TWISTORIAL MAPS BETWEEN QUATERNIONIC MANIFOLDS

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This paper is dedicated to the memory of Kris Galicki.

ABSTRACT
We introduce a natural notion of quaternionic map between almost quaternionic manifolds and we prove the following, for maps of rank at least one:

- A map between quaternionic manifolds endowed with the integrable almost twistorial structures is twistorial if and only if it is quaternionic.
- A map between quaternionic manifolds endowed with the nonintegrable almost twistorial structures is twistorial if and only if it is quaternionic and totally-geodesic.

As an application, we describe all the quaternionic maps between open sets of quaternionic projective spaces.

INTRODUCTION
An almost quaternionic structure on a manifold is a reduction of its frame bundle to the group $\text{Sp}(1) \cdot \text{GL}(m, \mathbb{H})$. The integrability condition for an almost quaternionic structure (that is, the condition that the corresponding reduction of the frame bundle be given by the cocycle determined by an atlas) is very restrictive [14]. Nevertheless, as $\text{Sp}(1) \cdot \text{GL}(m, \mathbb{H})$ is a Lie group of order two, there exists only one more general notion of integrability for an almost quaternionic structure, which amounts to the existence of a compatible torsion free connection (see [18]). In dimension at least eight, such a connection is called quaternionic whilst, in dimension four, a quaternionic connection is an anti-self-dual Weyl connection. A quaternionic manifold is a manifold endowed with an
almost quaternionic structure and a (compatible) quaternionic connection.

It is a basic fact that the problem of the existence of a quaternionic connection
on a manifold, endowed with an almost quaternionic structure, admits a twistor-
ial interpretation (see Remark 2.10(2), below).

In this paper, we introduce a natural notion of quaternionic map (Definition
2.4) with respect to which the class of quaternionic manifolds becomes a cate-
gory. Furthermore, we show that the quaternionic maps, of rank at least one, are
twistorial in a natural way; that is, they are characterised by the existence of a
holomorphic lift between the corresponding twistor spaces (Theorem 3.5).

The paper is organised as follows. In Section 1 we review some facts on quater-
nionic vector spaces (see [3]). In Section 2, after recalling the definition of almost
quaternionic structure, we introduce the notion of quaternionic map and we prove
its first properties (Proposition 2.6). Also, in Section 2 we recall the two almost
twistorial structures associated to a quaternionic manifold, one of which (Exam-
ple 2.9) is integrable, whilst the other one (Example 2.11) is nonintegrable.

In Section 3 we study twistorial maps between quaternionic manifolds. Besides
the above mentioned relation between quaternionic and twistorial maps, with
respect to the (integrable) twistorial structures, we prove that a map, of rank
at least one, is twistorial, with respect to the nonintegrable almost twistorial
structures, if and only if it is quaternionic and totally geodesic (Theorem 3.7).

Another result we obtain is that any quaternionic map is real-analytic, at least,
outside the frontier of the zero set of its differential (Corrolary 3.6).

Examples of quaternionic maps are given in Section 4. There, we, also, apply
results of Section 3 to describe all the quaternionic maps between open sets of
quaternionic projective spaces (Theorem 4.5).

Finally, in the Appendix we discuss how the quaternionic maps are related to
other, more or less similar, notions. We conclude that the quaternionic maps are
the natural morphisms of Quaternionic Geometry.

1. Quaternionic vector spaces

In this section, we review some facts, from [3], on quaternionic vector spaces
and quaternionic linear maps. Unless otherwise stated, all the vector spaces and
linear maps are assumed real.

**Definition 1.1.** Let $A$ and $B$ be (real or complex, unital) associative algebras.
Two morphisms $\rho, \sigma : A \to B$ are called $A$-equivalent if there exists an automor-
phism $\tau : A \to A$ such that $\sigma = \rho \circ \tau$. 

Let $\mathbb{H}$ be the division algebra of quaternions. The group of automorphisms of $\mathbb{H}$ is $\text{SO}(3)$, acting trivially on 1 and canonically on $\text{Im}\mathbb{H}(=\mathbb{R}^3)$; note that, all the automorphisms of $\mathbb{H}$ are interior.

The following definition is due to [3].

**Definition 1.2.** 1) A **linear hypercomplex structure** on a vector space $V$ is a morphism of associative algebras from $\mathbb{H}$ to $\text{End}(V)$. A vector space endowed with a linear hypercomplex structure is called a **hypercomplex vector space**.

2) A **linear quaternionic structure** on a vector space $V$ is an equivalence class of $\mathbb{H}$-equivalent morphisms of associative algebras from $\mathbb{H}$ to $\text{End}(V)$. Any representative of the class defining a linear quaternionic structure is called an **admissible** linear hypercomplex structure (of the given linear quaternionic structure). A vector space endowed with a linear quaternionic structure is called a **quaternionic vector space**.

Obviously, a hypercomplex vector space is just a left $\mathbb{H}$-module.

**Example 1.3.** The natural structure of left $\mathbb{H}$-module on $\mathbb{H}^m$ gives the (natural) linear hypercomplex structure of $\mathbb{H}^m$, $(m \geq 0)$. Moreover, any hypercomplex vector space is $\mathbb{H}$-linearly isomorphic to $\mathbb{H}^m$, for some $m \geq 0$.

The linear hypercomplex structure of $\mathbb{H}^m$ determines the (natural) linear quaternionic structure of $\mathbb{H}^m$.

A **hypercomplex linear map** $f : V \to W$ between hypercomplex vector spaces is an $\mathbb{H}$-linear map.

Let $V$ be a quaternionic vector space and let $\rho : \mathbb{H} \to \text{End}(V)$ be an admissible linear hypercomplex structure. As $\text{SO}(3)$ acting on $\mathbb{H}$, preserves 1 and $\text{Im}\mathbb{H}$, the vector spaces $Q_V = \rho(\text{Im}\mathbb{H})$ and $\bar{Q}_V = \rho(\mathbb{H})$ depend only of the linear quaternionic structure induced by $\rho$ on $V$. Furthermore, $\bar{Q}_V \subseteq \text{End}(V)$ is a division algebra (noncanonically) isomorphic to $\mathbb{H}$ and $Q_V$ is a three-dimensional oriented Euclidean vector space for which any oriented orthonormal basis $(I, J, K)$ satisfies the quaternionic identities (that is, $I^2 = J^2 = K^2 = IJK = -\text{Id}_V$). Similarly, the unit sphere $Z_V = \rho(S^2)$ is well-defined.

**Definition 1.4** (cf. [3]). Let $V$ and $W$ be quaternionic vector spaces and let $t : V \to W$ and $T : Z_V \to Z_W$ be maps.

We say that $t$ is a **quaternionic linear map, with respect to $T$**, if $t$ is linear and

$$t \circ J = T(J) \circ t,$$

for any $J \in Z_V$. 
Proposition 1.5. Let \( V \) and \( W \) be quaternionic vector spaces and let \( t : V \to W \) be a nonzero linear map.

(i) If \( t \) is quaternionic linear, with respect to \( T_1, T_2 : Z_V \to Z_W \), then \( T_1 = T_2 \).

(ii) If \( t \) is quaternionic linear, with respect to some map \( T : Z_V \to Z_W \), then \( T \) can be uniquely extended to an orientation preserving linear isometry from \( Q_V \) to \( Q_W \).

Proof. Let \( J \in Z_V \). As \( t \circ J = T_k(J) \circ t \), \((k = 1, 2)\), and \( t \neq 0 \) we have that the kernel of \( T_1(J) - T_2(J) \) is nonzero. But \( T_1(J) - T_2(J) \) is in \( \tilde{Q}_W \) which is a division algebra. Thus \( T_1(J) = T_2(J) \). This proves assertion (i).

To prove (ii) we, firstly, obtain, as above, that if \( (I, J, K) \) satisfy the quaternionic identities then, also, \((T(I), T(J), T(K))\) satisfy the quaternionic identities.

