HOLOMORPHIC MAPS BETWEEN GENERALIZED COMPLEX MANIFOLDS

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Abstract

We introduce a natural notion of holomorphic map between generalized complex manifolds and we prove some related results on Dirac structures and generalized Kähler manifolds.

Introduction

The generalized complex structures [9], [11] contain, as particular cases, the complex and symplectic structures. Although for the latter structures there exist well known definitions which give the corresponding morphisms (holomorphic maps and Poisson morphisms, respectively), it still lacks a suitable notion of holomorphic map with respect to which the class of generalized complex manifolds to become a category.

The aim of this paper is to introduce such a notion (Definition 4.3, below) based on the following considerations. Firstly, holomorphic maps between generalized complex manifolds should be invariant under $B$-field transformations. This is imposed by the fact that the group of (orthogonal) automorphisms of the Courant bracket (which defines the integrability in Generalized Complex Geometry) on a manifold is the semidirect product of the group of diffeomorphisms and the additive group of closed two-forms on the manifold [9]. Secondly, by [9], underlying any linear generalized complex structure there are a linear Poisson structure and a linear co-CR structure (that is, a linear CR structure on the dual vector space; see Section 3), which are preserved under linear $B$-field transformations. Moreover, these two structures determine, up to a (non-unique) linear $B$-field transformation, the given generalized linear complex structure; furthermore, if we choose a compatible linear $f$-structure (Definition 3.1) then there exists a distinguished linear $B$-field transformation with this property (see Proposition 3.3). It follows

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that a linear map is \textit{generalized complex} (Definition 3.4) if and only if, up to linear $B$-fields transformations, it is an $f$-linear Poisson morphism between linear generalized complex structures in normal form (Proposition 3.5). Now, to obtain the definition of holomorphic maps, between generalized complex manifolds, one just have to note that, essentially, all of the above mentioned (linear) facts hold, locally, on any regular generalized complex manifold (see Theorem 4.2).

The paper is organized as follows. In Section 1, after recalling [7] some basic facts on linear Dirac structures, we give explicit descriptions (Proposition 1.3) for the pull-back and push-forward of a linear Dirac structure, which we then use to show that any linear Dirac structure is, in a natural way, the pull-back of a linear Poisson structure (Corollary 1.5; cf. [5], [6]), which we call the \textit{canonical (linear) Poisson quotient} (cf. [6]), of the given linear Dirac structure. The smooth version (Theorem 2.3; cf. [7], [5], [6]) of this result is proved in Section 2 together with some other results on Dirac structures. For example, there we show (Corollary 2.5) that, locally, any regular Dirac structure is, up to a $B$-field transformation, of the form $\mathcal{V} \oplus \text{Ann}(\mathcal{V})$, where $\mathcal{V}$ is (the tangent bundle of) a foliation.

In Section 3, we introduce the notion of generalized complex linear map, along the above mentioned lines. It follows that two generalized linear complex structures $L_1$ and $L_2$, on a vector space $V$, can be identified if and only if $L_2$ is the linear $B$-field transform of the push-forward of $L_1$, through a linear isomorphism of $V$ (Corollary 3.6). Also, we explain (Remark 3.7) why another definition of the notion of generalized complex linear map is, in our opinion, inadequate.

In Section 4, we review some basic facts on generalized complex manifolds and we introduce the corresponding notion of holomorphic map. It follows that if a real analytic map $\varphi$, between real analytic regular generalized complex manifolds, is holomorphic then, locally, up to the complexification of a real analytic $B$-field transformation, the complexification of $\varphi$ descends to a complex analytic Poisson morphism between canonical Poisson quotients (Proposition 4.5). Also, we show that the pseudo-horizontally conformal submersions with minimal two-dimensional fibres, from Riemannian manifolds, provide natural constructions of generalized complex structures (Example 4.6).

In Section 5, we prove (Theorem 5.3) that if $(g,b,J_+,J_-)$ is the (special) bi-Hermitian structure corresponding to a generalized Kähler structure and we denote $\mathcal{H}^\pm = \ker(J_+ \mp J_-)$ then the following conditions are equivalent:

- $\mathcal{H}^\pm$ integrable;
- $\mathcal{H}^\pm$ geodesic;
- $\mathcal{H}^\pm$ holomorphic, with respect to $J_+$ or $J_-$. 

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It follows that, under natural conditions, the holomorphic maps between generalized Kähler manifolds descend to holomorphic maps between Kähler manifolds (Remark 5.6). Also, we classify the generalized Kähler manifolds \( M \) for which \( TM = \mathcal{H}^+ \oplus \mathcal{H}^- \) (Corollary 5.7; cf. [2, Theorem A]).

In Section 6, we describe, in terms of tamed symplectic manifolds (see Definition 6.1) the generalized Kähler manifolds for which either \( \mathcal{H}^+ \) or \( \mathcal{H}^- \) is zero; the obtained result (Theorem 6.2) also appears, in a different form, in [10]. Also, in Corollary 6.3, we prove a factorisation result for generalized Kähler manifolds with \( \mathcal{H}^+ \) an integrable distribution and \( \mathcal{H}^- = 0 \) (or \( \mathcal{H}^+ = 0 \) and \( \mathcal{H}^- \) an integrable distribution); see, also, Corollary 6.4 for a similar result and Theorem 6.10 for a generalization. Furthermore, we show how the associated holomorphic Poisson structures of [12] fit into our approach (Theorem 6.5, Remark 6.6) and we deduce some consequences for holomorphic maps (Corollaries 6.7 and 6.8).

1. Linear Dirac structures

In this section we recall ([7]; see [5], [6], [9]) some basic facts on linear Dirac structures.

Let \( V \) be a (real or complex, finite dimensional) vector space. The symmetric bilinear form \(< \cdot, \cdot >\) on \( V \oplus V^* \) defined by
\[
< u + \alpha, v + \beta > = \frac{1}{2} (\alpha(v) + \beta(u)),
\]
for any \( u + \alpha, v + \beta \in V \oplus V^* \), corresponds, up to the factor \( \frac{1}{2} \), to the canonical isomorphism \( V \oplus V^* \cong (V \oplus V^*)^\vee \) defined by \( u + \alpha \mapsto \alpha + u \), for any \( u + \alpha \in V \oplus V^* \). In particular, \(< \cdot, \cdot >\) is nondegenerate and, if \( V \) is real, its index is \( \dim V \). Thus, the dimension of the maximal isotropic subspaces of \( V \oplus V^* \) (endowed with \(< \cdot, \cdot >\)) is equal to \( \dim V \).

**Definition 1.1 ([7])**. A linear Dirac structure on \( V \) is a maximal isotropic subspace of \( V \oplus V^* \).

If \( b \) is a bilinear form on \( V \) then we shall denote by the same letter the corresponding linear map from \( V \) to \( V^* \); thus, \( b(u)(v) = b(u, v) \), for any \( u, v \in V \).

Let \( E \subseteq V \) be a vector subspace and let \( \varepsilon \in \Lambda^2 E^* \); denote
\[
L(E, \varepsilon) = \{ u + \alpha \mid u \in E, \alpha|_E = \varepsilon(u) \}.
\]
From the fact that \( \varepsilon \) is skew-symmetric it follows easily that \( L(E, \varepsilon) \) is isotropic. Also, \( L(E, 0) = E \oplus \text{Ann}(E) \), where \( \text{Ann}(E) = \{ \alpha \in V^* \mid \alpha|_E = 0 \} \).

We shall denote by \( \pi \) and \( ^\ast \pi \) the projections from \( V \oplus V^* \) onto \( V \) and \( V^* \),
respectively. Also, if \( L \subseteq V \oplus V^* \) then \( L^\perp \) denotes the ‘orthogonal complement’ of \( L \) with respect to \( \langle \cdot, \cdot \rangle \).

**Proposition 1.2 ([7]).** Let \( L \) be an isotropic subspace of \( V \oplus V^* \) and let \( E = \pi(L) \).

Then there exists a unique \( \varepsilon \in \Lambda^2 E^\ast \) such that \( L \subseteq L(E, \varepsilon) \). In particular, if \( L \) is a linear Dirac structure then \( L = L(E, \varepsilon) \). Furthermore, \( V \cap L = \ker \varepsilon \) and \( ^*\pi(L) = \text{Ann}(V \cap L) \).

Let \( L \) be a linear Dirac structure on \( V \). If \( ^*\pi(L) = V^* \) then \( L \) is called a linear Poisson structure (see [7]). By Proposition 1.2, if \( L \) is a linear Poisson structure then \( L = L(V^*, \eta) \) for some bivector \( \eta \in \Lambda^2 V^* \) (cf. [19]).

