The asymptotic limit of a bicontraction and related results

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Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the Banach algebra of all bounded linear operators on $\mathcal{H}$, while I is the identity operator on $\mathcal{H}$.
Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, $A \neq 0$. An operator $T \in \mathcal{B}(\mathcal{H})$ satisfying the operator inequality

$$T^*AT \leq A \tag{1}$$

is called an **A-contraction** on $\mathcal{H}$. Also, T is called an **A-isometry** if the equality in (1) occurs. It is easy to see from (1) that $\mathcal{N}(A)$ is an invariant subspace for T, and it is not invariant for T^*, in general.
An A-contraction T is **regular** (or T is **A-regular**) if

$$AT = A^{1/2}TA^{1/2}. \tag{2}$$

We know that if A is an orthogonal projection then any A-contraction is regular.

Let $T = (T_0, T_1)$ be a pair of commuting contractions on $\mathcal{H}$, that is $T_i \in \mathcal{B}(\mathcal{H})$, $||T_i|| \leq 1$ ($i = 0, 1$) and $T_0T_1 = T_1T_0$. Such T is called a **bicontraction** on $\mathcal{H}$, and when T_0 and T_1 are isometries, T is called a **bi-isometry** on $\mathcal{H}$.
Since T_i is a contraction, the **asymptotic limit** of T_i can be defined as

$$S_{T_i}h = \lim_{n\to\infty} T_i^{*n}T_i^n h \quad (h \in \mathcal{H}). \tag{3}$$

Clearly, $0 \leq S_{T_i} \leq T_i^*T_i$ and T_i is a S_{T_i}-isometry. Moreover, $\mathcal{N}(I - S_{T_i})$ is the maximum invariant subspace for T_i on which T_i is an isometry, while $\mathcal{N}(S_{T_i})$ is the maximum invariant subspace for T_i on which the sequence $\{T_i^n\}_{n\in\mathbb{N}}$ strongly converges to 0, for $i = 0, 1$.

We have that T_1 is a S_{T_0}-contraction. Thus, one can define the operator $S_{T_0,T_1} \in \mathcal{B}(\mathcal{H})$ by

$$S_{T_0,T_1}h = \lim_{m\to\infty} T_1^{*m} S_{T_0} T_1^m h = \lim_{m\to\infty} \lim_{n\to\infty} T_1^{*m} T_0^{*n} T_0^n T_1^m h \quad (4)$$

for $h \in \mathcal{H}$, and obviously $0 \leq S_{T_0,T_1} \leq S_{T_0}$.
By symmetry, T_0 is a S_{T_1}-contraction, and so can be defined the operator $S_{T_1,T_0} \in \mathcal{B}(\mathcal{H})$ by

$$S_{T_1,T_0}h = \lim_{n\to\infty} T_0^{*n} S_{T_1} T_0^n h = \lim_{n\to\infty} \lim_{m\to\infty} T_0^{*n} T_1^{*m} T_1^m T_0^n h. \quad (5)$$

We get $S_{T_0,T_1} = S_{T_1,T_0}$, and so the operator

$$S_T := S_{T_0,T_1} = S_{T_1,T_0} \quad (6)$$

can be defined by any of the iterated limits of the sequence

$$\{T_1^{*m} T_0^{*n} T_0^n T_1^m\}_{m,n\in\mathbb{N}}$$

in the strong topology of $\mathcal{B}(\mathcal{H})$. The operator S_T is called the **asymptotic limit** of the bicontraction T, and clearly, T_0 and T_1 are S_T-isometries.

Theorem

For a bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$ one has

$$\mathcal{N}(I - S_T) = \bigcap_{m,n\in\mathbb{N}} \mathcal{N}(I - T_1^{*m} T_0^{*n} T_0^n T_1^m), \tag{7}$$

and $\mathcal{N}(I - S_T)$ is the maximum invariant subspace for T_0 and T_1 on which T_0 and T_1 are isometries.

