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Wavelet FEM for option pricing in stochastic volatility models

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Outline

References

- Barndorff-Nielsen, O. E. and Shepard, N.: *Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics (with discussion).* J. R. Stat. Soc. B, **63**, 167–241 (2001).

- Bates, D. S.:*Jumps and stochastic volatility: the exchange rate process implicit in Deutsche Mark options* Review of Financial Studies, **9**, 69–107 (1996).

- von Petersdorff, T. and Schwab, C.: *Numerical solution of parabolic equations in high dimensions.* ESAIM: Mathematical Modelling and Numerical Analysis **38** N°1 (2004) 93–127.

Goal: Numerical pricing of options under a general class of SV models by the FE method

$(Z_t)_{t\geq 0}$ a $\mathbb{R}^d$-valued strong Markov process.

Find arbitrage-free price $u(z,t)$ of contract on Z,

$$u(z,t) = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}g(Z_T) \mid Z_t = z].$$

$u(z,t)$ is (viscosity) solution of

$$-\partial_t u + \mathcal{A}u + ru = 0, \qquad u(z,T) = g,$$

where $\mathcal{A}$ is the infinitesimal generator of Z.

Dynamics of $Z = (X, Y)$

X $\mathbb{R}^n$-valued, log-price dynamics of underlyings.

Y $\mathbb{R}^p$-valued, dynamics of volatility, $p \geq n$.

$$\begin{aligned} \mathrm{d}Z_t &= \gamma(Z_{t-}, t)\mathrm{d}t + \mathcal{Q}(Z_{t-}, t)\mathrm{d}W_t + \int_E \sigma(Z_{t-}, t, \xi)\widetilde{N}(\mathrm{d}t, \mathrm{d}\xi), \\ Z_0 &= z. \end{aligned}$$

- $\gamma : \mathbb{R}^{n+p} \times [0, T] \to \mathbb{R}^{n+p}$
- $\mathcal{Q} : \mathbb{R}^{n+p} \times [0, T] \to \mathbb{R}^{(n+p)\times m}$
- $\sigma : \mathbb{R}^{n+p} \times [0, T] \times \mathbb{R}^k \to \mathbb{R}^{(n+p)\times \ell}$
- W $\mathbb{R}^m$-valued Brownian motion, $\widetilde{N}$ compensated martingale measure of ℓ-dim. Poisson random measure

What is $\mathcal{A}$?

$$\mathcal{A} = \mathcal{A}_{\mathcal{Q}} + \mathcal{A}_{\gamma} + \mathcal{A}_{\nu}$$

$$\begin{aligned}
\mathcal{A}_{\mathcal{Q}}(u)(z) &:= -\frac{1}{2}\mathrm{tr}\left[(\mathcal{Q}\mathcal{Q}^{\top})(z)D^2u(z)\right] \\
\mathcal{A}_{\gamma}(u)(z) &:= -\langle\gamma(z), Du(z)\rangle \\
\mathcal{A}_{\nu}(u)(z) &:= -\sum_{j=1}^{\ell}\mathcal{A}_{\nu,j}(u)(z)
\end{aligned}$$

$$\mathcal{A}_{\nu,j}(u)(z) := \int_E \left[u(z+\sigma_j(z,\xi)) - u(z) - \langle\sigma_j(z,\xi), Du(z)\rangle\right]\nu_j(\mathrm{d}\xi)$$

Two examples ($n = p = 1$)

- Heston ('93)

$$\gamma(z) = \begin{pmatrix} -y/2 \\ \alpha(m-y) + \lambda(x,y) \end{pmatrix}, \quad \alpha, m > 0,$$

$$Q(z) = \begin{pmatrix} \sqrt{y} & 0 \\ \beta\rho\sqrt{y} & \beta\sqrt{1-\rho^2}\sqrt{y} \end{pmatrix}, \quad \beta > 0,\ \rho \in [-1,1]\,.$$

- Barndorff-Nielsen and Shepard ('01)

$$\gamma(z) = \begin{pmatrix} -y/2 + \alpha \\ -\lambda y + \beta \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R},\ \lambda > 0,$$

$$Q(z) = \begin{pmatrix} \sqrt{y} \\ 0 \end{pmatrix}, \quad \sigma(z,\xi) = \begin{pmatrix} \rho\xi \\ \xi \end{pmatrix}, \quad \rho \le 0\,.$$

Analysis and numerics depend on $\mathcal{Q}$

Three classes: $\mathcal{Q}\mathcal{Q}^\top$ is [1]