Let \( V \) and \( W \) be vector spaces endowed with linear Dirac structures \( L_V \) and \( L_W \), respectively, and let \( f : V \to W \) be a linear map. Denote
\[
 f_*(L_V) = \{ f(X) + \eta \mid X + f^*(\eta) \in L_V \}, \\
f^*(L_W) = \{ X + f^*(\eta) \mid f(X) + \eta \in L_W \}.
\]

**Proposition 1.3.** Let \( f : V \to W \) be a linear map. Let \( L(E, \varepsilon) \) and \( L(F, \eta) \) be linear Dirac structures on \( V \) and \( W \), respectively. Then
\[
 f_*(L(E, \varepsilon)) = L(f((E \cap \ker f)^{\perp \varepsilon}), \varepsilon) , \\
f^*(L(F, \eta)) = L(f^{-1}(F), f^*(\eta)) ,
\]
where \( \varepsilon \) is characterised by \( f^*(\varepsilon) = \varepsilon \) on \( (E \cap \ker f)^{\perp \varepsilon} \).

**Proof.** It is easy to prove that \( f_*(L_V) \) and \( f^*(L_W) \) are isotropic subspaces of \( W \oplus W^* \) and \( V \oplus V^* \), respectively.

Next, we show that there exists a unique two-form \( \varepsilon \) on \( f((E \cap \ker f)^{\perp \varepsilon}) \) such that \( f^*(\varepsilon) = \varepsilon \) on \( (E \cap \ker f)^{\perp \varepsilon} \). For this, it is sufficient to prove that if \( X_1, X_2 \in (E \cap \ker f)^{\perp \varepsilon} \) are such that \( f(X_1) = f(X_2) \) then \( \varepsilon(X_1, Y) = \varepsilon(X_2, Y) \), for any \( Y \in (E \cap \ker f)^{\perp \varepsilon} \). Now, if \( X_1, X_2 \in (E \cap \ker f)^{\perp \varepsilon} \), then \( X_1, X_2 \in E \) and, as \( X_1 - X_2 \in \ker f \), we have \( \varepsilon(X_1 - X_2, Y) = 0 \), for any \( Y \in (E \cap \ker f)^{\perp \varepsilon} \).

Thus, to complete the proof it is sufficient to show that
\[
 f_*(L(E, \varepsilon)) \supseteq L(f((E \cap \ker f)^{\perp \varepsilon}), \varepsilon) , \\
f^*(L(F, \eta)) \supseteq L(f^{-1}(F), f^*(\eta)) .
\]

Let \( Y + \xi \in L(f((E \cap \ker f)^{\perp \varepsilon}), \varepsilon) \); equivalently, there exists \( X \in (E \cap \ker f)^{\perp \varepsilon} \) such that \( f(X) = Y \) and \( \xi(f(X')) = \varepsilon(f(X), f(X')) \), for any \( X' \in (E \cap \ker f)^{\perp \varepsilon} \).

We claim that \( Y + \xi \in f_*(L(E, \varepsilon)) \); equivalently, there exists \( X \in (E \cap \ker f)^{\perp \varepsilon} \) such that \( f(X) = Y \) and \( \xi(f(X')) = \varepsilon(X, X') \), for any \( X' \in E \).
It is easy to prove that, if \( X \in (E \cap \ker f) \perp \varepsilon \) is such that \( f(X) = Y \), then \( \xi(f(X')) = \varepsilon(X, X') \), for any \( X' \in (E \cap \ker f) \cup (E \cap \ker f) \perp \varepsilon \).

It follows that, for any \( X \in (E \cap \ker f) \perp \varepsilon \) with \( f(X) = Y \), there exists \( X_1 \in \ker(\varepsilon|_{E \cap \ker f}) \) such that \( f(X_1) = X \), for any \( X' \in E \); as, then, we also have \( X_1 \in (E \cap \ker f) \perp \varepsilon \) and \( f(X_1) = 0 \), this shows that \( \varepsilon = \varepsilon(X + X_1, X') \), for any \( X' \in E \);

To prove the second relation of (1.1), let \( X + \xi \in L(f^{-1}(F), f^*(\eta)) \); equivalently, \( f(X) \in F \) and \( \xi(X') = \eta(f(X), f(X')) \) for any \( X' \in f^{-1}(F) \). As \( f^{-1}(F) \supseteq \ker f \), there exists \( \tilde{\xi} \) in the dual of \( f(V) \) such that \( \tilde{\xi} = f^*(\tilde{\eta}) \). Obviously, we can extend \( \tilde{\xi} \) to an one-form on \( W \), which we shall denote by the same symbol \( \tilde{\xi} \), such that \( \tilde{\xi}(Y) = \eta(f(X), Y) \), for any \( Y \in F \); equivalently, \( f(X) + \xi \in L(F, \eta) \). Therefore \( X + \xi = X + f^*(\tilde{\eta}) \in f^*(L(F, \eta)) \).

The proof is complete. \( \Box \)

**Definition 1.4** (see \[5\], \[6\], \[9\]). Let \( V \) and \( W \) be vector spaces endowed with linear Dirac structures \( L_V \) and \( L_W \), respectively, and let \( f : V \to W \) be a linear map. Then \( f_*(L_V) \) and \( f^*(L_W) \) are called the push forward and pull back, by \( f \), of \( L_V \) and \( L_W \), respectively.

Note that, if \( f : (V, L_V) \to (W, L_W) \) is a linear map between vector spaces endowed with linear Poisson structures then the following assertions are equivalent (see \[5\], \[6\]):

(i) \( f \) is a linear Poisson morphism (that is, \( f(\eta_V) = \eta_W \), where \( \eta_V \) and \( \eta_W \) are the bivectors defining \( L_V \) and \( L_W \), respectively; see \[16\]).

(ii) \( f_*(L_V) = L_W \).

From Proposition 1.3, we easily obtain the following result.

**Corollary 1.5** (cf. \[5\], \[6\]). Let \( V \) be a vector space endowed with a linear Dirac structure \( L = L(E, \varepsilon) \). Let \( W = \ker \varepsilon \) and denote by \( \varphi : V \to V/W \) the projection.

Then \( L = \varphi^*(\varphi_*(L)) \) and \( \varphi_*(L) \) is a linear Poisson structure on \( V/W \).

2. Dirac structures

In this section, we shall work in the smooth and (real or complex) analytic categories. All the notations of Section 1 will be applied to tangent bundles of manifolds and to (differentials of) maps between manifolds.
Definition 2.1 ([7]). An almost Dirac structure on a manifold \( M \) is a maximal isotropic subbundle of \( TM \oplus T^*M \).

An almost Dirac structure is integrable if its space of sections is closed under the Courant bracket defined by

\[
[X + \alpha, Y + \beta] = [X, Y] + \frac{1}{2} d(\iota_X \beta - \iota_Y \alpha) + \iota_X d\beta - \iota_Y d\alpha,
\]

for any sections \( X + \alpha \) and \( Y + \beta \) of \( TM \oplus \Lambda(T^*M) \), where \( \iota \) denotes the interior product.

A Dirac structure is an integrable almost Dirac structure.

Let \( L \) be a Dirac structure on \( M \). If \( \pi(L) = TM \) then \( L \) is a presymplectic structure whilst if \( *\pi(L) = T^*M \) then \( L \) is a Poisson structure ([7] (cf. [19]).

Recall [7, §4] that a point of a manifold endowed with an almost Dirac structure \( L \) is called regular if, in some open neighbourhood of it, \( \pi(L) \) and \( *\pi(L) \) are bundles.

The following result follows from the fact that it is sufficient to be proved for maps of constant rank between manifolds endowed with regular almost Dirac structures.

Proposition 2.2. Let \( M \) and \( N \) be manifolds endowed with the almost Dirac structures \( L_M \) and \( L_N \), respectively. Let \( \varphi : M \to N \) be a map which maps regular points of \( L_M \) to regular points of \( L_N \).

(i) If \( L_M \) is integrable, \( \varphi^*(L_M) = L_N \) and \( \varphi \) is surjective then \( L_N \) is integrable.

(ii) If \( L_N \) is integrable and \( \varphi^*(L_N) = L_M \) then \( L_M \) is integrable.

Next, we prove the following result.

Theorem 2.3 (cf. [7], [5], [6] ). Let \( L \) be a Dirac structure on \( M \) such that \( *\pi(L) \) is a subbundle of \( T^*M \). Then, locally, there exist submersions \( \varphi \) on \( M \) such that \( \varphi_*L \) is a Poisson structure and \( L = \varphi^*(\varphi_*(L)) \); moreover, these submersions are (germ) unique, up to Poisson diffeomorphisms of their codomains.

Proof. By hypothesis, \( TM \cap L \) is a subbundle of \( TM \). Furthermore, as \( L \) is integrable, \( TM \cap L \) is (the tangent bundle to) a foliation.

Let \( F = *\pi(L) \) and let \( \eta \) be the section of \( \Lambda^2 F^* \) such that \( L = L(F, \eta) \). Note that, \( F(= \text{Ann}(TM \cap L)) \) is locally spanned by the differentials of functions which are basic with respect to \( TM \cap L \).

Let \( f \) and \( g \) be functions, locally defined on \( M \), such that \( df \) and \( dg \) are sections of \( F \). Then there exists vector fields \( X \) and \( Y \), locally defined on \( M \), such that \( X + df \) and \( Y + dg \) are local sections of \( L \); in particular, we have \( \eta(df, dg) = X(g) = -Y(f) \). Hence \( [X + df, Y + dg] = [X, Y] + d(\eta(df, dg)) \) and
we deduce that $\eta(df, dg)$ is basic with respect to $TM \cap L$.

