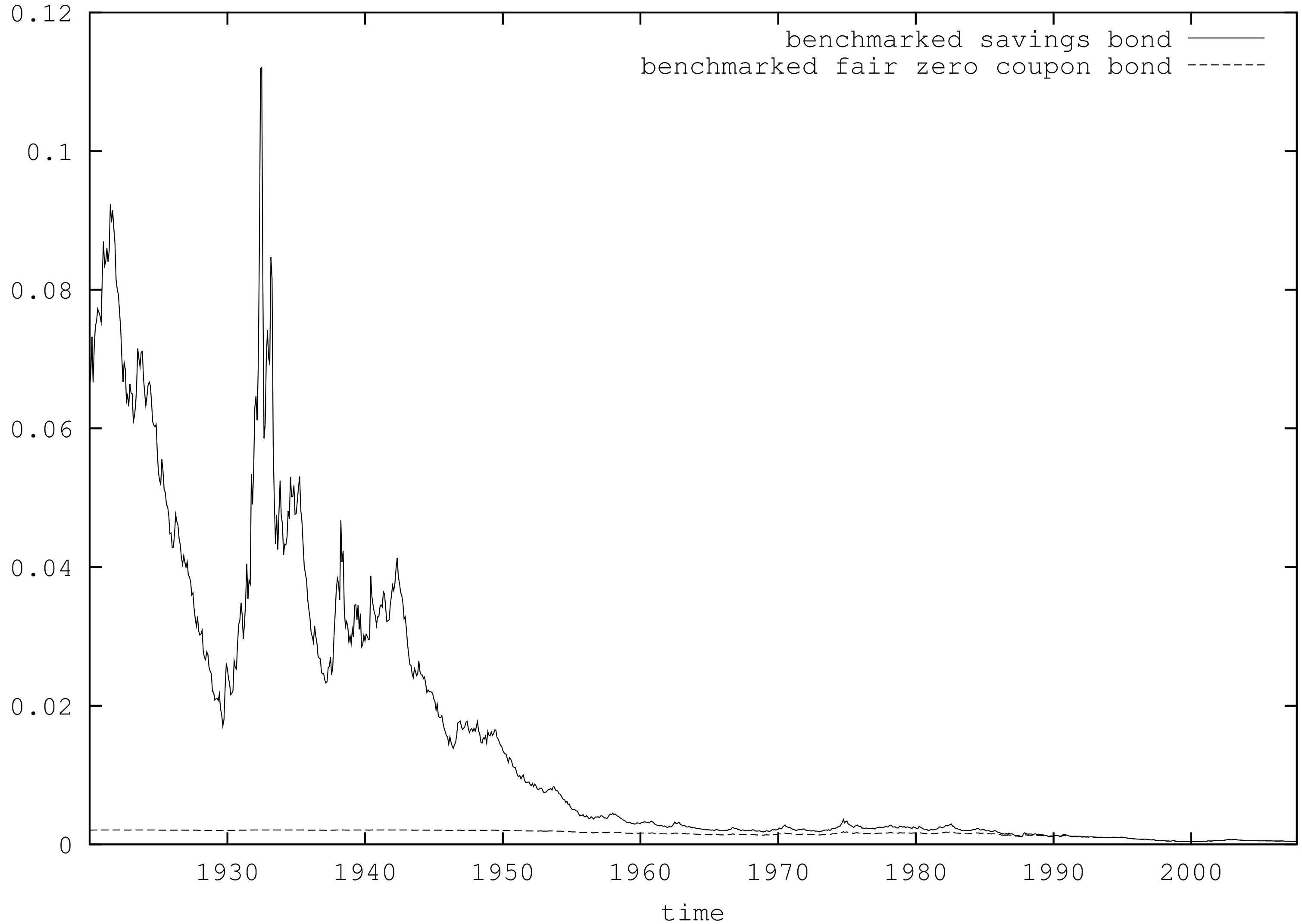

Figure 11: Benchmarked savings bond and benchmarked fair zero coupon bond.

