Criteria for orbital behavior of operators

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Other names

universal
topologically transitive i.e.

for any open sets U and V there is n such that $T^n(U) \cap V \neq \emptyset$

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same space (MacLane, 1957)

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$$\begin{pmatrix} 0 & 2 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 2 & \dots \\ \dots & & & & \end{pmatrix}$$

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If there there is and increasing sequence (k_n) of natural numbers, there are two dense sets, X_0 and Y_0, and there is a sequence of functions $(S_{k_n}) : Y_0 \longrightarrow Y_0$ (neither necessarily linear nor continuous) such that:

(i) $T^{k_n}x \longrightarrow 0$ for every $x \in X_0$;
(ii) $S_{k_n}y \longrightarrow 0$ for every $y \in Y_0$;
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then the operator T is hypercyclic.

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Not equivalent

C. J. Read & M. De La Rosa (2005, on l^1), F. Bayart & E. Matheron (2007 on l^2)

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First idea:

Look at the behavior of the sequence

$$||x||, ||Tx||, ||T^2x||, ||T^3x||, \ldots$$

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Six types:

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$\lim_n ||T^n x|| = 0$

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How different from hypercyclicity?

Posibilities:

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Hypercyclic restriction to an invariant subspace

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orbits dense in an one-dimensional subspace (can be done with forward weighted shifts)

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There is a nonhypercyclic weakly hypercyclic operator having all nonzero orbits increasing.

J - class vectors

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A vector $x \neq 0$ is called J class for the operator T if
$J_T(x) =$

$\{$ y : there are y$_n$ and k_n such that $y_n \rightarrow x$ and $T^{k_n} y_n \rightarrow y\}$

equals the space

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$J_T(x)$ is always closed

For $2B$ on l^∞ a vector x is J - class $\iff$ $x \in c_o$

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The operator has J - class irregular vectors.

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x is hypercyclic vector for T if it is J - class & irregular.

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Other restrictions may be needed.

Hypercyclic vs. irregular

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T invertible has property $\implies T^{-1}$ has the property

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Difficult for composition operators.

On the Hardy space a composition operator cannot have irregular vectors unless it is hypercyclic. Are there irregular non hyperciclic vectors?

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If there there is and increasing sequence (k_n) of natural numbers, there are two sets, X_0 and $Y_0 = \{y_1, y_2, y_3, \ldots\}$ such that $Y_0 \subset$ closure of X_0, $y_{2n-1} \to 0$, $||y_{2n}|| \to \infty$, and there is a sequence of functions $(S_{k_n}) : Y_0 \to Y_0$ (neither necessarily linear nor continuous) such that:

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then the operator T has irregular vectors.

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If there is $|\lambda| > 1$ and $x \in \ker T - \lambda$ such that $x \in$ closure Span $\{\ker(T - \alpha) : |\alpha| < 1\}$ then T has irregular vectors.

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Necessary but not sufficient
Satisfied by

$$\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

which does not have irregular vectors.