The proof follows quickly from Corollary 1.5 and Proposition 2.2.

Under the same hypotheses, as in Theorem 2.3, we call $\varphi_*(L)$ the canonical (local) Poisson quotient of $L$.

Next, we prove the following (cf. [7, Proposition 4.1.2]).

**Proposition 2.4.** Let $L = L(E, \varepsilon)$ be a Dirac structure on $M$ and let $x \in M$ be a regular point of $L$; denote by $P$ the leaf of $E$ through $x$.

Then for any submanifold $Q$ of $M$ transversal to $E$ and such that $\dim Q = \dim M - \dim P$ there exists a submersion $\rho$ from some open neighbourhood $U$ of $x$ in $M$ onto some open neighbourhood $V$ of $x$ in $P$ such that $\rho_* (L|_U) = L(TV, \varepsilon|_V)$ and the fibre of $\rho$ through $x$ is an open set of $Q$.

**Proof.** From Theorem 2.3 it follows that we may assume $L$ a Poisson structure.

If we ignore the fact that the fibre of $\rho$ through $x$ is fixed then the proposition is a consequence of [19, Corollary 2.3] and Proposition 1.3. To complete the proof just note that in the proof of [19, Theorem 2.1] (and, consequently, of [19, Corollary 2.3]), as well), at each step, the two functions involved may be assumed constant along $Q$.

Recall (see [9], [5]) that any closed two-form $B$ on $M$ corresponds to a $B$-field transformation which is the automorphism of $TM \oplus T^*M$, preserving the Courant bracket, defined by

$$\exp(B)(X + \alpha) = X + B(X) + \alpha$$

for any $X + \alpha \in TM \oplus T^*M$, where, as before, we have identified $B$ with the corresponding section of $\text{Hom}(TM, T^*M)$.

It is easy to prove that if $L = L(E, \varepsilon)$ is an almost Dirac structure on $M$ then $\exp(B)(L) = L(E, \varepsilon + B|_E)$.

**Corollary 2.5.** Let $L$ be a regular Dirac structure on $M$; denote $E = \pi(L)$. Then, locally, there exist two-forms $B$ on $M$ such that $\exp(B)(L) = E \oplus \text{Ann} E$.

**Proof.** By Proposition 2.4, locally, there exist submersions $\rho : M \to P$ onto presymplectic manifolds $(P, L(TP, \omega))$ such that $\rho_*(L) = L(TP, \omega)$.

Then $B = -\rho^*(\omega)$ is as required.

3. **Generalized complex linear maps**

Let $V$ be a (real) vector space. A linear CR structure on $V$ is a complex vector subspace $C$ of $V^C$ such that $C \cap \overline{C} = \{0\}$.

Dually, a linear co-CR structure on $V$ is a complex vector subspace $C$ of $V^C$
such that $C + \overline{C} = V_C$.

A complex vector subspace of $V_C$ is a linear co-CR structure if and only if its annihilator is a linear CR structure.

Note that, the eigenspaces of a linear complex structure are both linear CR and co-CR structures.

A linear $f$-structure on $M$ is an endomorphism $F$ of $V$ such that $F^3 + F = 0$. Any linear $f$-structure corresponds to a pair formed of a linear CR structure $C$ and a linear co-CR structure $D$, which are compatible \[13\]; these are given by $C = V^{1,0}$ and $D = V^0 \oplus V^{1,0}$, where $V^0$ and $V^{1,0}$ are the eigenspaces of $F$ corresponding to 0 and $i$, respectively.

Note that, a linear map $t : (V, F_V) \to (W, F_W)$ between vector spaces endowed with linear $f$-structures satisfies $t \circ F_V = F_W \circ t$ if and only if $t(C_V) \subseteq C_W$ and $t(D_V) \subseteq D_W$, where $C_V$ and $D_V$ ($C_W$ and $D_W$) are the linear CR and co-CR structures, respectively, corresponding to $F_V$ ($F_W$); equivalently, $t$ is $f$-linear if and only if it is CR linear and co-CR linear \[13\].

A linear generalized complex structure on $V$ is a maximal isotropic subspace $L = L(E, \varepsilon)$ of $V_C \oplus (V_C)^*$ such that $L \cap \overline{L} = \{0\}$ \[9\], \[11\]; equivalently, $E$ is a linear co-CR structure and $\text{Im}(\varepsilon|_{E \cap \overline{E}})$ is nondegenerate \[9\].

If $L = L(E, \varepsilon)$ is a linear generalized complex structure then we call $E$ and $L(E \cap \overline{E}, \text{Im}(\varepsilon|_{E \cap \overline{E}}))$ the associated linear co-CR and Poisson structures, respectively.

**Definition 3.1.** A linear $f$-structure $F$ and a two-form $\omega$ on $V$ are compatible if $\omega|_{V^0}$ is nondegenerate and $\ker \omega = V^{1,0} \oplus V^{0,1}$.

We say that a linear generalized complex structure $L$ on $V$ is in normal form if there exist a linear $f$-structure on $V$ and a compatible two-form $\omega$ with respect to which $L = L(V^0 \oplus V^{1,0}, i \omega)$.

**Remark 3.2.** 1) Let $L$ be a linear generalized complex structure on $V$. Denote by $J$ the linear complex structure on $V \oplus V^*$ whose eigenspace corresponding to $i$ is $L$. Then the bivector corresponding to the linear Poisson structure associated to $L$ is $\pi \circ (J|_{V^*})$ \[1\].

2) Let $L = L(V^0 \oplus V^{1,0}, i \omega)$ be a linear generalized complex structure in normal form, determined by the compatible linear $f$-structure $F$ and two-form $\omega$. Then the linear Poisson structure associated to $L$ is $L(V^0, \omega)$; denote by $\eta$ the corresponding bivector. Furthermore, if $J$ is the linear complex structure on $V \oplus V^*$ whose eigenspace corresponding to $i$ is $L$, then (cf. \[9\])

\[
J = \begin{pmatrix} F & \eta \\ -\omega & -F^* \end{pmatrix},
\]
where \((\cdot)^*\) denotes the transposition.

In the terminology of [9], we have that \(L\) is the \textit{product} of a complex vector space and a symplectic vector space. However, in the smooth category, the corresponding two notions are no longer equivalent.

The next result (which reformulates [9, Theorem 4.13]) shows that any linear generalized complex structure is determined, up to a linear \(B\)-field transformation, by its associated linear co-CR and Poisson structures.

**Proposition 3.3.** Let \(L\) be a linear generalized complex structure on \(V\) and let \(F\) be a linear \(f\)-structure on \(V\) such that \(\pi(L)\) is the linear co-CR structure associated to \(F\).

Then there exists a unique \(B \in \Lambda^2 V^*\) such that \((\exp B)(L)\) is in normal form, with \(F\) the corresponding linear \(f\)-structure.

**Proof.** If \(L = L(V^0 \oplus V^{1,0}, \varepsilon)\) then \(B\) is characterised by the relations \(B = -\text{Re} \varepsilon\), on \(V^0\), and \(B = -\varepsilon\), on \(V^{1,0}\) and \(V^0 \otimes V^{1,0}\). \(\square\)

Next, we make the following:

**Definition 3.4.** A linear map \(t : V \to W\), between vector spaces endowed with linear generalized complex structures \(L_V\) and \(L_W\), respectively, is \textit{generalized complex linear} if it is a co-CR linear Poisson morphism, with respect to the associated linear co-CR and Poisson structures.

Note that, Definition 3.4 is invariant under linear \(B\)-field transformations.

**Proposition 3.5.** Let \(t : V \to W\) be a linear map between vector spaces endowed with linear generalized complex structures \(L_V\) and \(L_W\), respectively.

Then the following assertions are equivalent:

(i) \(t\) is generalized complex linear.

(ii) Up to linear \(B\)-field transformations, \(L_V\) and \(L_W\) are in normal form and \(t\) is an \(f\)-linear Poisson morphism, with respect to the corresponding linear \(f\)-structures and Poisson structures, on \((V, L_V)\) and \((W, L_W)\).

(iii) Up to linear \(B\)-field transformations, \(t\) is the direct sum of a linear Poisson morphism, between symplectic vector spaces, and a complex linear map, between complex vector spaces.

**Proof.** From Proposition 1.3 it follows that it is sufficient to prove (i) \(\Rightarrow\) (ii).

Furthermore, if (i) holds then \(t(E_V \cap \overline{E_V}) = E_W \cap \overline{E_W}\). Hence,

\[
t^{-1}(E_W \cap \overline{E_W}) = \ker t + (E_V \cap \overline{E_V})
\]
and, consequently, there exist complementary vector spaces $V'$ and $W'$ of $E_V \cap \overline{E_V}$ and $E_W \cap \overline{E_W}$ in $V$ and $W$, respectively, such that $t(V') \subseteq W'$.