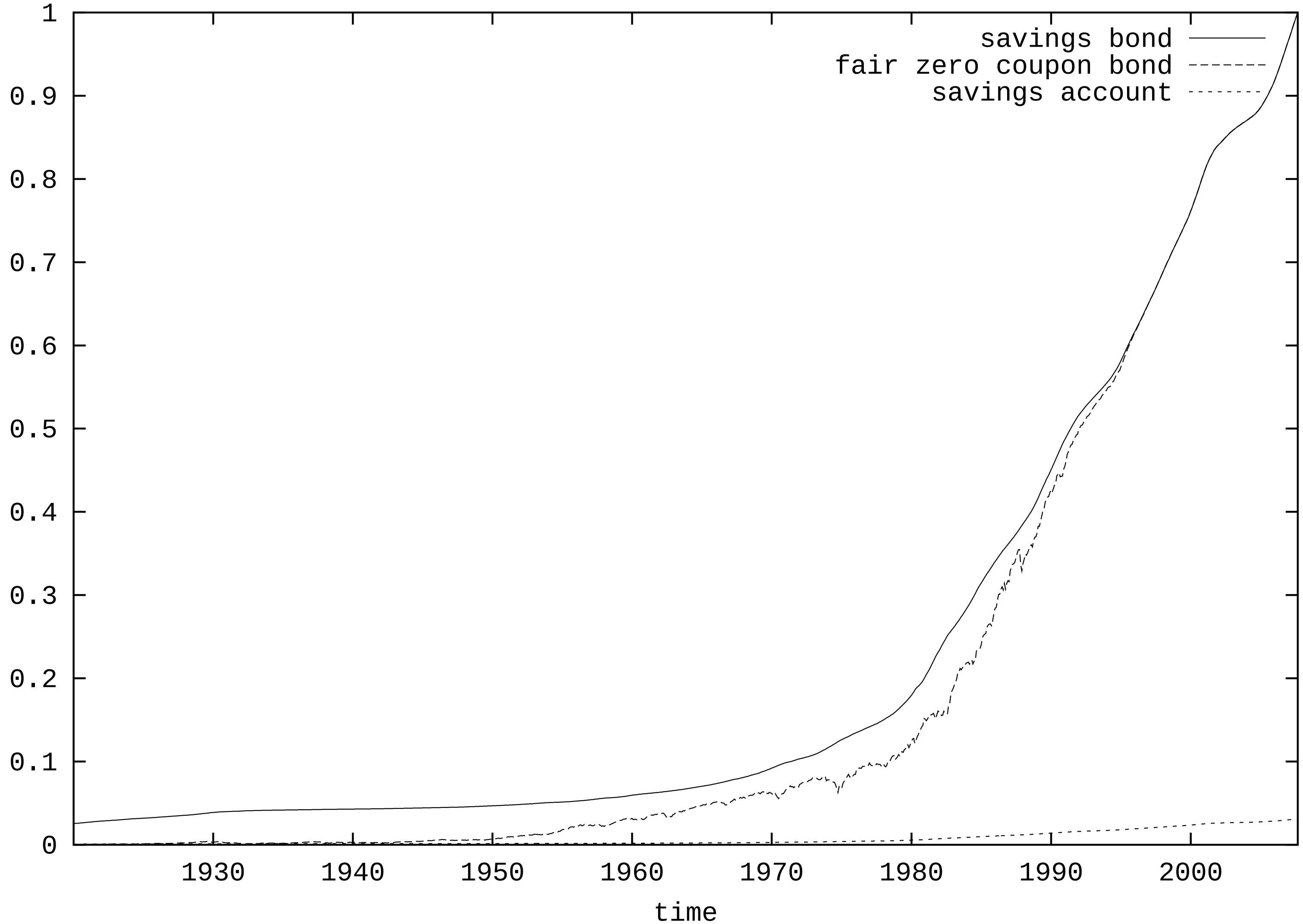

Figure 12: Savings bond, fair zero coupon bond and savings account.