Now, let \((a, b, c) \in S^2\). Then \( t \circ (aI + bJ + cK) = T(aI + bJ + cK) \circ t \). On the other hand, we have

\[
t \circ (aI + bJ + cK) = a t \circ I + b t \circ J + c t \circ K = (a T(I) + b T(J) + c T(K)) \circ t .
\]

Thus \( T(aI + bJ + cK) \circ t = (a T(I) + b T(J) + c T(K)) \circ t \) which, because \( t \neq 0 \), implies that \( T(aI + bJ + cK) = a T(I) + b T(J) + c T(K) \). The proof follows. \( \square \)

Next, we prove the following:

Proposition 1.6 (\( \mathbb{H} \)). (i) For any quaternionic vector space \( V \) there exists a quaternionic linear isomorphism from \( V \) to \( \mathbb{H}^m \) (endowed with its natural linear quaternionic structure), for some \( m \geq 0 \).

(ii) Any quaternionic linear map \( t : \mathbb{H}^m \to \mathbb{H}^n \) is given by \( t(X) = aXA \), \((X \in \mathbb{H}^m)\), for some \( a \in \mathbb{H} \) and an \( m \times n \) matrix \( A \), whose entries are quaternions.

Proof. Assertion (i) follows quickly from the fact that any hypercomplex vector space is \( \mathbb{H} \)-linearly isomorphic to \( \mathbb{H}^m \), for some \( m \geq 0 \).

Let \( t : \mathbb{H}^m \to \mathbb{H}^n \) be a quaternionic linear map, with respect to some map \( T : S^2(= Z_{\mathbb{H}^m}) \to S^2(= Z_{\mathbb{H}^n}) \).

If \( t = 0 \) then by taking, for example, \( a = 1 \) and \( A = 0 \) assertion (ii) is trivially satisfied. If \( t \neq 0 \) then, by Proposition 1.5(ii), there exists \( a \in \text{Sp}(1) \) such that \( T(i) = aia^{-1}, T(j) = aj a^{-1}, T(k) = ak a^{-1} \) and one checks immediately that \( t' = a^{-1}t \) is \( \mathbb{H} \)-linear.

Let \( A \) be the matrix of \( t' : \mathbb{H}^m \to \mathbb{H}^n \) with respect to the canonical bases of the free (left) \( \mathbb{H} \)-modules \( \mathbb{H}^m \) and \( \mathbb{H}^m \). Then \( t : \mathbb{H}^m \to \mathbb{H}^n \) is given by \( t(X) = aXA \), \((X \in \mathbb{H}^m)\), and the proof is complete. \( \square \)

From Proposition 1.6 we obtain the following result.
Corollary 1.7 \([1] \). The group of quaternionic linear automorphisms of \( \mathbb{H}^m \) is equal to \( \text{Sp}(1) \cdot \text{GL}(m, \mathbb{H}) \).

Let \( V \) be a quaternionic vector space and let \( \rho : \mathbb{H} \to \text{End}(V) \) be an admissible linear hypercomplex structure. Obviously, \( \rho \otimes \rho : \mathbb{H} \otimes \mathbb{H} \to \text{End}(V \otimes V) \) is also a morphism of associative algebras (the tensor products are taken over \( \mathbb{R} \)). As \( \text{SO}(3) \) acts on \( \mathbb{H} \) by isometries, \( \rho \otimes \rho \) maps the Euclidean structure

\[
1 \otimes 1 + i \otimes i + j \otimes j + k \otimes k
\]
on the (real) dual of \( \mathbb{H} \) onto an endomorphism \( \beta \) of \( V \otimes V \) which depends only of the linear quaternionic structure on \( V \). Let \( b \in \text{Hom}(V \otimes V, V \otimes V) \) be the composition of \( \beta \), to the left, with the projection \( V \otimes V \to V \otimes V \), where \( V \otimes V \) is the second symmetric power of \( V \). Note that, \( b \) is also characterised by

\[
b(X, Y) = \frac{1}{2} \sum_{i=0}^{3} (E_i(X) \otimes E_i(Y) + E_i(Y) \otimes E_i(X)),
\]

for any \( X, Y \in V \), where \( E_0 = \rho(1), E_1 = \rho(i), E_2 = \rho(j), E_3 = \rho(k) \).

Proposition 1.8. Let \( V \) be a quaternionic vector space. For any \( J \in Z_V \) we denote by \( V^{1,0;J} \) and \( V^{0,1;J} \) the eigenspaces of \( J \) with respect to \( i \) and \(-i \), respectively.

(i) The subspace \( b(V \otimes V) \) of \( V \otimes V \) is equal to the space of Hermitian contravariant symmetric 2-forms on \( V \) (that is, elements of \( V \otimes V \) invariant under \( J \otimes J \), for any \( J \in Z_V \).

(ii) For any \( J \in Z_V \) and \( \alpha \in V^* \) we have \( \iota_\alpha(b(V^{0,1;J}, V^{0,1;J})) = 0 \).

(iii) Let \( J, K \in Z_V \) be orthogonal on each other. Then for any \( X \in V^{1,0;J}, Y, Z \in V^{0,1;J} \) and \( \alpha \in V^* \) we have

\[
\iota_\alpha(b(X, Y)) = \alpha(X)Y + \alpha(Y)X + \alpha(KX)KY + \alpha(KY)KX.
\]

Proof. Assertion (i) follows, for example, from relation (1.1).

To prove (ii), let \( \alpha \in V^* \) and let \( I \in Z_V \) be included in any admissible hypercomplex basis \((I, J, K)\). Then

\[
2i_\alpha(b(X, Y)) = \alpha(X)Y + \alpha(Y)X + \alpha(IX)IY + \alpha(IY)IX
\]

\[
+ \alpha(JX)JY + \alpha(JY)JX + \alpha(KX)KY + \alpha(KY)KX.
\]

If \( X, Y \in V^{0,1;J} \) (that is, \( IX = -iX, IY = -iY \)) we have

\[
\alpha(IX)IY + \alpha(IY)IX = -(\alpha(X)Y + \alpha(Y)X),
\]

\[
\alpha(KX)KY + \alpha(KY)KX = \alpha(IJX)IJY + \alpha(IJY)IX
\]

\[
= -(\alpha(JX)JY + \alpha(JY)JX).
\]
The proof of (ii) follows. Assertion (iii) can be proved similarly.

\[\square\]

Remark 1.9. A result, similar to Proposition 1.8, can be straightforwardly established for $\beta$.

2. Quaternionic manifolds and maps

In this section we review some basic facts on (almost) quaternionic manifolds (see [3]) and we introduce the notion of quaternionic map.

Unless otherwise stated, all the manifolds and maps are assumed smooth.

Definition 2.1. A (fibre) bundle of associative algebras is a vector bundle whose typical fibre is a (finite-dimensional) associative algebra $A$ and whose structural group is the group of automorphisms of $A$.

Let $E$ and $F$ be bundles of associative algebras. A morphism of vector bundles $\rho : E \to F$ is called a morphism of bundles of associative algebras if $\rho$ restricted to each fibre is a morphism of associative algebras.

Next, we recall the definitions of almost quaternionic manifolds and almost hypercomplex manifolds.

Definition 2.2 ([5]). An almost quaternionic structure on a manifold $M$ is a pair $(E, \rho)$ where $E$ is a bundle of associative algebras, over $M$, with typical fibre $H$ and $\rho : E \to \text{End}(TM)$ is a morphism of bundles of associative algebras. An almost quaternionic manifold is a manifold endowed with an almost quaternionic structure.

An almost hypercomplex structure on a manifold $M$ is an almost quaternionic structure $(E, \rho)$ for which $E = M \times H$. An almost hypercomplex manifold is a manifold endowed with an almost hypercomplex structure.

It is well-known (see [3]) that there are other ways to define the almost quaternionic and hypercomplex manifolds.

Proposition 2.3. An almost quaternionic structure on a manifold $M$ corresponds to a reduction of the frame bundle of $M$ to $\text{Sp}(1) \cdot \text{GL}(m, H)$ (equivalently, to an $\text{Sp}(1) \cdot \text{GL}(m, H)$-structure).

An almost hypercomplex structure on a manifold $M$ corresponds to a reduction of the frame bundle of $M$ to $\text{GL}(m, H)$.

Proof. Let $M$ be a manifold endowed with an almost quaternionic structure $(E, \rho)$. At each $x \in M$, the morphism $\rho_x : E_x \to \text{End}(T_x M)$ determines a structure of quaternionic vector space on $T_x M$. 
Let $U$ be an open set of $M$ over which $E$ is trivial. Then, by passing to an open subset, if necessary, we can construct a local trivialization $h_U : U \times \mathbb{H}^m \to TM|_U$ of $TM$ which induces quaternionic linear isomorphisms on each fibre.