Then $(i) \implies (ii)$ and $(ii) \implies (iii)$ follow from Propositions 3.3 and 1.3, respectively, whilst $(iii) \implies (i)$ is trivial. □

Note that, by using Remark 3.2(2), assertion (ii) of Proposition 3.5 can be formulated in terms of the corresponding linear complex structures of $V \oplus V^*$ and $W \oplus W^*$.

The next result is an immediate consequence of Proposition 3.5.

**Corollary 3.6.** Let $t : V \to W$ be a linear isomorphism between vector spaces endowed with linear generalized complex structures $L_V$ and $L_W$, respectively.

Then the following assertions are equivalent:

(i) $t$ is generalized complex linear.

(ii) $t^\ast(L_V) = L_W$, up to linear $B$-field transformations.

We end this section with the following:

**Remark 3.7.** It has been proposed another definition for the notion of generalized complex linear map by imposing that the product of the graphs of the map and of its transpose be invariant under the product of the (endomorphisms corresponding to the) generalized linear complex structures, of the domain and codomain [8] (see [17]).

However this notion is not invariant under linear $B$-field transformations as we shall now explain.

Let $(V, J)$ be a complex vector space and let $b$ be a two-form on $V$; denote by $L_J$ the linear generalized complex structure corresponding to $J$. Then the map $\text{Id}_V : (V, L_J) \to (V, L_{(\exp b)(L_J)})$ satisfies the above mentioned condition if and only if $b$ is of type $(1, 1)$, with respect to $J$.

4. **Holomorphic maps between generalized complex manifolds**

From now on, unless otherwise stated, all the manifolds are assumed connected and smooth and all the maps are assumed smooth.

An almost (co-)CR structure on a manifold $M$ is a complex vector subbundle $C$ of $T^CM$ such that $C_x$ is a linear (co-)CR structure on $T_xM$, for any $x \in M$. An integrable almost (co-)CR structure is an almost (co-)CR structure whose space of sections is closed under the (Lie) bracket. A (co-)CR structure is an integrable almost (co-)CR structure.

Note that, the eigenbundles of a complex structure are both CR and co-CR structures.
Let $\varphi : M \to N$ be a submersion onto a complex manifold $(N, J)$; denote by $T^{1,0}N$ the eigenbundle of $J$ corresponding to $i$. Then $d\varphi^{-1}(T^{1,0}N)$ is a co-CR structure on $M$. Conversely, any co-CR structure is, locally, obtained this way.

An almost $f$-structure is a $(1, 1)$-tensor field $F$ such that $F^3 + F = 0$. Any almost $f$-structure on $M$ corresponds to a pair formed of an almost CR structure $\mathcal{C}$ and an almost co-CR structure $\mathcal{D}$, which are compatible \cite{13}; these are given by $\mathcal{C} = T^{1,0}M$ and $\mathcal{D} = T^0M \oplus T^{1,0}M$, where $T^0M$ and $T^{1,0}M$ are the eigenbundles of $F$ corresponding to 0 and $i$, respectively.

An almost $f$-structure is (co-)CR integrable if the associated almost (co-)CR structure is integrable. An (integrable almost) $f$-structure is an almost $f$-structure which is both CR and co-CR integrable \cite{13}.

A map between manifolds endowed with almost (co-)CR structures ($f$-structures) is (co-)CR holomorphic ($f$-holomorphic) if, at each point, its differential is (co-)CR linear ($f$-linear).

A generalized almost complex structure on $M$ is a complex vector subbundle $L$ of $T^CM \oplus (T^CM)^\ast$ such that $L_x$ is a linear generalized complex structure on $T_xM$, for any $x \in M$. An integrable generalized complex structure is a generalized almost complex structure whose space of sections is closed under the (complexification of the) Courant bracket; a generalized (almost) complex manifold is a manifold endowed with a generalized (almost) complex structure \cite{9, 11}.

A point $x$ of a generalized almost complex manifold $(M, L)$ is regular if it is regular for the associated almost Poisson structure; equivalently, in some open neighbourhood of $x$, $\pi(L)$ is a complex vector subbundle of $T^CM$ (note that, then $\pi(L)$ is an almost co-CR structure on $M$).

An almost $f$-structure $F$ and a two form $\omega$ on $M$ are compatible if $\omega$ is nondegenerate on $T^0M$ and $i_X\omega = 0$, for any $X \in T^{1,0}M \oplus T^{0,1}M$.

A generalized (almost) complex structure $L$ on $M$ is in normal form if $L = L(T^0M \oplus T^{0,1}M, i\omega)$ for some compatible almost $f$-structure and two-form $\omega$ on $M$. Note that, a generalized almost complex structure in normal form is regular.

**Proposition 4.1.** Let $L = L(T^0M \oplus T^{1,0}M, i\omega)$ be the generalized almost complex structure in normal form, corresponding to the compatible almost $f$-structure $F$ and two-form $\omega$ on $M$.

Then the following assertions are equivalent:

(i) $L$ is integrable.

(ii) $F$ is integrable, $L(T^0M, \omega)$ is a Poisson structure and $\omega$ is invariant under the parallel displacement of $T^{1,0}M \oplus T^{0,1}M$. 
Proof. From \cite[Proposition 4.19]{9} it follows quickly that assertion (i) is equivalent to the fact that $F$ is co-CR integrable and $(d\omega)|_{T^0 M @ \pi T^1.0 M} = 0$. Assuming $F$ co-CR integrable, the latter condition is equivalent to the fact that $L(T^0 M, \omega)$ is a Poisson structure, $F$ is CR integrable and $(L_X \omega)|_{T^0 M} = 0$ for any vector field $X$ tangent to $T^{1,0} M \oplus T^{0,1} M$, where $L$ denotes the Lie derivative. \hfill \Box

The next result follows from the proof of \cite[Theorem 4.35]{9}.

**Theorem 4.2.** Let $L$ be a regular generalized almost complex structure on $M$ and let $L'$ be the associated almost Poisson structure.

Then the following assertions are equivalent:

(i) $L$ is integrable.

(ii) $\pi(L)$ and $L'$ are integrable and, locally, for any submersion $\rho: M \to P$, with $\dim P = \text{rank}(\rho(L'))$ and $\rho_*(L')$ a symplectic structure on $P$, we have that, up to a B-field transformation, $L$ is in normal form with respect to the $f$-structure on $M$ determined by $\pi(L)$ and $\pi(L) \cap \text{ker} d\rho$.

We can now formulate the notion of holomorphic map between generalized complex manifolds.

**Definition 4.3.** 1) A map between generalized almost complex manifolds is almost holomorphic if, at each point, its differential is generalized complex linear.

2) A map between generalized complex manifolds is holomorphic if, on an open neighbourhood of each regular point of its domain, up to B-field transformations, it is an $f$-holomorphic Poisson morphism between generalized complex manifolds in normal form.

The following facts are immediate consequences of the definitions.

**Remark 4.4.** 1) A map between regular generalized almost complex manifolds is almost holomorphic if and only if it is a co-CR Poisson morphism, with respect to the associated almost co-CR and Poisson structures.

2) Any holomorphic map, between generalized complex manifolds, is almost holomorphic.

3) Any holomorphic map, between generalized complex manifolds, maps regular points to regular points.

4) Let $\varphi : (M, L_M) \to (N, L_N)$ be a diffeomorphism between generalized complex manifolds. Then $\varphi$ is holomorphic if and only if, in an open neighbourhood of each regular point of $M$, we have $\varphi_* (L_M) = L_N$, up to B-field transformations.

5) The composition of two (almost) holomorphic maps, between generalized (almost) complex manifolds is (almost) holomorphic.
6) Let \( L = L(E, \varepsilon) \) be a regular generalized complex structure on \( M \). Then the holomorphic (local) functions on \((M, L)\) are just the co-CR holomorphic functions on \((M, E)\). Equivalently, if \( E \) is locally defined by the submersion \( \varphi : M \to (N, J) \) onto the complex manifold \((N, J)\) (that is, \( E = d\varphi^{-1}(T^{1,0}N) \)) then, locally, any holomorphic function on \((M, L)\) is the composition of \( \varphi \) followed by a holomorphic function on \((N, J)\).

From Theorem 2.3 we obtain the following result.

**Proposition 4.5.** Let \((M, L_M)\) and \((N, L_N)\) be regular real analytic generalized complex manifolds and let \( \varphi : M \to N \) be a real analytic map.

If \( \varphi \) is holomorphic then, locally, up to the complexification of a real analytic B-field transformation, the complexification of \( \varphi \) descends to a complex analytic Poisson morphism between canonical Poisson quotients.

Let \( L(E, i\varepsilon) \) be a generalized complex structure in normal form on a Riemannian manifold \((M, g)\).

Then \( E \) is coisotropic (that is, \( E^\perp \) is isotropic), with respect to \( g \), if and only if \( E \cap E^\perp \) is locally defined by pseudo-horizontally conformal submersions onto complex manifolds (a map from a Riemannian manifold to an almost complex manifold is pseudo-horizontally conformal if it pulls back \((1,0)\)-forms to isotropic forms).

Also, if \( \varepsilon^k \) has constant norm, with respect to \( g \), where \( \dim(E \cap E^\perp) = 2k \), then the leaves of \( E \cap E^\perp \) are minimal submanifolds of \((M, g)\).

