Stock Market Insider Trading in Continuous Time with Imperfect Dynamic Information

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Outline

1. Model Setup
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3. Examples
Related literature

- *Linear rational expectations equilibrium approach*
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  - Presence of the insider is able to explain some stylized facts of stock prices behavior Kyle (1985) O’Hara (1995)
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- **Nonlinear rational expectations equilibrium approach**
  - Derivation of the insider optimal strategy: Biagini and Øksendal (2005), Imkeller et all (2004)
A Dynamic Information Model: goals and results

The goals of this work is to

- Investigate how presence of the insider influences market efficiency.

The results of this work are:

- Existence of unique markovian inconspicuous equilibrium pricing rule and characterization of the optimal insider's strategy when variance of the private signal is lower than variance of noisy demand at the end of trading period.

- Existence of non markovian inconspicuous equilibrium and characterization of optimal insider's strategies.
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A Dynamic Information Model: Market structure

- Time horizon is finite: [0, 1]
- Market consists of:
  - One riskless asset which has a constant growth rate \( r = 0 \).
  - A risky asset with final payoff \( f(Z_1) \), where \( f \) is a strictly increasing function and \( Z_t \), the time \( t \) profit of the firm which issued the stock, is given by

\[
Z_t = \nu + \int_0^t \sigma_Z(s) dB_s^1,
\]

where \( B_t^1 \) is a standard Brownian motion and \( \nu \) is \( N(0, \sigma) \) independent of \( \mathcal{F}_1^{B^1} \).
- \( P_t \) denotes the price of the risky asset at time \( t \).
A Dynamic Information Model: Market participants

There are three types of agents on the market:

- **Noisy/liquidity traders**: their total demand by time $t$ is given by standard Brownian motion $B^2_t$ independent of $B^1_t$, $\nu$.

- **Informed investor**: observes $\mathcal{F}^l_t = \mathcal{F}^{B^1_t,B^2_t,\nu}_t$ and is risk-neutral, i.e. she solves

$$
\sup_{\theta} \mathbb{E}[X^\theta_1] = \sup_{\theta} \mathbb{E}[(f(Z_1) - P_1)\theta_1 + \int_0^1 \theta_s - dP_s]
$$

- **Market maker**: observes $\mathcal{F}^M_t = \mathcal{F}^Y_t$ where $Y_t = \theta_t + B^2_t$ is the total order process and sets the price according to

$$
P_t = \mathbb{E}[f(Z_1)|\mathcal{F}^M_t]
$$
Admissible Pricing Rule and Trading Strategy

**Definition:** A pair of measurable functions, \( w : [0, 1] \to \mathbb{R}_+ \setminus \{0\} \) and \( H \in C^{2,1}(\mathbb{R} \times [0, 1]), H : \mathbb{R} \times [0, 1] \to \mathbb{R} \), increasing in the first variable, is an admissible pricing rule \(((H, w) \in \mathcal{H})\) if:

1. The weighting function, \( w(t) \), is given by
   \[
   w(t) = \sum_{i=1}^{n} \alpha_i 1_{\{t \in (t_{i-1}, t_i]\}}
   \]
   and it satisfies \( \int_{0}^{1} w^2(t) dt = 1 \).

2. \( E \left[ \int_{0}^{1} H^2(\int_{0}^{t} w(s) dB_s^2, t) dt \right] + H^2(\int_{0}^{1} w(s) dB_s^2, 1) \] < \( \infty \).

3. \( E \left( h_i^{-1} \left( \mathbb{E} \left[ f(Z_1) | \mathcal{F}_{t_i}^Z \right] \right) \right)^2 \) < \( \infty \) where \( h_i^{-1}(H(y, t_i)) = y \).

Moreover, \((H, w)\) is a rational pricing rule if it satisfies

\[
H(\int_{0}^{t} w(s) dY_s, t) = P(Y_{[0,t]}, t) = \mathbb{E} \left[ f(Z_1) | \mathcal{F}_t^M \right].
\]

**Definition:** A strategy \( \theta_t \in \mathcal{A}(H, w) \) if: it is \( \mathcal{F}_t^I \)-semimartingale,

\[
\mathbb{E} \left[ \int_{0}^{1} H^2 \left( \int_{0}^{t} w(s) d\theta_s + \int_{0}^{t} w(s) dB_s^2, t \right) dt \right] < \infty.
\]

Moreover, the insider’s trading strategy is inconspicuous if \( Y_t = \theta_t + B_t^2 \) is Brownian motion on its own filtration \( \mathcal{F}_t^Y \).
**Definition of Equilibrium**