If $h_U$ and $h_V$ are two such local trivializations, with $U \cap V \neq \emptyset$, then, by Corollary 1.7, we have that $((h_V)^{-1} \circ h_U)(x, q) = (x, a(x)q)$ for some map $a : U \cap V \to \text{Sp}(1) \cdot \text{GL}(m, \mathbb{H})$, $(x \in U \cap V, q \in \mathbb{H}^m)$.

Conversely, if the frame bundle of $M$ admits a reduction to $\text{Sp}(1) \cdot \text{GL}(m, \mathbb{H})$ then the morphism of Lie groups $\text{Sp}(1) \cdot \text{GL}(m, \mathbb{H}) \to SO(3)$, $a \cdot A \mapsto \text{Ad}a$ determines an oriented Riemannian vector bundle $Q$ of rank three and an injective morphism of vector bundles $Q \hookrightarrow \text{End}(TM)$ with the property that any positive local orthonormal frame of $Q$ satisfies the quaternionic identities. Let $\tilde{Q}$ be generated by $Q$ and $\text{Id}_{TM}$. Then $\tilde{Q} \hookrightarrow \text{End}(TM)$ is a subbundle of associative algebras and its typical fibre is $\mathbb{H}$.

The proof for almost hypercomplex manifolds is similar. \qed

Let $M$ be a manifold endowed with an almost quaternionic structure $(E, \rho)$. Then, as each fibre of $E$ is an associative algebra isomorphic to $\mathbb{H}$, there exists an oriented Riemannian vector subbundle of rank three $\text{Im}E \subseteq E$ with the property that any positive local orthonormal frame of it satisfies the quaternionic identities. Let $Q_M = \rho(\text{Im}E)$ and $\tilde{Q}_M = \rho(E)$. Then $\tilde{Q}_M \subseteq \text{End}(TM)$ is a subbundle of associative algebras and its typical fibre is $\mathbb{H}$. Also, $Q_M$ is an oriented Riemannian vector bundle of rank three with the property that any positive local orthonormal frame of it satisfies the quaternionic identities; denote by $Z_M$ the sphere bundle of $Q_M$.

Note that, any almost quaternionic manifold $M$, $(\dim M = 4m)$, is oriented; at each $x \in M$, the orientation of $T_xM$ is given by any $J \in (Z_M)_x$. Denote by $L$ the line bundle of $M$; that is, the line bundle over $M$ associated to the frame bundle of positive frames through the morphism of Lie groups $\text{GL}(4m, \mathbb{R})_0 \to (0, \infty)$, $a \mapsto (\det a)^{1/(4m)}$; see [13]. (Sometimes, $(L^*)^{4m}$ is called ‘the bundle of densities’ of $M$ whilst $L$ is called ‘the bundle of densities of weight 1’ or, even, ‘the weight bundle’ of $M$.)

Also, as $\text{Sp}(1) \cdot \text{GL}(1, \mathbb{H})$ is equal to the connected component of the identity of $\text{CO}(4)$, a four-dimensional almost quaternionic manifold is just an oriented conformal manifold.

**Definition 2.4.** Let $\varphi : M \to N$ be a map between almost quaternionic manifolds and let $\Phi : Z_M \to Z_N$ be such that $\pi_N \circ \Phi = \varphi \circ \pi_M$, where $\pi_M : Z_M \to M$ and $\pi_N : Z_N \to N$ are the projections.

Then $\varphi$ is a *quaternionic map, with respect to $\Phi$, if $d\varphi_{\pi_M(J)} \circ J = \Phi(J) \circ d\varphi_{\pi_M(J)}$. \text{ }}
for any $J \in Z_M$.

A quaternionic immersion/submersion/diffeomorphism is a quaternionic map which is an immersion/submersion/diffeomorphism.

An injective quaternionic immersion is called an almost quaternionic submanifold.

Remark 2.5. 1) For immersions (and, in particular, diffeomorphisms) our definition of quaternionic map particularizes to give notions already in use (see, for example, [1]).

2) Let $M$, $N$ and $P$ be almost quaternionic manifolds, and let $\varphi : M \to N$ and $\psi : N \to P$ be quaternionic maps, with respect to some maps $\Phi : Z_M \to Z_N$ and $\Psi : Z_N \to Z_P$, respectively. Then, obviously, $\psi \circ \varphi$ is quaternionic, with respect to $\Psi \circ \Phi$.

The following result follows quickly from Proposition 1.5.

Proposition 2.6. Let $M$ and $N$ be almost quaternionic manifolds and let $\varphi : M \to N$ be a map of rank at least one.

(i) If $\varphi$ is quaternionic, with respect to $\Phi_1, \Phi_2 : Z_M \to Z_N$, then $\Phi_1 = \Phi_2$.

(ii) If $\varphi$ is quaternionic, with respect to $\Phi : Z_M \to Z_N$, then $\Phi$ induces an isomorphism of $\text{SO}(3)$-bundles $Q_M = \varphi^*(Q_N)$.

Let $M$ be an almost quaternionic manifold, $\dim M = 4m$. An almost quaternionic connection on $M$ is a connection $\nabla$ which induces a connection on $Q_M$ (that is, if $J$ is a section of $Q_M$ and $X$ is a vector field on $M$ then $\nabla_X J$ is a section of $Q_M$); equivalently, $\nabla$ induces a connection on the reduction to $\text{Sp}(1) \cdot \text{GL}(m, \mathbb{H})$ of the frame bundle of $M$, corresponding to the almost quaternionic structure. If $m \geq 2$, a quaternionic connection on $M$ is a torsion-free almost quaternionic connection. If $m = 1$, a quaternionic connection on $M$ is an anti-self-dual Weyl connection.

Definition 2.7 (18; cf. 15). A quaternionic manifold is an almost quaternionic manifold endowed with a quaternionic connection.

The set of quaternionic connections on a quaternionic manifold is well-understood.

Proposition 2.8 (16; see 3). Let $M$ be a quaternionic manifold, $\dim M = 4m$. The set of quaternionic connections on $M$ is an affine space, over the vector space of 1-forms on $M$, isomorphic to the affine space of connections on $L$: if $\alpha$ is the difference between the connections induced on $L^{\frac{2m}{m+1}}$ by two quaternionic
connections $\nabla^2$ and $\nabla^1$ on $M$ then
\[
\nabla^2_X Y = \nabla^1_X Y + \alpha(X)Y + \alpha(Y)X - \iota_a(b(X,Y)) ,
\]
for any vector fields $X$ and $Y$ on $M$.

Next, we recall the natural ‘almost twistorial structures’ of a quaternionic manifold (see [13] for the general notion of ‘almost twistorial structure’).

**Example 2.9** ([18]). Let $M$ be a quaternionic manifold. The quaternionic connection of $M$ induces a connection $H \subseteq TZ_M$ on $Z_M$. Let $H^{1,0}, H^{0,1}$ be the complex subbundle of $H^C$ such that $d\pi_M(H^{1,0})$ is the eigenspace corresponding to $i$ of $J \in \text{End}(T\pi_M(Z_M))$, for any $J \in Z_M$, where $\pi_M : Z_M \to M$ is the projection.

Let $J_M$ be the almost complex structure on $Z_M$ whose eigenbundle corresponding to $i$ is equal to $H^{1,0} \oplus (\ker d\pi_M)^{1,0}$.

We have that $J_M$ is integrable (this can be proved by using [17, Theorem 1.1]). Furthermore, $J_M$ does not depend of the quaternionic connection on $M$ (this can be proved by using Propositions 1.8(ii) and 2.8).

We call $\tau_M = (Z_M, M, \pi_M, J_M)$ the twistorial structure of $M$.

From the integrability of $\tau_M$ it follows that there exists a unique real-analytic structure on $M$ with respect to which the following conditions are satisfied:

a) The almost quaternionic structure of $M$ is real-analytic;

b) Locally, there exist real-analytic quaternionic connections on $M$.

Note that, the given quaternionic connection on $M$ is not necessarily real-analytic.