Conversely, we have the following:

**Example 4.6.** Let \( \varphi : (M, g) \to (N, J) \) be a pseudo-horizontally conformal submersion from a Riemannian manifold onto an almost complex manifold, with \( \dim M = \dim N + 2 \).

Denote \( \mathcal{V} = \ker d\varphi \), \( \mathcal{H} = \mathcal{V}^\perp \) and let \( \omega \) be the volume form of \( \mathcal{V} \). Also, let \( F \) be the unique skew-adjoint almost f-structure on \( M \) such that \( \ker F = \mathcal{V} \) and, with respect to which, \( \varphi \) is co-CR holomorphic. Obviously, \( F \) and \( \omega \) are compatible; denote by \( L \) the corresponding generalized almost complex structure in normal form.

From Proposition 4.1 it follows that \( L \) is integrable if and only if \( J \) is integrable, the fibres of \( \varphi \) are minimal and the integrability tensor of \( \mathcal{H} \) is of type \((1, 1)\); note that, if \( \dim M = 4 \) then this is equivalent to the condition that \( \varphi \) is a harmonic morphism (see [4]), where \( N \) is endowed with the conformal structure with respect to which \( J \) is a Hermitian structure.

Moreover, any generalized complex structure, in normal form, on a Riemannian...
manifold such that the corresponding $f$-structure is skew-adjoint, the associated Poisson structure has rank two and its symplectic form has norm 1 is, locally, obtained this way.

The pseudo-horizontally conformal submersions with totally-geodesic fibres onto complex manifolds, for which the integrability tensor of the horizontal distribution is of type $(1, 1)$, admit a twistorial description from which it follows that they abound on Riemannian manifolds of constant curvature [14] (cf. [4]).

Also, see [3] for a study of the harmonic pseudo-horizontally conformal submersions with minimal fibres and [4] for twistorial constructions of harmonic morphisms with two-dimensional fibres on four-dimensional Riemannian manifolds.

5. Generalized Kähler manifolds

We start this section by recalling from [9] a few facts on generalized Kähler manifolds.

A generalized (almost) Kähler manifold is a manifold $M$ endowed with two generalized (almost) complex structures such that the corresponding sections $J_1$ and $J_2$ of $\text{End}(TM \oplus T^*M)$ commute and $J_1J_2$ is negative definite.

Any generalized almost Kähler structure $(L_1, L_2)$ on a manifold $M$ corresponds to a quadruple $(g, b, J^+, J^-)$ where $g$ is a Riemannian metric, $b$ is a two-form and $J^\pm$ are almost Hermitian structures on $(M, g)$. The (bijective) correspondence is given by

$L_1 = L^+ \oplus L^-,$  $L_2 = L^+ \oplus L^-$,

where $L^\pm_j = \{ X + (b \pm g)(X) \mid X \in V^\pm \}$

with $V^\pm$ the eigenbundles of $J^\pm$ corresponding to $i$.

According to [9, Theorem 6.28], the following assertions are equivalent:

(i) $L_1$ and $L_2$ are integrable.

(ii) $L_+^\pm$ and $L_-^\pm$ are integrable.

(iii) $J_\pm$ are integrable and parallel with respect to $\nabla^\pm = \nabla^g \pm \frac{1}{2} g^{-1} h$, where $\nabla^g$ is the Levi-Civita connection of $g$ and $h = db$ (equivalently, $J_\pm$ are integrable and $d^\pm J_\pm = \mp h$, where $\omega_\pm$ are the Kähler forms of $J_\pm$).

Now, if we (pointwisely) denote $E_j = \pi(L_j), (j = 1, 2)$, then $E_1 = V^+ + V^-$ and $E_2 = V^+ + 2^-$. Hence, $E_1^\perp = V^+ \cap V^-$, $E_2^\perp = V^+ \cap \overline{V^-}$ and, therefore, $E_1$ and $E_2$ are coisotropic.

Remark 5.1. Let $(M, L_1, L_2)$ be a generalized Kähler manifold.

1) The (skew-adjoint) almost $f$-structures $F_j$ determined by $E_j$ and $E_j^\perp$ are integrable; we call $F_j$ the $f$-structures of $L_j, (j = 1, 2)$. 

2) The holomorphic functions of \((M, L_1)\) and \((M, L_2)\) are the bi-holomorphic functions of \((M, J_+, J_-)\) and \((M, J_+, -J_-)\), respectively.

Let \(\mathcal{H}^\pm = \ker(J_+ \mp J_-)\). Then \(\mathcal{H}^+\) and \(\mathcal{H}^-\) are orthogonal; this follows from \(\mathcal{H}^+ = (V^+ \cap V^-) \oplus (V^+ \cap \overline{V^-})\) and \(\mathcal{H}^- = (V^+ \cap \overline{V^-}) \oplus (V^+ \cap \overline{V^-})\). Denote \(\mathcal{V} = (\mathcal{H}^+ \oplus \mathcal{H}^-)\).

Note that, \(\mathcal{H}^+, \mathcal{H}^-\) and \(\mathcal{V}\) are invariant under \(J_+\) and \(J_-\). Also, \(J_+ - J_-\) and \(J_+ + J_-\) are invertible on \(\mathcal{V}\).

**Proposition 5.2.** The following assertions are equivalent:

(i) \(L_1\) and \(L_2\) are regular.

(ii) \(\mathcal{H}^+\) and \(\mathcal{H}^-\) are distributions on \(M\).

(iii) \(\mathcal{V}\) is a distribution on \(M\).

**Proof.** The obvious relations

\[
E_1 = (V^+ \cap V^-)^\perp = (V^+ \cap V^-) \oplus \mathcal{H}^- \oplus \mathcal{V},
\]

\[
E_2 = (V^+ \cap \overline{V^-})^\perp = (V^+ \cap \overline{V^-}) \oplus \mathcal{H}^+ \oplus \mathcal{V}
\]

imply

\[
E_1 \cap \overline{E_1} = \mathcal{H}^- \oplus \mathcal{V} = (\mathcal{H}^+)\perp,
\]

\[
E_2 \cap \overline{E_2} = \mathcal{H}^+ \oplus \mathcal{V} = (\mathcal{H}^-)\perp
\]

which show that (i) \(\iff\) (ii).

Also, as the dimensions of \(\mathcal{H}^+\) and \(\mathcal{H}^-\) are upper semicontinuous functions on \(M\), assertion (ii) holds if and only if \(\mathcal{H}^+ \oplus \mathcal{H}^- (= \mathcal{V})\) is a distribution on \(M\). \(\square\)

Next, we prove the following result.

**Theorem 5.3.** Let \((M, L_1, L_2)\) be a generalized Kähler manifold with \(L_1\) regular.

Then the following assertions are equivalent:

(i) \(\mathcal{H}^+\) is integrable.

(ii) \(\mathcal{H}^+\) is geodesic.

(iii) \(\mathcal{H}^+\) is a holomorphic distribution on \((M, J_+).\)

(iv) \(\mathcal{H}^+\) is a holomorphic distribution on \((M, J_-).\)

Furthermore, if (i), (ii), (iii) or (iv) holds then \(\mathcal{H}^+\) is a holomorphic foliation on \((M, J_{\pm})\) and its leaves, endowed with \((g, J_{\pm})\), are Kähler manifolds.

To prove Theorem 5.3 we need some preparations.

Let \(\mathcal{H}\) be a distribution on a Riemannian manifold \((M, g)\) endowed with a linear connection \(\nabla\); denote \(\mathcal{V} = \mathcal{H}\perp.\)
The second fundamental form of $\mathcal{H}$, with respect to $\nabla$, is the $\mathcal{V}$-valued symmetric two-form $B^\mathcal{H}$ on $\mathcal{H}$ defined by $B^\mathcal{H}(X, Y) = \frac{1}{2} \mathcal{V}(\nabla_X Y + \nabla_Y X)$; then $\mathcal{H}$ is geodesic, with respect to $\nabla$, if and only if $B^\mathcal{H} = 0$ (cf. [4]).

The next result follows from a straightforward calculation.

**Lemma 5.4** (cf. [18]). Let $(M, g, J)$ be a Hermitian manifold endowed with a distribution $\mathcal{H}$ and a conformal connection $\nabla$ such that $\nabla J = 0$.

If $\mathcal{V}$ is integrable and $J$-invariant then the following relation holds:

$$2g(B^\mathcal{H}(JX, Y), V) + g(I^\mathcal{H}(X, Y), JV) = g(T(V, JX), Y) + g(T(V, X), JY),$$

for any $X, Y \in \mathcal{H}$ and $V \in \mathcal{V}$, where $T$ is the torsion of $\nabla$ and $I^\mathcal{H}$ is the integrability tensor of $\mathcal{H}$, defined by $I^\mathcal{H}(X, Y) = -[X, Y]$, for any sections $X$ and $Y$ of $\mathcal{H}$.

To prove Theorem 5.3 we also need the following lemma.