**Definition** A pair \(((H^*, w^*), \theta^*)\) is an equilibrium if \((H^*, w^*)\) is an admissible pricing rule, \(\theta^*\) is an admissible strategy, and:

1. Given \(\theta^*\), \((H^*, w^*)\) is a rational pricing rule, i.e. it satisfies

\[
H^*(\xi^*_t, t) = \mathbb{E}[f(Z_1)|\mathcal{F}^M_t]
\]

where \(\xi^*_t = \int_0^t w^*(s)dY^*_s\)

2. Given \((H^*, w^*)\), \(\theta^*\) solves the optimization problem

\[
\sup_{\theta \in \mathcal{A}} \mathbb{E}[(f(Z_1) - H^*(\xi^*_1, 1))\theta_1 + \int_0^1 \theta_s - dH^*(\xi^*_s, s)]
\]

Moreover, a pricing rule \((H^*(\xi, t), w^*(t))\) is an inconspicuous equilibrium pricing rule if there exists an inconspicuous insider trading strategy \(\theta^*\) such that \(((H^*, w^*), \theta^*)\) is an equilibrium.
Assumptions

I assume that the following conditions hold:

1. \( \Sigma_z(t) = \int_0^t \sigma^2_Z(s)ds < \infty \) thus can rescale: \( \sigma^2 = 1 - \Sigma_z(1) \).
2. \( F(Z_t, t) = \mathbb{E} \left[ f(Z_1) \mid \mathcal{F}_t^Z \right] \), the fundamental value of the risky asset, is a square integrable martingale.
3. There exists an increasing admissible weighting function \( g(t) = \sum_{i=1}^n \alpha_i 1_{\{t \in (t_{i-1}, t_i]\}} \) such that:
   \[
   \Delta(t) > 0 \text{ for all } t \in [0, 1] \setminus \{t_i\}_{i=1}^n
   \]
   \[
   \Delta(t_i) = 0 \text{ for all } t_i
   \]
   \[
   \int_{t_{i-1}}^{t_i} \Delta^{-2}(s)ds < \infty \text{ for all } t \in [t_{i-1}, t_i)
   \]
   \[
   \lim_{t \to t_i} \int_{t_{i-1}}^{t_i} \Delta^{-1}(s)ds = \infty.
   \]

where \( \Delta(t) = \Sigma_z(t) + \sigma^2 - \int_0^t g^2(s)ds \).
Proposition Suppose that the assumptions are satisfied. Then, given an admissible pricing rule \((H, w) \in \mathcal{H}\) such that \(\int_0^t w^2(s)ds\) is convex function, \(w(t) = \sum_{i=1}^n \alpha_i 1_{\{t \in (t_{i-1}, t_i]\}}\), and \(H\) satisfies the partial differential equation

\[ H_t(y, t) + \frac{w^2(t)}{2} H_{yy}(y, t) = 0, \]  

an admissible trading strategy \(\theta^* \in \mathcal{A}(H, w)\) is optimal for the insider if and only if:

1. The process \(\theta^*_t\) is continuous and has bounded variation.
2. The weighted total order, \(\xi^*_t = \int_0^t w(s)d\theta^*_s + \int_0^t w(s)dB^2_s\) satisfies

\[ h_i \left( \xi^*_t \right) = H(\xi^*_t, t_i) = F \left( Z_{t_i}, t_i \right). \]
Sketch of the proof, \( w(t) \equiv 1 \) case

Consider a system of PDEs:

\[
V_t + \frac{1}{2} V_{yy} + \frac{\sigma_z^2(t)}{2} V_{zz} = 0 \tag{3}
\]
\[
V_y + F(z, t) - H(y, t) = 0 \tag{4}
\]

where \( F(Z_t, t) = \mathbb{E}[f(Z_1)|\mathcal{F}_t^l]. \)

If \( H \) satisfies (1) then there exists \( V \) - a solution of (3) and (4) such that for some \( y^* \) we will have for any \( y, z \in \mathbb{R}, y \neq y^* \)

\[
V(y, z, 1) > V(y^*, z, 1) = 0
\]

where \( y^*(z) \) solves \( H(y^*, 1) = f(z). \)

Since for any admissible trading strategy we have

\[
\mathbb{E}[X_1^\theta] = \mathbb{E}\left[ \int_0^1 (F(Z_s, s) - H(Y_{s-}, s)) d\theta_s + \int_0^1 \theta_s dF + [\theta, F - H]_1 \right]
\]
Sketch of the proof