**Remark 2.10** (cf. [14]). Let $M$ be an almost quaternionic manifold and let $\nabla$ be an almost quaternionic connection on $M$. Then, similarly to Example 2.9, we construct an almost complex structure $J^\nabla$ on $Z_M$ and an almost twistorial structure $\tau^\nabla = (Z_M, M, \pi_M, J^\nabla)$.

1) Let $J$ be a (local) admissible almost complex structure on $M$ and let $s^J$ be the section of $Z_M$ corresponding to $J$. Then any two of the following assertions imply the third:

(i) $J$ is integrable;

(ii) $s^J : (M, J) \to (Z_M, J^\nabla)$ is holomorphic;

(iii) $T^\nabla(A^2(T^{0,1}:J_M)) \subseteq T^{0,1}:J_M$, where $T^\nabla$ is the torsion of $\nabla$.

2) The almost twistorial structure $\tau^\nabla$ is integrable if and only if $M$ admits a quaternionic connection and $\tau^\nabla = \tau_M$. Indeed, if $J^\nabla$ is integrable then, locally, there exist many admissible almost complex structures $J$ which satisfy assertion (ii), above; moreover, as $J^\nabla$ is integrable, any such $J$ is integrable. If $\dim M = 4$, it is fairly well-known that, then, any Weyl connection is anti-self-dual, whilst,
if \( \dim M \geq 8 \) then, by [4, Theorem 2.4], \( M \) admits a quaternionic connection. Also, we have that \( T^{1,0}\mathcal{J}^\nabla Z_M \) is, pointwisely, generated by \((\ker d\pi)^{1,0}\) and the holomorphic tangent bundles to the images of the local sections of \( Z_M \) corresponding to admissible local complex structures on \( M \). As this, also, holds for any quaternionic connection on \( M \), we obtain \( \mathcal{J}^\nabla = \mathcal{J}_M \).

**Example 2.11** (cf. [7]). With the same notations as in Example 2.11, let \( \mathcal{J}'_M \) the almost complex structure on \( Z_M \) whose eigenbundle corresponding to \( i \) is equal to \( (H_1,0) \oplus (\ker d\pi)^{0,1} \).

We have that \( \mathcal{J}'_M \) is nonintegrable (that is, always not integrable). Furthermore, from Propositions 1.8(iii) and 2.8 it follows that \( \mathcal{J}'_M \) determines the quaternionic connection on \( M \).

We call \( \tau'_M = (Z_M, M, \pi_M, \mathcal{J}'_M) \) the nonintegrable almost twistorial structure of \( M \).

We end this section with a well-known fact which will be used later on; for the reader’s convenience we also supply a proof.

**Proposition 2.12** (see [2]). Any almost quaternionic submanifold \( N \) of a quaternionic manifold \( M \) is totally-geodesic with respect to any quaternionic connection \( \nabla \) on \( M \). Moreover, \( \nabla \) induces a quaternionic connection on \( N \).

**Proof.** We have \( J(T_{\pi_M(J)}N) \subseteq T_{\pi_M(J)}N \), for any \( J \in (Q_M)|_N \), where \( \pi_M : Q_M \to M \) is the projection. Thus, the vector bundle \((TM|_N)/TN\) admits a unique reduction to \( \text{Sp}(1) \cdot \text{GL}(m-n, \mathbb{H}) \) such that the projection \( \pi : TM \to (TM|_N)/TN \) is quaternionic linear on each fibre, where \( \dim N = 4n \) and \( \dim M = 4m \).

Let \( B \) be the ‘second fundamental form’ of \( N \subseteq M \) with respect to \( \nabla \); that is, \( B(X,Y) = \pi(\nabla_X Y) \), for any vector fields \( X, Y \) on \( N \). We have to prove that \( B = 0 \).

Let \( J \) be a section of \( Z_M \) over some open set \( U \) of \( M \) which intersects \( N \). Then, for any \( x \in U \cap N \) and \( X \in T_x N \), we have that \( \nabla_X J \in (Q_M)_x \) and, consequently, \((\nabla_X J)(T_x N) \subseteq T_x N \). Therefore, \( \nabla_X (JY) - J(\nabla_X Y) \) is a vector field on \( U \cap N \), for any vector fields \( X, Y \) on \( U \cap N \).

Hence, for any \( J \in (Z_M)|_N \) and \( X, Y \in TN \), we have \( B(X,JY) = JB(X,Y) \); as \( B \) is symmetric, this is equivalent to \( B(JX,JY) = -B(X,Y) \). By applying this property to a positive orthonormal frame of \( Q_M \) the proof follows quickly. \( \square \)

**Remark 2.13.** Proposition 2.12 motivates the use of the term ‘quaternionic submanifold’, instead of ‘almost quaternionic submanifold’, when dealing with an ambient quaternionic manifold.
3. Twistorial maps between quaternionic manifolds

The following definition is a particular case of [13, Definition 4.1].

**Definition 3.1.** Let $M$ and $N$ be quaternionic manifolds and let $\varphi : M \to N$ be a map. Suppose that there exists a map $\Phi : Z_M \to Z_N$ such that $\pi_N \circ \Phi = \varphi \circ \pi_M$.

Denote by $\tau_M = (Z_M, M, \pi_M, J_M)$ and $\tau_N = (Z_N, N, \pi_N, J_N)$ the twistorial structures of $M$ and $N$, respectively. Then $\varphi : (M, \tau_M) \to (N, \tau_N)$ is a twistorial map, with respect to $\Phi$, if $\Phi : (Z_M, J_M) \to (Z_N, J_N)$ is holomorphic.

Similarly, $\varphi : (M, \tau'_M) \to (N, \tau'_N)$ is a twistorial map, with respect to $\Phi$, if $\Phi : (Z_M, J'_M) \to (Z_N, J'_N)$ is holomorphic, where $\tau'_M = (Z_M, M, \pi_M, J'_M)$ and $\tau'_N = (Z_N, N, \pi_N, J'_N)$ are the nonintegrable almost twistorial structures of $M$ and $N$, respectively.

Let $M$ be an almost quaternionic manifold and let $J \in Z_M$. We denote $T^{1,0;J}M$ and $T^{0,1;J}M$ the eigenspaces of $J$ corresponding to $i$ and $-i$, respectively.

**Proposition 3.2.** Let $M$ and $N$ be quaternionic manifolds; denote by $\nabla^M$ and $\nabla^N$ the quaternionic connections of $M$ and $N$, respectively.

Let $\varphi : M \to N$ be a quaternionic map with respect to some map $\Phi : Z_M \to Z_N$; suppose that $\varphi$ is of rank at least one.

Then the following assertions are equivalent:

(i) $\varphi : (M, \tau_M) \to (N, \tau_N)$ is twistorial, with respect to $\Phi$.

(ii) $(\nabla_\Phi)^{(1,0;J)}M, T^{1,0;J}M) \subseteq T^{0,1;J}N$, for any $J \in Z_M(= \varphi^*(Z_N))$, where $\nabla$ is the connection, on $\text{Hom}(M, \varphi^*(TN))$, induced by $\nabla^M$ and $\nabla^N$.

(iii) $A(JX)J = J \circ (A(X)J)$, for any $J \in Z_M$ and $X \in T_{\pi_M(J)M}$, where $A$ is the difference between the connections, on $Q_M(= \varphi^*(Q_N))$, induced by $\nabla^M$ and $\nabla^N$.

**Proof.** To prove (i) $\iff$ (iii), note that it is sufficient to consider $\varphi$ of constant rank (there exists a dense open subset of $M$ such that on each of its connected components $\varphi$ has constant rank). As then, locally, the image of $\varphi$ is a quaternionic submanifold of $N$, Proposition 2.12 implies that we can further assume $\varphi$ submersive.

Let $J_0 \in Z_M$ and let $x_0 = \pi_M(J_0)$. Also, let $S$ be (the image of) a local section of $\varphi$, containing $x_0$, such that $T_{x_0}S$ is preserved by $J_0$. Locally, we may extend $J_0$ to a section $J$ of $Z_M$, over $S$. Then there exists a local section $\tilde{J}$ of $Z_N$ such that $J = \varphi^*(\tilde{J})$ (equivalently, $\varphi \circ J = \tilde{J} \circ \varphi$); denote $J_0 = J_{\varphi(x_0)}$.