**Lemma 5.5.** Let $(M, J)$ be a complex manifold and let $\mathcal{H}$ be a complex vector subbundle of $(TM, J)$. The following assertions are equivalent:

(i) $\mathcal{H}$ is integrable.

(ii) $\mathcal{H}^1,0$ is a CR structure and $\mathcal{H}$ is a holomorphic distribution on $(M, J)$.

Proof. This follows from the fact that assertion (ii) holds if and only if for any sections $X, Y$ of $\mathcal{H}^1,0$ and $Z$ of $T^{0,1}M$ we have that $[X, Y]$ is a section of $\mathcal{H}^1,0$ and $[X, Z]$ is a section of $\mathcal{H}^1,0 \oplus T^{0,1}M$. □

Proof of Theorem 5.3 We may assume that, also, $L_2$ is regular.

Obviously, the second fundamental form of $\mathcal{H}^+$, with respect to $\nabla^g$, is equal to the second fundamental forms of $\mathcal{H}^+$, with respect to $\nabla^\pm$.

As $L_1$ and $L_2$ are integrable we have that $E_1$ and $E_2$ are integrable and, consequently, $\mathcal{H}^+ \oplus \mathcal{V}$ and $\mathcal{H}^- \oplus \mathcal{V}$ are integrable; in particular, the integrability tensor of $\mathcal{H}^+$ takes values in $\mathcal{V}$. Furthermore, $\mathcal{H}^+ \oplus \mathcal{V}$ and $\mathcal{H}^- \oplus \mathcal{V}$ are holomorphic with respect to both $J_+$ and $J_-$.

Now, by applying Lemma 5.4 to $\mathcal{H} = \mathcal{H}^+$ twice, with respect to $\nabla^+$ and $\nabla^-$, we quickly obtain

$$4g(B^\mathcal{H}^+(J_+X, Y), V) = -g(I^\mathcal{H}^+(X, Y), (J_++J_-)(V)),$$

for any $X, Y \in \mathcal{H}^+$ and $V \in \mathcal{H}^- \oplus \mathcal{V}$. As $J_++J_-$ is invertible on $\mathcal{V}$, we obtain that $(i) \iff (ii)$.

The equivalences $(iii) \iff (i) \iff (iv)$ follow quickly from Lemma 5.5 and the fact that the eigenbundles of $J_+|_{\mathcal{H}^+}$ corresponding to $i$ are equal to $V^+ \cap V^-$ which is integrable.
To complete the proof just note that if $\mathcal{H}^+$ is integrable then $(g, b, J_+, J_-)$ induces, by restriction, a generalized Kähler structure on each leaf $L$ of $\mathcal{H}^+$ and $J_+ = J_-$ on $L$.

**Remark 5.6.** 1) Let $(M, L_1, L_2)$ be a generalized Kähler manifold with $L_1$ regular. If $\mathcal{H}^+$ is integrable then, by Theorem 5.3, the co-CR structure associated to $L_1$ (that is, $E_1$) is, locally, given by holomorphic Riemannian submersions from $(M, g, J_{\pm})$ onto Kähler manifolds $(P, h, J)$; in particular, the leaves of $\mathcal{H}^+$, endowed with $(g, J_{\pm})$ can be, locally, identified with $(P, h, J)$.

2) If $(M, L_1^M, L_2^M)$ and $(N, L_1^N, L_2^N)$ are generalized Kähler manifolds with $\mathcal{H}_M^+$ and $\mathcal{H}_N^+$ integrable distributions then any almost holomorphic map $\varphi : (M, L_1^M) \to (N, L_1^N)$ descends, locally (with respect to the Riemannian submersions of Remark 5.6(1)), to a holomorphic map between Kähler manifolds.

Let $(M_j, g_j, J_j)$ be Kähler manifolds, $(j = 1, 2)$. Then on $M_1 \times M_2$ there are two nonequivalent natural generalized Kähler structures: the *first product* is just the Kähler product structure whilst the *second product* is given by $L_1 = L(T^{1,0}M_1 \times TM_2, i\omega_2)$ and $L_2 = L(T^{1,0}M_2 \times TM_1, i\omega_1)$, where $\omega_j$ are the Kähler forms of $J_j$ , $(j = 1, 2)$; see Section 6, below, for the corresponding definitions in a more general setting. Note that, both $L_1$ and $L_2$ are in normal form; moreover, the corresponding almost $f$-structures are skew-adjoint (and, thus, unique with this property).

We end this section with the following consequence of Theorem 5.3 (cf. [2, Theorem A]).

**Corollary 5.7.** Any generalized Kähler manifold with $\mathcal{V} = 0$ is, up to a unique $B$-field transformation, locally given by the second product of two Kähler manifolds.

**Proof.** Let $(M, L_1, L_2)$ be a generalized Kähler manifold with $\mathcal{V} = 0$. Then, Proposition 5.2 implies that $\mathcal{H}^\pm$ are complementary orthogonal distributions on $M$.

As $L_1$ and $L_2$ are integrable, we have $\mathcal{H}^\pm$ integrable. Furthermore, by Theorem 5.3, we have that $\mathcal{H}^\pm$ are geodesic foliations which are holomorphic with respect to both $J_{\pm}$; moreover, $(g, J_{\pm})$ induce, by restriction, Kähler structures on their leaves.

If $L_2 = L(E_2, \varepsilon_2)$ then, from the definitions it follows that $\varepsilon_2 = (b - i\eta)|_{E_2}$, where $\eta$ is the two-form on $M$ characterised by $\iota_X\eta = 0$ if $X \in \mathcal{H}_-$ and $\eta|_{\mathcal{H}_+}$ is the Kähler form of $J_+|_{\mathcal{H}_+}$. As $(L_X\eta)(Y, Z) = 0$ for any sections $X$ of $\mathcal{H}_-$ and $Y$, $Z$ of $\mathcal{H}^+$, and $(d\varepsilon_2)(X, Y, Z) = 0$ for any $X, Y, Z \in E_2$, we obtain that
\((db)(X, Y, Z) = 0\) for any \(X \in V^+ \cap \mathcal{H}^- (= E_2 \cap \mathcal{H}^-)\) and \(Y, Z \in V^+ \cap \mathcal{H}^+\). Furthermore, from Lemma 5.4, applied to \(\mathcal{H} = \mathcal{H}^+\) with \(J = J_+\) and \(\nabla = \nabla^+\), we obtain \((db)(X, Y, Z) = 0\) for any \(X \in \mathcal{H}^-\) and \(Y, Z \in V^+ \cap \mathcal{H}^+\).

It follows that \(db = 0\) and the proof is complete. \(\square\)

6. Tamed symplectic and generalized Kähler manifolds

The following definition is fairly standard.

**Definition 6.1.** A tamed almost symplectic manifold is a manifold \(M\) endowed with a nondegenerate two-form \(\varepsilon\) and an almost complex structure \(J\) such that \(\varepsilon(JX, X) > 0\) for any nonzero \(X \in TM\).

A tamed symplectic manifold is a tamed almost symplectic manifold \((M, \varepsilon, J)\) such that \(J\) and \(\varepsilon^{-1} J^* \varepsilon\) are integrable and \(d\varepsilon = 0\).

Obviously, \((M, \varepsilon, J)\) is a tamed symplectic manifold if and only if \(\varepsilon\) is a symplectic form, \(T^{1,0}M\) and \((T^{1,0}M)^{\perp} \varepsilon\) are integrable, and \(\varepsilon(JX, X) > 0\), for any nonzero \(X \in TM\).

The next result also appears, in a different form, in [10].

**Theorem 6.2.** Let \(M\) be a manifold endowed with a nondegenerate two-form \(\varepsilon\) and an almost complex structure \(J\); denote \(J_+ = J\) and \(J_- = -\varepsilon^{-1} J^* \varepsilon\). Let \(g\) and \(b\) be the symmetric and skew-symmetric parts, respectively, of \(\varepsilon J\).

Then the following assertions are equivalent:

(i) \((M, \varepsilon, J)\) is a tamed symplectic manifold.

(ii) \((g, b, J_+, J_-)\) defines a generalized Kähler structure such that \(J_+ + J_-\) is invertible.

Moreover, up to a unique B-field transformation, any generalized Kähler structure, on \(M\), with \(J_+ + J_-\) invertible is obtained this way from a tamed symplectic structure.

**Proof.** Firstly, note that \(\varepsilon(J_+ X, Y) = -\varepsilon(X, J_- Y)\), for any \(X, Y \in TM\). This implies that

\[
\begin{align*}
    g(X, Y) &= \frac{1}{2} \varepsilon((J_+ + J_-)(X), Y), \\
    b(X, Y) &= \frac{1}{2} \varepsilon((J_+ - J_-)(X), Y),
\end{align*}
\]

(6.1)

for any \(X, Y \in TM\).

Therefore \((M, \varepsilon, J)\) is a tamed almost symplectic manifold if and only if the quadruple \((g, b, J_+, J_-)\) defines a generalized almost Kähler manifold with \(J_+ + J_-\) invertible.