- **self-financing hedge portfolio** hedge ratio

$$\delta_t^* = \frac{\partial P(t,T)}{\partial \bar{S}_t^*}$$

$$= P^*(0,T)\,\exp\left\{\frac{-2\,\eta\,\bar{S}_t^*}{\alpha\,(\exp\{\eta\,T\} - \exp\{\eta\,t\})}\right\}$$

$$\times\,\frac{2\,\eta}{\alpha\,(\exp\{\eta\,T\} - \exp\{\eta\,t\})}$$

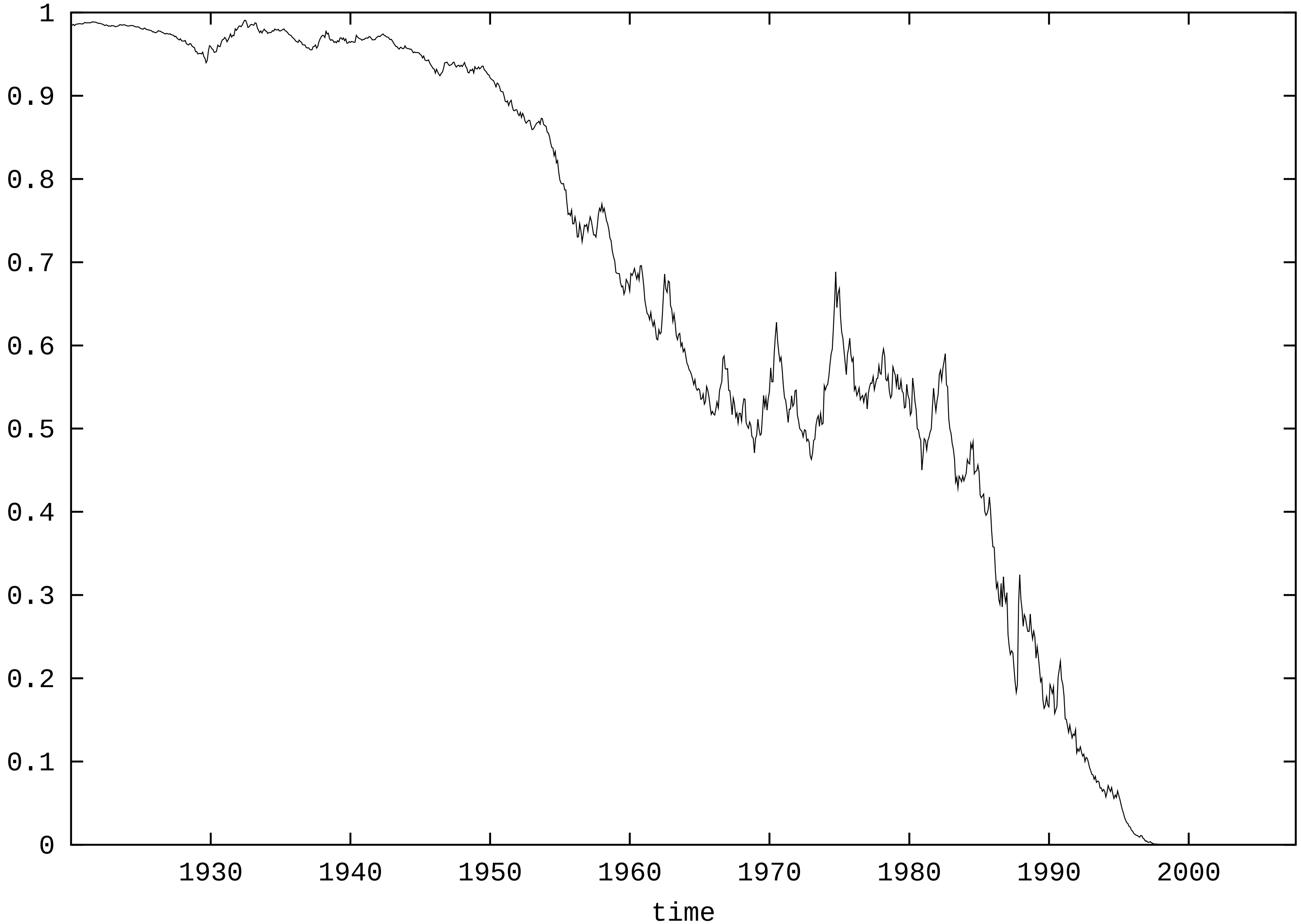

Figure 13: Fraction invested in the index.

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