Now, the differential of $J$ at $x_0$ is a complex linear map from $(T_{x_0}S, J_0|_{T_{x_0}S})$ to $(T_{J_0}Z_M, (J_M)_{J_0})$ if and only if $\nabla^M_{J_0X}J = J_0 \circ \nabla^M_XJ$, for any $X \in T_{x_0}S$. Similarly, the differential of $\tilde{J}$ at $\varphi(x_0)$ is a complex linear map from $(T_{\varphi(x_0)}N, J_0)$ to
\( (T_{\mu}Z_N, (J_N)_{\mu}) \) if and only if \( \nabla^N_{j_{\mu}X}j = j_0 \circ \nabla_X^Nj \), for any \( X \in T_{\varphi(\mu)}N \).

It follows that \( d\Phi_{j_0} : (T_{\mu}Z_M, (J_M)_{\mu}) \to (T_{\mu}Z_N, (J_N)_{\mu}) \) is complex linear if and only if \( A(j_0X)j_0 = j_0 \circ (A(X)j_0) \), for any \( X \in T_{\tau_0}M \).

To prove (ii) \( \iff \) (iii), firstly, note that, by considering \( d\varphi \) as a section of \( \text{Hom}(TM, \varphi^*(TN)) \), we have \( d\varphi \circ J = j \circ d\varphi \), for any \( J \in Z_M \). By taking the covariant derivative of this equality we obtain

\[
(\nabla_X d\varphi) \circ J - J \circ (\nabla_X d\varphi) = (A(X)J) \circ d\varphi,
\]

for any \( J \in Z_M \) and \( X \in T_{\pi_M(J)}M \). The proof follows. \( \square \)

**Remark 3.3.** The method of Proposition 3.2 can be applied in several other contexts. For example, let \((M^m, c, D)\) be a Weyl space, \( \dim M = m \), and let \( 1 \leq r \leq \frac{1}{2} m \). If \( r < \frac{1}{2} m \) let \( \pi : P \to M \) be the bundle of skew-adjoint \( f \)-structures on \((M^m, c)\) whose kernels have dimension \( m - 2r \) (any \( F \in P \) is a skew-adjoint linear map on \((T_{\pi(F)}M, c_{\pi(F)})\) such that \( F^3 + F = 0 \) and \( \dim(\ker F) = m - 2r \)). If \( r = \frac{1}{2} m \), \( (m \text{ even}) \), let \( P \) be the bundle of positive orthogonal complex structures on \((M^m, c)\).

Then \( D \) induces a connection \( \mathcal{H} \subseteq TP \) on \( P \). Define \( \mathcal{H}^0 \subseteq \mathcal{H} \) such that \( d\pi(\mathcal{H}^0) = T^0_{\pi(F)}PM \), where \( T^0_{\pi(F)}PM \) is the eigenspace of \( F \) corresponding to 0, \( (F \in P) \). Also, define \( \mathcal{H}^{0,1} \subseteq \mathcal{H}^C \) such that \( d\pi(\mathcal{H}^{0,1}) = T^{0,1}_{\pi(F)}PM \), where \( T^{0,1}_{\pi(F)}PM \) is the eigenspace of \( F \) corresponding to \(-i\), \( (F \in P) \).

Let \( F \) be the almost \( f \)-structure on \( P \) whose eigendistributions corresponding to 0 and \(-i\) are \( \mathcal{H}^0 \) and \( \mathcal{H}^{0,1} \oplus (\ker d\pi_M)^{0,1} \), respectively. Then \( \tau_{m,r} = (P, M, \pi, F) \) is an almost twistorial structure on \( M \); see [17] for the characterisation of the integrability of \( \tau_{m,r} \).

Now, let \((M^{2n}, c_M, D^M)\) and \((N^{2n-1}, c_N, D^N)\) be Weyl spaces; denote by \( \tau^M_{2n,n} = (P_M, M, \pi_M, F) \) and \( \tau^N_{2n-1,n-1} = (P_N, N, \pi_N, F) \) the corresponding almost twistorial structures.

Let \( \varphi : (M^{2n}, c_M) \to (N^{2n-1}, c_N) \) be a horizontally conformal submersion. There exists a unique map \( \Phi : P_M \to P_N \) such that \( \pi_N \circ \Phi = \varphi \circ \pi_M \) and

\[
d\varphi(T^0_{\pi_M(J)}M) = T^0_{\pi_N(\Phi(J))}N \oplus T^{0,1}_{\pi_N(\Phi(J))}N,
\]

for any \( J \in P_M \).

The following assertions are equivalent:

(i) \( \varphi : (M^{2n}, \tau^M_{2n,n}) \to (N^{2n-1}, \tau^N_{2n-1,n-1}) \) is twistorial, with respect to \( \Phi \) (that is, \( \Phi : (P_M, F) \to (P_N, F) \) is holomorphic; equivalently, \( d\Phi(T^{0,1}P_M) \subseteq T^0P_N \oplus T^{0,1}P_N \).

(ii) \( (Dd\varphi)(T^0_{\pi_M(J)}M, T^{0,1}_{\pi_M(J)}M) \subseteq d\varphi(T^0_{\pi_N(\Phi(J))}N), \) for any \( J \in P_M \), where \( D \)
is the connection, on $\text{Hom}(TM, \varphi^*(TN))$, induced by $D^M$ and $D^N$.

In the particular case $n = 2$, from the above equivalence it follows quickly the known (see [13] and the references therein) characterisation of twistorial maps from four-dimensional conformal manifolds to three-dimensional Weyl spaces; also, recall that then, if (i) or (ii) holds, $(M^4, c_M)$ is anti-self-dual if and only if $(N^3, c_N, D^N)$ is Einstein–Weyl.

If $n \geq 3$ and (i) or (ii) holds then $(M^{2n}, c_M)$ is flat if and only if $D^N$ is the Levi-Civita connection of constant curvature local representatives of $c_N$.

Similarly to Proposition 3.2, we obtain the following result.

**Proposition 3.4.** Let $M$ and $N$ be quaternionic manifolds; denote by $\nabla^M$ and $\nabla^N$ the quaternionic connections of $M$ and $N$, respectively.

Let $\varphi : M \to N$ be a quaternionic map with respect to some map $\Phi : Z_M \to Z_N$; suppose that $\varphi$ is of rank at least one.

Then the following assertions are equivalent:

(i) $\varphi : (M, \tau_M) \to (N, \tau_N)$ is twistorial, with respect to $\Phi$.

(ii) $(\nabla d\varphi)(T^{0,1;J}_{\pi_M(J)}M, T^{0,1;J}_{\pi_M(J)}M) \subseteq T^{0,1;J}_{\varphi(\pi_M(J))}N$, for any $J \in Z_M (= \varphi^*(Z_N))$, where $\nabla$ is the connection, on $\text{Hom}(TM, \varphi^*(TN))$, induced by $\nabla^M$ and $\nabla^N$.

(iii) $A(JX)J = -J \circ (A(X)J)$, for any $J \in Z_M$ and $X \in T_{\pi_M(J)}M$, where $A$ is the difference between the connections, on $Q_M (= \varphi^*(Q_N))$, induced by $\nabla^M$ and $\nabla^N$.

Next, we prove the following result.

**Theorem 3.5.** Let $\varphi : M \to N$ be a map between quaternionic manifolds and let $\Phi : Z_M \to Z_N$ be such that $\pi_N \circ \Phi = \varphi \circ \pi_M$.

If the zero set of the differential of $\varphi$ has empty interior then the following assertions are equivalent:

(i) $\varphi : (M, \tau_M) \to (N, \tau_N)$ is twistorial, with respect to $\Phi$.

(ii) $\varphi : M \to N$ is quaternionic, with respect to $\Phi$.

**Proof.** Obviously, (i) $\implies$ (ii). Thus, it is sufficient to prove (ii) $\implies$ (i).

Let $F \subseteq M$ be the zero set of the differential of $\varphi$. As $M \setminus F$ is dense in $M$ and $\pi_M$ is open, we have $\pi_M^{-1}(M \setminus F)$ dense in $Z_M$. Thus, we may suppose that, at each point, $\varphi$ has rank at least one.