Now, with respect to \(J_\pm\), we have \(\omega_\pm = -\varepsilon^{1,1}, b^{1,1} = 0\) and \(b^{2,0} = \pm i \varepsilon^{2,0}\). It
quickly follows that if $J_\pm$ are integrable then $d\varepsilon = 0$ if and only if $d^2 \omega_\pm = \mp db$.

We have thus proved that $(i) \iff (ii)$.

Suppose that $(g, b, J_+, J_-)$ corresponds to the generalized Kähler structure $(L_1, L_2)$ on $M$. Then $J_+ + J_-$ is invertible if and only if $\pi(L_2) = TM$. Hence, if $J_+ + J_-$ is invertible then, up to a unique $B$-field transformation, we have $L_2 = L(TM, i\varepsilon)$ for some symplectic form $\varepsilon$ on $M$ and, consequently,

$$
i \varepsilon (X - iJ_+X, Y) = (b + g)(X - iJ_+X, Y),$$

$$(6.2) \quad i \varepsilon (X + iJ_-X, Y) = (b - g)(X + iJ_-X, Y),$$

for any $X, Y \in TM$. By using the fact that $J_+ + J_-$ is invertible, from (6.2) we quickly obtain that $g$ and $b$ satisfy (6.1). Together with the fact that $g$ and $b$ are symmetric and skew-symmetric, respectively, this shows that $J_- = -\varepsilon^{-1}J_+^*\varepsilon$ and the proof follows. \hfill \square

It is easy to rephrase Theorem 6.2 so that to obtain the description of generalized Kähler manifolds with $J_+ - J_-$ invertible.

Let $(M, L_1^M, L_2^M)$ and $(N, L_1^N, L_2^N)$ be generalized Kähler manifolds corresponding to the tamed symplectic manifolds $(M, \varepsilon_M, J_M)$ and $(N, \varepsilon_N, J_N)$, respectively. Then $(M \times N, L_1^M \times L_1^N, L_2^M \times L_2^N)$ and $(M \times N, L_1^M \times L_2^N, L_2^M \times L_1^N)$ are called the first and second product of $(M, L_1^M, L_2^M)$ and $(N, L_1^N, L_2^N)$, respectively; note that, the first product is the generalized Kähler manifold corresponding to $(M \times N, \varepsilon_M + \varepsilon_N, J_M \times J_N)$.

**Corollary 6.3.** Any generalized Kähler manifold with $\mathcal{H}^+$ an integrable distribution and $\mathcal{H}^- = 0$ is, up to a unique $B$-field transformation, locally given by the first product of a Kähler manifold and a generalized Kähler manifold for which both $J_+ + J_-$ and $J_+ - J_-$ are invertible.

**Proof.** Let $(M, L_1, L_2)$ be a generalized Kähler manifold with $\mathcal{H}^+$ a distribution and $\mathcal{H}^- = 0$. Then, by Theorem 6.2, up to a unique $B$-field transformation, we have that $(M, L_1, L_2)$ corresponds to the tamed symplectic manifold $(M, \varepsilon, J)$.

Thus, by (6.1), we have $\iota_X b = 0$ for any $X \in \mathcal{H}^+$ and $\varepsilon = \eta + \varepsilon'$ where $\eta$ and $\varepsilon'$ are the two-forms on $M$ characterised by $\iota_X \eta = 0$, $(X \in \mathcal{V})$, $\iota_X \varepsilon' = 0$, $(X \in \mathcal{H}^+)$, $\eta = \omega_+$ on $\mathcal{H}^+$, and $\varepsilon' = \varepsilon$ on $\mathcal{V}$.

If, further, $\mathcal{H}^+$ is integrable then, by Theorem 5.3, it is also geodesic and its leaves endowed with $(g, J)$ are Kähler manifolds; in particular, $d\eta = 0$ on $\mathcal{H}^+$. As, also, $\mathcal{H}^+$ and $\mathcal{V}$ are holomorphic foliations, it quickly follows that $\mathcal{L}_X \eta (Y, Z) = 0$ for any sections $X$ of $\mathcal{V}$ and $Y$, $Z$ of $\mathcal{H}^+$; consequently, $d\eta = 0$.

We have thus obtained $d\varepsilon' = 0$ which implies $(\mathcal{L}_X \varepsilon')(Y, Z) = 0$ for any sections $X$ of $\mathcal{H}^+$ and $Y$, $Z$ of $\mathcal{V}$. Together with (6.1), this gives $(\mathcal{L}_X b)(Y, Z) = 0$ and
(L_X g)(Y, Z) = 0 for any sections X of \( \mathcal{H}^+ \) and Y, Z of \( \mathcal{V} \); in particular, this shows that \( \mathcal{V} \) is geodesic. The proof follows. \( \square \)

Obviously, a result similar to Corollary 6.3 holds for any generalized Kähler manifold with \( \mathcal{H}^+ = 0 \) and \( \mathcal{H}^- \) an integrable distribution.

**Corollary 6.4.** Let \((M, L_1, L_2)\) be a generalized Kähler manifold such that \(L_2\) is in normal form with respect to its f-structure and the two-form \(\epsilon\) on \(M\).

Then, in a neighbourhood of each regular point of \(L_1\), we have that \((M, L_1, L_2)\) is the second product of a Kähler manifold and a generalized Kähler manifold determined by a tamed symplectic manifold.

**Proof.** Assume \(L_1\) regular. Define \(\epsilon_{\pm}\) to be the (complex linear) two-forms on \(T^{1,0}_M + T^{-1,0}_M\) such that \(\epsilon_{\pm} = \epsilon\) on \(T^{1,0}_M\) and \(i_X \epsilon_{\pm} = 0\) if \(X \in T^{1,0}_M\).

Obviously, \(d\epsilon_{\pm} = 0\) on \(T^{1,0}_M\). Also, from the fact that \(i_X \epsilon_{\pm} = 0\) if \(X \in T^{1,0}_M\) it quickly follows that if \(X_{\pm}, Y_{\pm}, Z_{\pm} \in T^{1,0}_M\) then \(d\epsilon_{\pm}(X_{\pm}, Y_{\pm}, Z_{\pm}) = 0\); together with the fact that \(\epsilon = \epsilon_{+} + \epsilon_{-}\) on \(T^{1,0}_M + T^{-1,0}_M\), this implies that \(d\epsilon_{\pm}(X_{\pm}, Y_{\pm}, Z_{\pm}) = 0\). Thus, we have proved that \(d\epsilon_{\pm} = 0\) on \(T^{1,0}_M + T^{-1,0}_M\).

Therefore \(\ker \epsilon_{\pm} = (T^{1,0}_M) \oplus (T^{0,1}_M \cap \mathcal{H}^-)\) is integrable which implies that \(\mathcal{H}^-\) is an antiholomorphic distribution on \((M, J_\mp)\). Hence, by Lemma 5.5, we have that \(\mathcal{H}^-\) is integrable and the proof follows from Theorem 5.3 and the fact that \(\ker \epsilon = \mathcal{H}^-\). \(\square\)

Let \((M, \epsilon, J)\) be a tamed almost symplectic manifold. With the same notations as in Corollary 6.4, if \((M, L_1, L_2)\) is the generalized Kähler manifold determined by \((M, \epsilon, J)\) then, from (6.1), it follows that \(L_1 = L(T^{1,0}_M + T^{-1,0}_M, i \epsilon_{+} - i \epsilon_{-})\).

**Theorem 6.5 (cf. [12]).** Let \((M, \epsilon, J)\) be a tamed almost symplectic manifold and let \((M, L_1, L_2)\) be the corresponding generalized almost Kähler manifold; denote by \(\rho^\pm : T^\mathbb{C} M \rightarrow T^{1,0}_M\) the projections.

If \((M, L_1, L_2)\) is generalized Kähler then \(J_{\pm}\) are integrable and \(\rho^\pm(L_2)\) are holomorphic Poisson structures on \((M, J_{\pm})\), respectively. Furthermore, the converse holds if also \(J_{+} - J_{-}\) is invertible; moreover, in this case, if \((M, L_1, L_2)\) is generalized Kähler then \(\rho_{\pm}(L_2)\) are holomorphic symplectic structures on \((M, J_{\pm})\), respectively.

**Proof.** Assume, for simplicity, that \((M, \epsilon, J)\) is real analytic. Also, we may assume \(L_1\) regular. If \((M, L_1, L_2)\) is generalized Kähler then, by passing to the complexification of \((M, \epsilon, J)\), from Proposition 1.3 and the proof of Corollary 6.4 we obtain that \(\rho^\pm(L_2)\) are the canonical Poisson quotients of \(L(T^{0,1}_M + T^{-1,0}_M, i \epsilon_{\pm})\).

If \(J_{+} \pm J_{-}\) are invertible and \(J_{\pm}\) are integrable then \(\rho^\pm(L_2)\) are holomorphic Poisson structures on \((M, J_{\pm})\) if and only if \(d\epsilon_{\pm} = 0\). \(\square\)
We call the $\rho^\pm_2(L_2)$ of Theorem 6.5 the *holomorphic Poisson structures associated to* $(M, L_1, L_2)$.