(A) positive definite: $\exists c > 0 \ : \ z^\top(\mathcal{Q}\mathcal{Q}^\top)(z)z \geq cz^\top z$, $\forall z \in \mathbb{R}^d$

(B) positive semidefinite: $\exists z \in \mathbb{R}^d \ : \ \mathrm{rank}((\mathcal{Q}\mathcal{Q}^\top)(z)) < d$

(C) positive semidefinite: $\forall z \in \mathbb{R}^d \ : \ \mathrm{rank}((\mathcal{Q}\mathcal{Q}^\top)(z)) < d$

- (B) & (C) $\rightarrow$: P(I)DE with nonnegative characteristic form
- typically, (B) holds exactly for one z, usually $z = 0$
- (C) means that there is at least one coordinate direction without diffusion component

[1] $d := n + p$

Analytical framework for models of class (B)

Given $f \in L^2(0,T;V^*)$, find $u \in L^2(0,T;V) \cap H^1(0,T;V^*)$ such that

$$\langle \partial_t u, v\rangle + \underbrace{\langle \mathcal{A}u, v\rangle}_{a(u,v)} = \langle f, v\rangle, \quad \forall v \in V$$

Typically, V is a weighted H^1-Sobolev space.

Theorem (Heston, change of var. $\tilde{u}(x,\tilde{y}) := u(x, 1/4\tilde{y}^2)$)

Assume $1 - 4\sqrt{2}|2\alpha\beta^{-2} - 1/2| > \rho^2$. *Then* $a(\cdot,\cdot)$ *satisfies:* $\exists C, c, c_0 > 0$ *such that for all* $u, v \in V$

$$|a(u,v)| \le C\|u\|_V\|v\|_V, \quad a(u,u) \ge c\|v\|_V^2 - c_0\|v\|^2,$$

where $V = \overline{C_0^\infty(\mathbb{R}^2)}^{\|\cdot\|_V}$, $\|v\|_V^2 := \|y\partial_x v\|^2 + \|\partial_y v\|^2 + \|v\|^2$.

Analytical framework for models of class (C)

Treat time t as additional space variable. Consider

$$\begin{aligned} -\mathrm{div}(\mathcal{Q}(z)\nabla u) + \langle \gamma(z), \nabla u\rangle + cu + \mathcal{A}_\nu(u) &= f \quad \text{in } \mathcal{D} \subset \mathbb{R}^d \\ u &= 0 \quad \text{in } \Gamma_0 \cup \Gamma_- \end{aligned}$$

$\mathcal{V} := \{u \in H^1(\mathcal{D}) \;:\; \gamma_0 u\,|_{\Gamma_0} = 0\}$. The weak form reads: Find $u \in \mathcal{H}$ such that

$$B(u,v) = (f,v), \qquad \forall v \in \mathcal{V}$$

Bilinear form $B : \mathcal{H} \times \mathcal{V} \to \mathbb{R}$

$$B(u,v) = (\mathcal{Q}\nabla u, \nabla v) - (u, \mathrm{div}(\gamma v)) + c(u,v) + \langle u, v\rangle_+ + a_\nu(u,v).$$

Example: BNS model

Theorem

Assume the Lévy measure has the form $\nu(\mathrm{d}\xi) = Ce^{-\kappa\xi}\xi^{-1-\alpha}$, $C, \kappa > 0$, $\alpha \in (0,1)$, $\xi \in \mathbb{R}_+$. *Then the bilinear form associated to* $\mathcal{A}_\nu$ *satisfies:* $\exists C_1, C_2, C_3 > 0$ *such that for all* $u, v \in V = \widetilde{H}^{\alpha/2}(\mathcal{D})$

$$|a_\nu(u,v)| \leq C_1 \|u\|_V \|v\|_V, \quad a_\nu(u,u) \geq C_2 \|v\|_V^2 - C_3 \|v\|^2,$$

Theorem

Let $\omega > 0$ *satisfy* $c > \lambda/2 + C_3 + \omega$. *Then* $\exists$ *Hilbert space* $\widehat{\mathcal{H}} \subset \mathcal{H}$ *and unique* $u \in \widehat{\mathcal{H}}$ *solving* $B(u,v) = (f,v)$, $\forall v \in \mathcal{V}$. *Here*

$$\mathcal{H} := \overline{\mathcal{V}}^{\|\cdot\|_H}, \ \|w\|_{\widetilde{\mathcal{H}}}^2 := (x_2 \partial_{x_1} w, \partial_{x_1} w) + (w,w) + \langle w, w \rangle_{\Gamma_+} + (w,w)_{\alpha/2}.$$