By Proposition 3.2, it is sufficient to prove that if $\varphi$ is quaternionic then

$$(\nabla d\varphi)(T^{0,1;J}_{\pi_M(J)}M, T^{0,1;J}_{\pi_M(J)}M) \subseteq T^{0,1;J}_{\varphi(\pi_M(J))}N, \quad \text{for any } J \in Z_M (= \varphi^*(Z_N)).$$

As in the proof of Proposition 3.2, we may suppose $\varphi$ submersive. Denote
\( \mathcal{V} = \ker d\varphi \) and let \( \mathcal{H} \) be a distribution on \( M \), complementary to \( \mathcal{V} \), and which is preserved by \( Z_M \) (for example, let \( \mathcal{H} \) be the orthogonal complement of \( \mathcal{V} \) with respect to some Hermitian metric on \( M \)); as usual, we identify \( \mathcal{H} = \varphi^*(TN) \).

Let \( J \) be an admissible complex structure (locally) defined on \( M \). As \( \mathcal{V} \) and \( \mathcal{H} \) are invariant under \( J \), we have decompositions
\[
\mathcal{V}^C = \mathcal{V}_0^J \oplus \mathcal{V}_1^J \quad \text{and} \quad \mathcal{H}^C = \mathcal{H}_0^J \oplus \mathcal{H}_1^J.
\]

Because \( \varphi \) is quaternionic, we have \( \mathcal{H}_0^J = \varphi^* (T^0\mathcal{N}^J) \).

Let \( V \) and \( X \) be sections of \( \mathcal{V}_0^J \) and \( T^0\mathcal{N}^J \) respectively. As the section of \( Z_M \) corresponding to \( J \) is a holomorphic map from \((M,J)\) to \((Z_M,J_M)\), we have that
\[
(\nabla d\varphi)(V,X) = -d\varphi(\nabla V X)
\]
is a section of \( \mathcal{H}_0^J \).

From the fact that \( \nabla M \) and \( \nabla N \) are torsion-free it follows that there exists a section \( \alpha \) of \( \mathcal{H}_0^J \) such that
\[
(\nabla d\varphi)(X,Y) = S_\alpha(X,Y),
\]
for any \( X,Y \in \mathcal{H} \) (cf. Proposition 2.8). Hence, by Proposition 1.8, \((\nabla d\varphi)(\mathcal{H}_0^J, \mathcal{H}_0^J) \subseteq \mathcal{H}_0^J \).

The proof is complete.

From Theorem 3.5 we obtain the following result (which, also, holds for a more general class of twistorial maps).

**Corollary 3.6.** Any quaternionic map between quaternionic manifolds is real-analytic, at least, outside the frontier of the zero set of its differential.

A condition equivalent to assertion (ii) of the following result is used in [11], for maps between quaternionic Kähler manifolds.

**Theorem 3.7.** Let \( \varphi : M \to N \) be a map between quaternionic manifolds and let \( \Phi : Z_M \to Z_N \) be such that \( \pi_N \circ \Phi = \varphi \circ \pi_M \).

If the zero set of the differential of \( \varphi \) has empty interior then the following assertions are equivalent:

(i) \( \varphi : (M, \tau'_M) \to (N, \tau'_N) \) is twistorial, with respect to \( \Phi \).

(ii) \( \varphi \) is quaternionic, with respect to \( \Phi \), and the connections on \( Q_M (= \varphi^*(Q_N)) \), induced by the quaternionic connections of \( M \) and \( N \), are equal.

(iii) \( \varphi \) is a totally geodesic map which is quaternionic, with respect to \( \Phi \).

**Proof.** This can be proved as follows. By Proposition 3.4, we have (ii)\(\Longrightarrow\)(i) and (iii)\(\Longrightarrow\)(ii). Thus, it remains to prove (ii)\(\Longrightarrow\)(iii).

If (i) holds then \( \varphi \) is quaternionic and, by Theorem 3.5, \( \varphi : (M, \tau'_M) \to (N, \tau'_N) \) is twistorial. Thus, assertion (iii) of Proposition 3.2 and assertion (iii) of Proposition 3.4 are both satisfied. This shows that (i)\(\Longrightarrow\)(ii).

Finally, from Proposition 3.4, it follows quickly that the (1,1)-component of \( \nabla d\varphi \) is zero. This implies \( \nabla d\varphi = 0 \) and the proof is complete. \(\square\)
Next, we explain why, in Theorems 3.5 and 3.7, the assumption on the zero set of the differential of the map cannot be weakened.

**Remark 3.8.** Let $M$ and $N$ be quaternionic manifolds. Suppose that $Z_M = M \times S^2$ is the trivial bundle and let $\pi : Z_M \to S^2$ be the projection.

Let $y \in N$ and let $T : S^2 \to (Z_N)_y$ be an orientation preserving isometry. Then the constant map $\varphi : M \to N, x \mapsto y, (x \in M)$, is quaternionic, with respect to $\Phi = T \circ \pi$.

On the other hand, $\varphi$ is twistorial, with respect to $\Phi$, if and only if $M$ is hypercomplex.

Note that, Theorems 3.5 and 3.7 hold for any nonconstant real-analytic map (without any assumption on the zero set of the differential of the map).

4. Examples and further results

Firstly, we mention that, as any quaternionic submanifold corresponds to an injective quaternionic immersion, in [21] can be found many examples of quaternionic maps. For example, we have the following:

**Example 4.1.** Let $Gr_2(m + 2, \mathbb{C})$ be the Grassmannian manifold of complex vector subspaces, of dimension 2, of $\mathbb{C}^{m+2}$, $(m \geq 1)$.

This is a quaternionic manifold of (real) dimension $4m$. Its twistor space is the flag manifold $F_{1,m+1}(m + 2, \mathbb{C})$ of pairs $(l, p)$ with $l$ and $p$ complex vector subspaces of dimensions 1 and $m + 1$, respectively, such that $l \subset p$. The projection $\pi : F_{1,m+1}(m + 2, \mathbb{C}) \to Gr_2(m + 2, \mathbb{C})$ is defined by $\pi(l, p) = l \oplus p^\perp$, for any $(l, p) \in F_{1,m+1}(m + 2, \mathbb{C})$, where $p^\perp$ is the orthogonal complement of $p$ with respect to the canonical Hermitian product on $\mathbb{C}^{m+2}$.

Any injective complex linear map $A : \mathbb{C}^{m+2} \to \mathbb{C}^{n+2}$, $(1 \leq m \leq n)$, induces, canonically, a quaternionic map $\varphi^A : Gr_2(m+2, \mathbb{C}) \to Gr_2(n+2, \mathbb{C})$, with respect to the map $\Phi^A : F_{1,m+1}(m + 2, \mathbb{C}) \to F_{1,n+1}(n + 2, \mathbb{C})$ defined by $\Phi^A(l, p) = (A(l), A(p) \oplus q)$, for any $(l, p) \in F_{1,m+1}(m + 2, \mathbb{C})$, where $q \subset \mathbb{C}^{n+2}$ is a fixed complement of $\text{im}A$ in $\mathbb{C}^{n+2}$. (Note that, if we choose another complement of $\text{im}A$ in $\mathbb{C}^{n+2}$ then $\Phi^A$ changes by a composition, to the left, with a holomorphic diffeomorphism of $F_{1,n+1}(n + 2, \mathbb{C})$.)

The next example shows that, besides quaternionic immersions, there are many other quaternionic maps.

**Example 4.2.** Let $\mathbb{H}P^m$ be the left quaternionic projective space of (real) dimension $4m$, $(m \geq 1)$. This is a quaternionic manifold (see [14]). Its twistor
space is \( \mathbb{CP}^{2m+1} \), where the projection \( \pi : \mathbb{CP}^{2m+1} \to \mathbb{HP}^m \) is induced by the identification \( \mathbb{H}^{m+1} = \mathbb{C}^{2m+2} \), through the morphism of Lie groups \( \mathbb{C}^* \to \mathbb{H}^* \).

Let \( A : \mathbb{H}^{m+1} \to \mathbb{H}^{m+1} \) be a left \( \mathbb{H} \)-linear map, \( (m,n \geq 1) \). Then \( A \) induces two maps \( \varphi^A : \mathbb{HP}^m \setminus P_{\mathbb{H}}(\ker A) \to \mathbb{HP}^m \) and \( \Phi^A : \mathbb{CP}^{2m+1} \setminus P_{\mathbb{C}}(\ker A) \to \mathbb{CP}^{2m+1} \), where \( P_{\mathbb{H}}(\ker A) \) and \( P_{\mathbb{C}}(\ker A) \) are the quaternionic and complex projective spaces, respectively, determined by \( \ker A \).