Then by applying Itô formula we get:

\[
\mathbb{E}[X_1^\theta] = \mathbb{E}[V(0, v, 0) - V(Y_1, Z_1, 1)] - \int_0^1 \frac{V_{yy}(Y_s^-, Z_s, s)}{2} d\langle \theta^c \rangle_s
\]

\[
+ \sum_{s \leq 1} [\Delta V(Y_s, Z_s, s) - V_y(Y_s, Z_s, s) \Delta Y_s]
\]

\[
+ \int_0^1 (F(Z_s, s) - H(Y_s^-, s)) dB^2_s
\]

due to properties of \( V \) we are done
Theorem Suppose that the assumptions are satisfied. Then there exists an equilibrium and it is given by the weighting function

\[ w^*(s) = g(s), \]

the pricing rule

\[ H^*(\xi, t) = \mathbb{E} \left[ f \left( \xi + \int_t^1 g(s) dB_s^2 \right) \right], \]

and the trading strategy \( \theta^*_t \) satisfying \( \theta^*_0 = 0 \) and

\[
d\theta^*_t = \begin{cases} 1 \{t \in (0, t_1]\} \frac{(Z_t - \alpha_1 Y_t) \alpha_1}{\Sigma Z(t) + \sigma^2 - \int_0^t g^2(s) ds} dt \\ + \sum_{i=1}^{n-1} 1 \{t \in (t_i, t_{i+1}]\} \frac{(Z_t - Z_{t_i} - \alpha_{i+1} (Y_t - Y_{t_i})) \alpha_{i+1}}{\Sigma Z(t) + \sigma^2 - \int_0^t g^2(s) ds} dt \end{cases}
\]
Example 1: Let $\sigma = 1$ and $\sigma_z(t) = 0$, thus $\Sigma_z(t) + \sigma^2 - t = 1 - t$ is strictly positive on $[0, 1)$ and $\int_0^t \frac{ds}{(\Sigma_z(s) + \sigma^2 - s)^2} = \frac{1}{1-t} < \infty$. Therefore optimal informed trader’s strategy is given by

$$d\theta_t^* = \frac{Z_1 - Y_t^*}{1 - t} dt$$

Example 2: Consider $\sigma = \sqrt{1 - \rho^2}$ and $\sigma_z(t) = \rho \in [0, 1)$ then $\Sigma_z(t) + \sigma^2 - t = (1 - \rho^2)(1 - t)$ is strictly positive on $[0, 1)$ and $\int_0^t \frac{ds}{(\Sigma_z(s) + \sigma^2 - s)^2} = \frac{1}{(1-\rho^2)^2(1-t)} < \infty$. Therefore optimal informed trader’s strategy is given by

$$d\theta_t^* = \frac{Z_t - Y_t^*}{(1 - \rho^2)(1 - t)} dt$$
Example 2 ctd: limiting case $\rho \to 1$

**Proposition**  Consider a sequence of $\rho_n \to 1$ and

$$
Y^n_t = \int_0^t \frac{Z^n_s - Y^n_s}{(1 - \rho_n^2)(1 - s)} \, ds + B^2_t \\
Z^n_t = \sqrt{1 - \rho_n^2} v + \rho_n B^1_t
$$

Then $Y^n - Z^n \to 0$ weakly in Skorokhod topology.

**Proof**  Let $B^n_t = \frac{B^2_t - \rho_n B^1_t}{\sqrt{1 + \rho_n^2}}$ and consider

$$
X^n_t = (1 - t)^{1 - \rho_n^2} \int_0^t (1 - s)^{-1 - \rho_n^2} \, dB^n_t
$$

Since $X$ is continuous and for any $\epsilon > 0$

$$
\mathbb{E} |X^n_t - X^n_s|^{2m} \leq C_{2m}(\epsilon)|t - s|^m
$$

for any $s, t \in [0, 1 - \epsilon]$ and some constant $C_{2m}(\epsilon)$, we have that $X^n$ is tight on $[0, 1)$. Since $X^n_1 = 0$ for all $n$, $X^n$ is tight on $[0, 1]$. 

Work in Progress and Future Research

Work in Progress

1. Uniqueness of inconspicuous equilibrium pricing rule in the non Markovian case.
2. Existence of an equilibrium when the market parameters do not satisfy the assumptions.
3. Extension of this model to include potential bankruptcy of the firm issuing the stock (with Campi and Çetin)

Future Extensions

1. More general utility functions.