FEM for models of class (B)

$V^L \subset V$, $\dim V^L < \infty$. The Galerkin approximation: Find $u^L \in L^2(0,T;V^L) \cap H^1(0,T,(V^L)^*)$ such that

$$\langle \partial_t u^L, v\rangle + \langle \mathcal{A} u^L, v\rangle = \langle f, v\rangle, \quad \forall v \in V^L$$

is equivalent to the ODE: Find $\underline{U}_L(t) \in \mathbb{R}^{\dim V^L}$ such that

$$\mathbf{M}\underline{\dot{U}}_L(t) + \mathbf{A}\underline{U}_L(t) = \underline{F}(t)$$

where, with $V^L = \operatorname{span}\{\Phi_j\}_{j=1}^{\dim V^L}$,

$$\mathbf{M} = \big((\Phi_i, \Phi_j)\big)_{1\le i,j\le \dim V^L}, \quad \mathbf{A} = \big(a(\Phi_i, \Phi_j)\big)_{1\le i,j\le \dim V^L}.$$

Note: V^L is a sparse tensor product wavelet space, with $\dim V^L = O(2^L L^{d-1})$ in contrast to classical FE spaces $\dim V_c^L = O(2^{dL})$.

Convergence rates

hp-discontinuous Galerkin discretization in time +
preconditioned GMRES to approximatively solve linear systems
$\Rightarrow$

Theorem

The fully discrete Galerkin scheme with polynomial degree r of wavelets yields approximate option prices at maturity $U_L(T)$ with accuracy ($N_L = \dim V^L$, $d = n + p$)

$$\|U_L(T) - u(T)\| \leq C N_L^{-s} (\log N_L)^{(d-1)s}, \quad s = r + \frac{r}{d(r+1) - 1}$$

and can be computed with at most $O(N_L (\log N_L)^7)$ operations (if $\nu \equiv 0$).

Note: for classical FE spaces we have $s = \frac{r+1}{d}$

SDFEM for models of class (C)

Basic idea of streamline diffusion FEM: add extra diffusion in the direction of streamline: take test functions $v + \delta_L \langle \gamma, \nabla v \rangle$ rather than just v. δ_L: SD parameter to be chosen later.

Find $u_L \in V^L$ such that

$$B_\delta(u_L, v) = \ell_\delta(v), \quad \forall v \in V^L,$$

where

$$\begin{aligned} B_\delta(\varphi, \phi) &= B(\varphi, \phi) + \delta_L \sum_{\kappa \in \mathcal{T}^L} \big(-\operatorname{div}(\mathcal{Q} \nabla \varphi) + c\varphi, \langle \gamma, \nabla \phi \rangle \big)_\kappa \\ &\quad + \delta_L \big(\langle \gamma, \nabla \varphi \rangle, \langle \gamma, \nabla \phi \rangle \big)_S + \delta_L a_\nu(\varphi, \langle \gamma, \nabla \phi \rangle), \\ \ell_\delta(\phi) &= (f, \phi) + \delta_L \sum_{\kappa \in \mathcal{T}^L} \big(f, \langle \gamma, \nabla \phi \rangle \big)_\kappa \end{aligned}$$

Choice of δ_L

Assume $\exists c_0 \;:\; c - 1/2\,\mathrm{div}\gamma - C_3 \geq c_0 > 0$.

Streamline diffusion norm $\|\cdot\|_{\mathrm{SD}}$

$$\|v\|^2_{\mathrm{SD}} := \|\sqrt{\mathcal{Q}}\nabla v\|^2 + c_0\|v\|^2 + \|v\|^2_{\Gamma_-\cup\Gamma_+} + C_2\|v\|^2_s + \delta_L\|\langle\gamma,\nabla v\rangle\|^2_s$$

Theorem

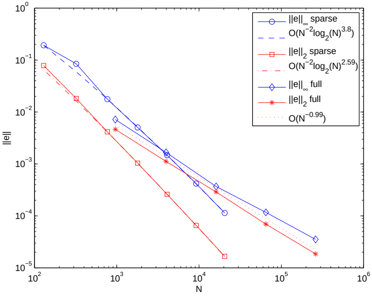
Let δ_L satisfy

$$0 \leq \delta \leq 1/8 \min\left\{\frac{h_L^2}{(n+p)r^4|\sqrt{\mathcal{Q}}|^2}, 3\frac{C_2}{C_1^2}, 3\frac{c_0}{c^2}\right\}.$$

Then $B_\delta(\cdot,\cdot)$ is coercive on $V^L \times V^L$, i.e. $B_\delta(v,v) \geq \frac{1}{4}\|v\|^2_{\mathrm{SD}}$.