Then \( \varphi^A \) is a quaternionic map, with respect to \( \Phi^A \) (just note, for example, that \( \Phi^A \) is holomorphic).

**Example 4.3.** Let \( \pi : \mathbb{H}^{m+1} \setminus \{0\} \to \mathbb{HP}^m \) be the Hopf fibration. Then \( \pi \) is a quaternionic map, with respect to the canonical projection
\[
\Pi : 2(m+1)\mathcal{O}(1) \setminus 0 \to P(2(m+1)\mathcal{O}(1)) = \mathbb{CP}^{2m+1},
\]
where \( \mathcal{O}(1) \) is the dual of the tautological line bundle over \( \mathbb{CP}^1 \).

The following example, based on a construction of [20], is, essentially, a generalization of Example 4.3.

**Example 4.4.** Let \( M \) be a quaternionic manifold of dimension \( 4m \), \( (m \geq 1) \), and let \( (P,M,\text{Sp}(1) \cdot \text{GL}(m,\mathbb{H})) \) be its bundle of quaternionic frames (that is, quaternionic linear isomorphisms from \( \mathbb{H}^m \) to \( T_xM \), \( (x \in M) \)).

Define \( \rho : \text{Sp}(1) \cdot \text{GL}(m,\mathbb{H}) \to \mathbb{H}^*/\{\pm 1\} \) by \( \rho(a \cdot A) = \pm a \), for any \( a \cdot A \in \text{Sp}(1) \cdot \text{GL}(m,\mathbb{H}) \), where \( \mathbb{H}^* = \mathbb{H} \setminus \{0\} \).

Denote \( E = \rho(P) \). Then \( (E, M, \mathbb{H}^*/\{\pm 1\}) \) is a principal bundle. Furthermore, the quaternionic connection of \( M \) induces a principal connection \( \mathcal{H} \subseteq TE \). Let \( \mathcal{V} = \ker \pi \), where \( \pi : E \to M \) is the projection.

Let \( q \in S^2 \subseteq \text{Im} \mathbb{H} \). The multiplication to the right by \( -q \) defines a negative (linear) complex structure on \( \mathbb{H} \) which, obviously, is invariant under the left action of \( \mathbb{H}^* \) on \( \mathbb{H} \). Thus, \( q \) induces on \( \mathcal{V} \) a structure of complex vector bundle \( J_q \mathcal{V} \).

As \( \tilde{Q}_M \) is a bundle associated to \( E \), we have \( \pi^*(\tilde{Q}_M) = E \times \mathbb{H} \). Together with the fact that \( \mathcal{H} = \pi^*(TM) \) this induces a left action of \( \mathbb{H}^* \) on \( \mathcal{H} \). In particular, \( q \) induces on \( \mathcal{H} \) a structure of complex vector bundle \( J_q \mathcal{H} \).

Obviously, \( J_q = J_q^{\mathcal{V}} \oplus J_q^{\mathcal{K}} \) is an almost complex structure on \( E \).

The morphism of Lie groups \( \mathbb{C}^* \to \mathbb{H}^* \) given by \( a + bi \mapsto a + bq \), \( (a,b \in \mathbb{R}) \), induces a right action of \( \mathbb{C}^*/\{\pm 1\} \) on \( E \). Furthermore, the quotient of \( E \) through this action is \( Z_M \) and the projection \( \psi^q : (E, J_q) \to (Z_M, J^M) \) is holomorphic.

From the fact that \( J^M \) is integrable it follows that \( J^q \) is integrable. Thus, \( (J^q, J^l, J^k) \) defines a hypercomplex structure on \( E \). The complex structure \( J_E \) of its twistor space \( Z_E(= E \times S^2) \) is characterised by the following: \( J_E|_{\{e\} \times S^2} \) is
the canonical complex structure of $S^2$ whilst $\mathcal{J}_E|_{E \times \{q\}} = J^q$, for any $e \in E$ and $q \in S^2$.

Note that, $Z_M = E \times \chi S^2$, where $\chi : (\mathbb{H}^*/\{\pm 1\}) \times S^2 \to S^2$ is defined by $\chi(\pm p, q) = pqp^{-1}$, for any $\pm p \in \mathbb{H}^*/\{\pm 1\}$ and $q \in S^2$. Let $\Pi : Z_E \to Z_M$ be the projection. Alternatively, $\Pi$ can be defined by $\Pi(e, q) = \psi^q(e)$, for any $(e, q) \in Z_E$.

Then $\Pi : (Z_E, \mathcal{J}_E) \to (Z_M, \mathcal{J}_M)$ is holomorphic and $\pi : E \to M$ is a quaternionic submersion, with respect to $\Pi$.

Next, we prove that Example 4.2 gives all the quaternionic maps between open sets of quaternionic projective spaces.

**Theorem 4.5.** Let $U$ be a connected open set of $\mathbb{H}P^m$ and let $\varphi : U \to \mathbb{H}P^n$ be a quaternionic map, $(m, n \geq 1)$.

Then there exists an $\mathbb{H}$-linear map $A : \mathbb{H}^{m+1} \to \mathbb{H}^{n+1}$ such that $\varphi = \varphi^A|_U$ and, in particular, $U \cap P_\mathbb{H}(\ker A) = \emptyset$.

To prove Theorem 4.5, we need to lemmas, the first of which is, most likely, known but we do not have a reference for it.

**Lemma 4.6.** Let $M$ be a quaternionic manifold and let $\tau_M = (Z_M, M, \pi_M, \mathcal{J}_M)$ be its twistorial structure.

Then a function $f : M \to \mathbb{C}$ is constant if and only if $f \circ \pi_M : (Z_M, \mathcal{J}_M) \to \mathbb{C}$ is holomorphic.

**Proof.** If $f \circ \pi_M : (Z_M, \mathcal{J}_M) \to \mathbb{C}$ is holomorphic then $f$ is holomorphic with respect to any (local) admissible complex structure $J$ on $M$; equivalently, the differential of $f$ is zero on $T^{0,1}; J M$.

It follows that $df = 0$ and the lemma is proved. \hfill $\square$

**Lemma 4.7** (cf. [3]). Let $A$ be a complex linear map from $\mathbb{H}^{m+1} = \mathbb{C}^{2m+2}$ to $\mathbb{H}^{n+1} = \mathbb{C}^{2n+2}$, of complex rank at least 4, $(m, n \geq 1)$. Suppose that there exists an open set $U \subseteq \mathbb{H}^{m+1}$ such that $A$ maps the intersection of any quaternionic line (through the origin) with $U$ into a quaternionic line.

Then $A$ is quaternionic linear.

**Proof.** The (germ unique) complexification of $\mathbb{H}P^m$ is $\text{Gr}_2(2m + 2, \mathbb{C})$ (to show this, use the fact that $\text{GL}(m + 1, \mathbb{H})$ acts transitively on $\mathbb{H}P^m$).

Clearly, $A$ determines a holomorphic map $\psi^A : \mathcal{A} \to (\mathbb{H}P^n)^\mathbb{C}$, where $\mathcal{A}$ is the (open) subset of $\text{Gr}_2(2m + 2, \mathbb{C})$ formed of those two-dimensional complex vector subspaces of $\mathbb{C}^{2m+2}$ whose intersection with $\ker A$ is $\{0\}$. Note that, $\psi^A$ has a pole along $(\mathbb{H}P^n)^\mathbb{C} \setminus \mathcal{A}$.
From the hypothesis, it follows that the restriction of $\psi^A$ to some open set intertwines the conjugations $C_m$ and $C_n$ of $(\mathbb{H}P^m)^C$ and $(\mathbb{H}P^n)^C$, respectively. By analyticity, we obtain that $\psi^A$ and $C_n \circ \psi^A \circ C_m$ determine a holomorphic map on $A \cup C_m(A)$. Consequently, $C_m(A) = A$ (otherwise, $\psi^A$ could be holomorphically extended over points of $(\mathbb{H}P^m)^C \setminus A)$ and $C_n \circ \psi^A = \psi^A \circ C_m$. Hence, $A$ maps any quaternionic line whose intersection with $\ker A$ is $\{0\}$ onto a quaternionic line.