**Remark 6.6** (cf. [12], [10]). 1) Let $(M, L_1, L_2)$ be a generalized Kähler manifold with $J_+ + J_-$ invertible. Denote by $\eta_{\pm}$ be the (real) bivectors on $M$ which determine the holomorphic Poisson structures on $(M, J_\pm)$, respectively, associated to $(M, L_1, L_2)$; that is, with respect to $J_\pm$, we have $\eta_{\pm}^{1,1} = 0$ and the holomorphic bivectors corresponding to $\rho^\pm_2(L_2)$ are $\eta_{\pm}^{2,0}$, respectively.

It quickly follows that

$$\eta_+ = -\eta_+ = \frac{1}{2}(J_+ - J_-) \varepsilon^{-1} = \frac{1}{4}[J_+, J_-]g^{-1},$$

where $(M, \varepsilon, J)$ is the tamed symplectic manifold associated to $(M, L_1, L_2)$.

Hence, the symplectic foliation associated to $\eta_+$ is given by $\mathcal{Y} (= \text{im}(J_+ - J_-))$.

2) If the generalized almost Kähler structure $(L_1, L_2)$ on $M$ corresponds to the quadruple $(g, b, J_+, J_-)$ then $(L_2, L_1)$ corresponds to $(g, b, J_+, -J_-)$. Assume that $(M, L_1, L_2)$ is a generalized Kähler manifold with $J_+ + J_-$ and $J_+ - J_-$ invertible and let $\eta_+$ and $\eta'_+$ be the bivectors which determine, as in (1), the holomorphic symplectic structures associated to $(M, L_1, L_2)$ and $(M, L_2, L_1)$, respectively. Then (6.3) implies that $\eta'_+ = -\eta_+$.

Next, we prove some results on holomorphic maps between generalized Kähler manifolds.

**Corollary 6.7.** Let $(M, L_1, L_2)$ be a generalized almost Kähler manifold with $J_+ + J_-$ and $J_+ - J_-$ invertible.

If $\varphi : M \to M$ is a diffeomorphism then any two of the following assertions imply the third:

(i) $\varphi : (M, L_1) \to (M, L_2)$ is holomorphic.

(ii) $\varphi : (M, L_2) \to (M, L_2)$ is holomorphic.

(iii) $[d\varphi, J_+ J_-] = 0$.

**Proof.** Let $L = L(T^1_+M + T^1_-M ; \varepsilon_1)$. By using the first relation of (6.1), we obtain

$$\text{(Im } \varepsilon_1)(J_+ - J_-) = \varepsilon(J_+ + J_-),$$

which, firstly, shows that if (iii) holds then (i) $\iff$ (ii).

Furthermore, (6.3) implies that $\varepsilon^{-1}(\text{Im } \varepsilon_1)$ is skew-adjoint, with respect to $g$, and, consequently, $\varepsilon - \text{Im } \varepsilon_1$ is invertible. This fact together with (6.3) proves that (i), (ii)$\implies$ (iii).
Corollary 6.8. Let \((M, L^M_1, L^M_2)\) and \((N, L^N_1, L^N_2)\) be generalized Kähler manifolds, with \(J^M_+ + J^M_-\) and \(J^N_+ + J^N_-\) invertible, and let \(\varphi : M \rightarrow N\) be a map.

(i) If \(\varphi : (M, L^M_1) \rightarrow (N, L^N_1)\) and \(\varphi : (M, J^M) \rightarrow (N, J^N)\) are almost holomorphic then \(\varphi\) is a holomorphic Poisson morphism between the corresponding associated holomorphic Poisson manifolds; moreover, the converse holds if \(\varphi\) is an immersion.

(ii) If \(\varphi : (M, L^M_2) \rightarrow (N, L^N_2)\) and, either, \(\varphi : (M, J^M) \rightarrow (N, J^N)\) or \(\varphi : (M, J^M) \rightarrow (N, J^N)\) are holomorphic maps then \(\varphi\) is a holomorphic Poisson morphism between the associated holomorphic Poisson structures.

Proof. Assertion (i) follows from Proposition \[1.3\] and the proof of Theorem \[6.5\].

To prove (ii), note that if \(\varphi : (M, L^M_2) \rightarrow (N, L^N_2)\) is holomorphic then \(\varphi : (M, J^M) \rightarrow (N, J^N)\) is holomorphic if and only if \(\varphi : (M, J^M) \rightarrow (N, J^N)\) is holomorphic. The proof quickly follows from Remark \[6.6(1)\].

If \((g, J_\pm)\) are Kähler structures on \(M\) then \((g, 0, J_+, J_-)\) corresponds to a generalized Kähler structure \((L_1, L_2)\) on \(M\); furthermore, if \(b\) is a closed two-form on \(M\) then \((g, b, J_+, J_-)\) corresponds to \(((\exp b)(L_1), (\exp b)(L_2))\).

Example 6.9 (cf. \[12\]). Let \((M, g, I, J, K)\) be a hyper-Kähler manifold. Denote by \(\omega_I, \omega_J, \omega_K\) the Kähler forms of \(I, J, K\), respectively, and let \(\varepsilon = - (\omega_J + \omega_K)\).

Then \((M, \varepsilon, J)\) is a tamed symplectic manifold. The corresponding generalized Kähler structure \((L_1, L_2)\) is given by \((g, b, J_+, J_-)\), where \(b = \omega_I, J_+ = J\) and \(J_- = K\). Also, \(L_1 = L(T^\mathbb{C}M, 2\omega_I - i(\omega_J - \omega_K))\), \(L_2 = L(T^\mathbb{C}M, -i(\omega_J + \omega_K))\) and \(\varepsilon_+ = -i(\omega_I - i\omega_J), \varepsilon_- = -(\omega_K - i\omega_I)\).

We end with a generalization of Corollaries \[5.7\] and \[6.3\].

Theorem 6.10. Let \((M, L_1, L_2)\) be a generalized Kähler manifold. Then the following assertions are equivalent:

(i) \(\mathcal{H}^+ \oplus \mathcal{H}^-\) is an integrable distribution.

(ii) Locally, up to a B-field transformation, \((M, L_1, L_2)\) is the first product of a generalized Kähler manifold for which \(J_+ J_-\) are invertible and the second product of two Kähler manifolds.

Proof. By applying Lemma \[5.4\] to \(\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-\) twice, with respect to \(\nabla^+\) and \(\nabla^-\), we obtain

\[
2g(B^{\mathcal{H}^+ \oplus \mathcal{H}^-}(J_+ X_+, X_-), V) + g(I^{\mathcal{H}^+ \oplus \mathcal{H}^-}(X_+, X_-), J_+ V) = (db)(V, J_+ X_+, X_-) + (db)(V, X_+, J_+ X_-),
\]

\[
2g(B^{\mathcal{H}^+ \oplus \mathcal{H}^-}(J_+ X_+, X_-), V) + g(I^{\mathcal{H}^+ \oplus \mathcal{H}^-}(X_+, X_-), J_- V) = -(db)(V, J_+ X_+, X_-) + (db)(V, X_+, J_+ X_-),
\]

(6.4)
for any $X_{\pm} \in \mathcal{H}^{\pm}$ and $V \in \mathcal{V}$. Consequently, we, also, have

$$g(I^+_{\mathcal{H}^+}\oplus I^-_{\mathcal{H}^-}(X_+, X_-), (J_+-J_-)(V)) = 2 \, db(V, J_+X_+, X_-),$$

for any $X_{\pm} \in \mathcal{H}^{\pm}$ and $V \in \mathcal{V}$.

Suppose that (i) holds. Then, by (6.5), we have $db(V, X_+, X_-) = 0$, for any $X_{\pm} \in \mathcal{H}^{\pm}$ and $V \in \mathcal{V}$. Moreover, from Corollaries 5.7 and 6.3 it follows that $db(X, Y, Z) = 0$ if $X, Y, Z \in \mathcal{H}^+ \oplus \mathcal{H}^-$ or $X \in \mathcal{H}^\pm$ and $Y, Z \in \mathcal{V} \oplus \mathcal{H}^\pm$.

As $d(db) = 0$, this shows that $db$ is basic with respect to $\mathcal{H}^+ \oplus \mathcal{H}^-$. Hence, locally, there exists a two-form $b'$, basic with respect to $\mathcal{H}^+ \oplus \mathcal{H}^-$, such that $db = db'$.

Furthermore, from (6.4) and (6.5) we obtain $B^\mathcal{H}^+\oplus \mathcal{H}^-(X_+, X_-) = 0$, for any $X_{\pm} \in \mathcal{H}^{\pm}$. Together with Theorem 5.3 and Corollary 6.3, this shows that $\mathcal{V}$ and $\mathcal{H}^+ \oplus \mathcal{H}^-$ are geodesic foliations on $(M, g)$.

Thus, we have proved that $(M, L_1, L_2)$ is the first product of a generalized Kähler manifold with $\mathcal{H}^+ = 0 = \mathcal{H}^-$ and a generalized Kähler manifold with $\mathcal{V} = 0$. Hence, by Corollary 5.7, assertion (ii) holds.

The implication (ii)$\implies$(i) is trivial. $\square$

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