Convergence rates: European call in Heston model

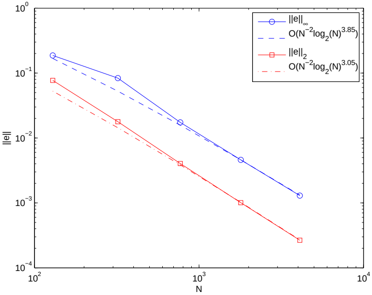
Set $(\alpha, \beta, \rho, m) = (2.5, 0.5, -0.5, 0.025)$, $T = 0.5$, $K = 1$, $e_L := U_L(T) - u(T)$. For $q \in \{2, \infty\}$ we find



$$\|e_L\|_q = O(N_L^{-2} (\log N_L)^{c_q}), \quad c_2 \approx 2.59, \quad c_\infty \approx 3.8.$$

Convergence rates: European call in Bates model

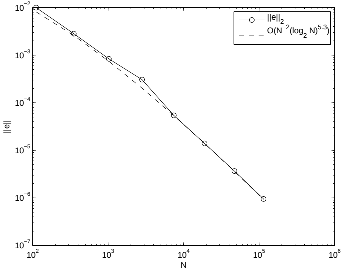
$\mathcal{A}_\nu(u)(x) := -\lambda \int_{\mathbb{R}} \left[u(x_1+\xi, x_2) - u(x)\right] k(\xi) \mathrm{d}\xi$. Set
$(\alpha, \beta, \rho, m, \lambda) = (2.5, 0.5, -0.5, 0.025, 0.5)$, $T = 0.5$, $K = 1$

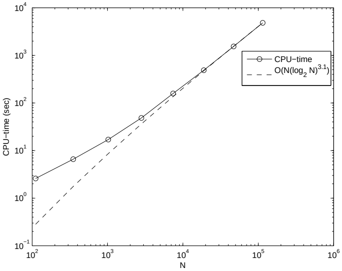


$$\|e_L\|_q = O(N_L^{-2}(\log N_L)^{c_q}), \quad c_2 \approx 3.05, \quad c_\infty \approx 3.85.$$

Linear complexity of pricing methodology

Example: twoscale SV (pure diffusion) model, i.e. $n = 1,\ p = 2$





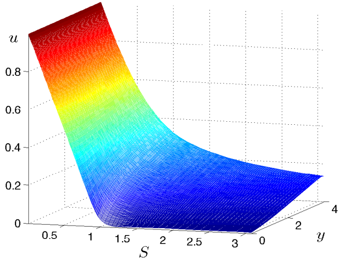
$$\|e_L\|_2 = O(N_L^{-2}(\log N_L)^{5.3}), \quad t_{\mathrm{CPU}} = O(N_L(\log N_L)^{3.1})^2$$

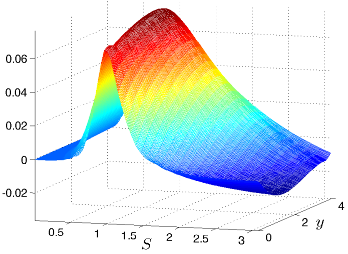
[2]Matlab, 2GHz CPU, 2G RAM

European put in BNS model

- $T = 0.5$, $K = 1$, $r = 0$, $\lambda = 2.5$, IG(γ, δ)-Lévy kernel

$$k(\xi) = \frac{\delta}{2\sqrt{2\pi}} \xi^{-\frac{3}{2}} (1 + \gamma\xi) e^{-\frac{1}{2}\gamma\xi}, \qquad \gamma = 2, \delta = 0.0872.$$





Left: Price for $\rho = -4$.
Right: Difference of prices for $\rho = -0.01$ and $\rho = -4$.

Conclusion

Wavelet FEM for stochastic volatility models

- Derivative pricing
- PIDEs of mixed parabolic/hyperbolic type
- Sparse tensor product spaces
- Stabilization of method due absence of diffusion
- Optimal convergence rates (up to log-terms)
- Linear complexity of numerical scheme (up to log-terms)