Note that, there are no quaternionic lines which intersect $\ker A$ along complex vector spaces of dimension 1 (otherwise, the map $\Phi^A$ from $\mathbb{C}P^{2m+1} \setminus P_{\mathbb{C}}(\ker A)$ to $\mathbb{C}P^{2n+1}$, determined by $A$, would induce a continuous extension of $\psi^A$ over points of $(\mathbb{H}P^m)^C \setminus A)$).

Thus, $A$ maps any quaternionic line into a quaternionic line and, by [3, Theorem 1.1], the proof of the lemma is complete. □

**Remark 4.8.** With the same notations as in Lemma 4.7 and its proof, let $\varphi^A : \mathbb{H}P^m \setminus P_{\mathbb{H}}(\ker A) \to \mathbb{H}P^n$ be the quaternionic map determined by $A$. Then $\psi^A$ is the complexification of $\varphi^A$.

**Proof of Theorem 4.5.** Let $\varphi$ be a quaternionic map from a connected open set of $\mathbb{H}P^m$ to $\mathbb{H}P^n$.

We, firstly, assume the differential of $\varphi$ nowhere zero.

Then, by Corollary 3.6, $\varphi$ is real-analytic. Therefore, it is sufficient to find an $\mathbb{H}$-linear map $A$ such that $\varphi = \varphi^A$ on some open set. Moreover, similarly to the proof of Proposition 3.2, we may assume $\varphi$ submersive, surjective and with connected fibres.

By Theorem 3.3, $\varphi$ is twistorial, with respect to some holomorphic map $\Phi$ between open sets $U$ and $V$ of $\mathbb{C}P^{2m+1}$ and $\mathbb{C}P^{2n+1}$, respectively. Also, $U$ and $V$ contain families of projective lines (the twistor lines) which are mapped one onto another by $\Phi$. Moreover, as the complexification of $\mathbb{H}P^m$ contains many complex-quaternionic submanifolds, the preimage through $\Phi$ of a hyperplane of $\mathbb{C}P^{2n+1}$ (not disjoint from $V$) is an open subset of a hyperplane of $\mathbb{C}P^{2m+1}$.

An argument similar to the one used in [9, page 65] shows that the nonhomogeneous components of $\Phi$ divided by suitable linear functions are constant along the twistor lines. Hence, by Lemma 4.6, these meromorphic functions are constant.

We have thus proved that $\Phi$ is induced by some complex linear map from $\mathbb{C}^{2m+2}$ to $\mathbb{C}^{2n+2}$. Then the proof (under the assumption that the differential of $\varphi$ is nowhere zero) follows from Lemma 4.7.

Finally, note that, if an $\mathbb{H}$-linear map $A : \mathbb{H}^{m+1} \to \mathbb{H}^{n+1}$ induces a nonconstant (quaternionic) map $\varphi^A : \mathbb{H}P^m \setminus P_{\mathbb{H}}(\ker A) \to \mathbb{H}P^n$ then its real rank is, at least, eight. Hence, at each point, the rank of the differential of $\varphi^A$ is at least four.
It follows quickly that, the differential of any nonconstant quaternionic map, between open sets of quaternionic projective spaces, is nowhere zero. The proof is complete.

**Remark 4.9.** Theorem 4.5 can be, also, proved by applying an inductive argument, based on a result of [14], to show that in terms of non-homogeneous quaternionic projective coordinates \((x_j)_{j=1,...,m}\), and \((y_\alpha)_{\alpha=1,...,n}\), on \(\mathbb{H}P^m\) and \(\mathbb{H}P^n\), respectively, any quaternionic map \(\varphi : \mathbb{H}P^m \to \mathbb{H}P^n\) is given by

\[
y_\alpha = (x_j a^j_0 + a^0_\alpha)^{-1}(x_j a^j_\alpha + a^0_\alpha), \quad (\alpha = 1, \ldots, n),
\]

where the coefficients \(a^j_\alpha\) are constant quaternions (and the Einstein summation convention is used).

We end this section with the following immediate consequence of Theorem 4.5.

**Corollary 4.10.** Any (globally defined) quaternionic map from \(\mathbb{H}P^m\) to \(\mathbb{H}P^n\) is induced by an injective \(\mathbb{H}\)-linear map \(\mathbb{H}^{m+1} \to \mathbb{H}^{n+1}\); in particular, \(m \leq n\).

**Appendix A. Comparison with other notions of quaternionicity**

Firstly, we mention the fairly standard notion of ‘hypercomplex (triholomorphic) map’ between almost hypercomplex manifolds. Obviously, any such map is quaternionic with respect to the induced almost quaternionic structures on its domain and codomain.

Secondly, there have been studied maps, between quaternionic Kähler manifolds \(M\) and \(N\), which pull-back Kähler forms of elements of \(Z_N\) to (the space of) Kähler forms of elements of \(Z_M\) (see [12]). As the Kähler forms are nondegenerate, this condition applies only to immersions and to constant maps, and, therefore, it is too restrictive (also, the presence of a Riemannian metric is required).

Thirdly, there exists the notion of ‘regular quaternionic function’, of one quaternionic variable, introduced in [8] (see [19] for a modern presentation and further results) and later generalised to maps between hyper-Kähler manifolds (see [6], [10] and the references therein):

**Definition A.1.** Let \(V\) and \(W\) be quaternionic vector spaces and let \(T : Z_V \to Z_W\) be an orientation preserving isometry.

We say that a map \(t : V \to W\) is linear Fueter-quaternionic, with respect to \(T\), if \(t\) is real linear and for some (and, consequently, any) positive orthonormal basis \((I,J,K)\) of \(Q_V\) we have \(t = T(I) \circ t \circ I + T(J) \circ t \circ J + T(K) \circ t \circ K\).

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With the same notations as in Definition A.1, let \( C_T \) be the endomorphism defined by \( C_T(t) = T(I) \circ t \circ I + T(J) \circ t \circ J + T(K) \circ t \circ K \), \( t \in \text{Hom}_R(V,W) \), where \( (I,J,K) \) is a positive orthonormal basis of \( Q_V \). A straightforward calculation shows that \( C_T \) does not depend on the positive orthonormal basis \( (I,J,K) \) and, in particular, the notion of ‘Fueter-quaternionic map’, between almost quaternionic manifolds, is well-defined.

Furthermore, \( C_T \) satisfies the equation \( (C_T)^2 + 2C_T - 3 = 0 \). Let \( \mathscr{F}_T \) and \( \mathscr{Q}_T \) be the eigenspaces of \( C_T \) corresponding to 1 and \(-3\), respectively. Then \( \text{Hom}_R(V,W) = \mathscr{Q}_T \oplus \mathscr{F}_T \) and \( \mathscr{F}_T \) is the space of linear Fueter-quaternionic maps, with respect to \( T \), whilst \( \mathscr{Q}_T \) is the space of linear quaternionic maps, with respect to \( T \) \[^{10}\]. Apparently, this would suggest that Fueter-quaternionic maps are ‘anti-quaternionic’. In fact, by reformulating results mentioned in \[^{10}\] and \[^{10}\], the following proposition can be easily obtained.

**Proposition A.2.** Let \( V \) and \( W \) be quaternionic vector spaces and let \( T \) be an orientation preserving isometry from \( Z_V \) to \( Z_W \).

Then, for any line through the origin \( d \subseteq Q_V \), we have \( \mathscr{Q}_{T \circ S_d} \subseteq \mathscr{F}_T \), where \( S_d \) is the symmetry in \( d \). Moreover, \( \mathscr{F}_T \) is generated by \( \bigcup_d \mathscr{Q}_{T \circ S_d} \).

Finally, let \( U, V \) and \( W \) be quaternionic vector spaces and let \( T' : Z_U \to Z_V \) and \( T'' : Z_V \to Z_W \) be orientation preserving isometries. Also, let \( d \subseteq Q_V \) be a line through the origin. If \( t' : U \to V \) and \( t'' : V \to W \) are linear quaternionic maps, with respect to \( S_d \circ T' \) and \( T'' \circ S_d \), respectively, then, by Proposition A.2, \( t' \) and \( t'' \) are Fueter-quaternionic, with respect to \( T' \) and \( T'' \), respectively. However, \( t'' \circ t' \) is Fueter-quaternionic, with respect to \( T'' \circ T' \), if and only if \( t'' \circ t' = 0 \).

**References**


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