Let $\widehat{T}_{1-i} \in \overline{\mathcal{R}(S_{T_i})}$ is the operator satisfying

$$\widehat{T}_{1-i} S_{T_i}^{1/2} = S_{T_i}^{1/2} T_{1-i}.$$

Theorem

For a bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$ the following statements are equivalent :
(i) $S_T T_1 = T_1 S_T$;
(ii) T_1 is S_T-regular and $\mathcal{N}(S_T)$ reduces T_1;
(iii) T_1^* is a regular S_T-contraction;
(iv) T_1^* is a S_T-contraction and either T_1 or T_1^* is S_T-regular.
Moreover, if T_1 is S_{T_0}-regular then the conditions $(i) - (iv)$ are also equivalent to
(v) $S_{\widehat{T}_1} = S^2_{\widehat{T}_1}$ and $R_1 S_T = 0$, if T_1 on $\mathcal{H} = \overline{\mathcal{R}(S_{T_0})} \oplus \mathcal{N}(S_{T_0})$ has the operator matrix representation

$$T_1 = \begin{pmatrix} \widehat{T}_1 & 0 \\ R_1 & Q_1 \end{pmatrix}. \tag{8}$$

In addition, when T_1 is S_{T_0}-regular, we have $S_T = S_T^2$ if and only if $S_{\widehat{T}_1} = S^2_{\widehat{T}_1}$ and $S_{T_0} h = S^2_{T_0} h$ for $h \in \mathcal{R}(S_T)$.

Remark. We derive that the condition $S_T = S_T^2$ implies $S_T T_1 = T_1 S_T$ and, by symmetry $S_T T_0 = T_0 S_T$.
Since $S_T = S_{T_0}^{1/2} S_{\widehat{T}_1} S_{T_0}^{1/2}$ we have $S_T^2 = S_{T_0}^{1/2} S_{\widehat{T}_1} S_{T_0} S_{\widehat{T}_1} S_{T_0}^{1/2}$, and so $S_T = S_T^2$ if and only if $S_{\widehat{T}_1} = S_{\widehat{T}_1} S^0 S_{\widehat{T}_1}$, $S^0 = S_{T_0}|_{\overline{\mathcal{R}(S_{T_0})}}$.
The last equality implies that $S_{\widehat{T}_1}$ has a generalized inverse, or equivalently, that $\mathcal{R}(S_{\widehat{T}_1})$ is closed.

Corollary

Let $T = (T_0, T_1)$ be a bicontraction on $\mathcal{H}$ such that $T_1 S_{T_0} = S_{T_0} T_1$. Then

$$S_T T_1 = T_1 S_T \Leftrightarrow S_T = S_T^2 \Leftrightarrow S_{\widehat{T}_1} = S_{\widehat{T}_1}^2$$

Theorem

Let $T = (T_0, T_1)$ *be a bicontraction on* $\mathcal{H}$ *such that* $S_{T_0} = S_{T_0}^2$*.*
The following statements are equivalent :
(i) $S_T = S_T^2$;
(ii) $S_{\widehat{T}_1} = S_{\widehat{T}_1}^2$;
(iii) $T_1^*|_{\overline{\mathcal{R}(S_T)}}$ *is a coisometry.*

Theorem

Let $T = (T_0, T_1)$ *be a bicontraction on* $\mathcal{H}$ *such that* $S_{T_0} = S_{T_0}^2$. *If* $T_1 S_{T_0} = S_{T_0} T_1$ *then the following statements hold :*
(i) $S_{T_1} = S_{T_1}^2$ *if and only if* $S_T = S_T^2$ *and* $S_{Q_1} = S_{Q_1}^2$, *where* $Q_1 = T_1|_{\mathcal{N}(S_{T_0})}$.
(ii) $\mathcal{R}(S_T) = \mathcal{R}(S_{T_0}) \cap \mathcal{R}(S_{T_1})$, $\overline{\mathcal{R}(S_T)} = \mathcal{N}(I - S_{T_0}) \cap \overline{\mathcal{R}(S_{T_1})}$, *hence if* $\mathcal{R}(S_{T_1})$ *is closed then* $\mathcal{R}(S_T)$ *is closed, too.*

Remark. The previous theorem shows that, in certain conditions, if S_{T_0} and S_{T_1} are orthogonal projection, then S_T is also an orthogonal projection. But, when S_{T_0} and S_T are orthogonal projections, S_{T_1} is not necessarily an orthogonal projection, in general.
For instance, suppose that T_1 is a S_{T_0}-isometry, that is $T_1^* S_{T_0} T_1 = S_{T_0}$, which yields $S_T = S_{T_0}$. Hence, if $S_T = S_T^2$ then $T_1 S_{T_0} = T_1 S_T = S_T T_1 = S_{T_0} T_1$ and $\widehat{T}_1$ is an isometry, therefore $S_{\widehat{T}_1} = I$. In this case we have $S_{T_1} = S_{T_1}^2$ if and only if $S_{Q_1} = S_{Q_1}^2$, where $Q_1 = T_1|_{\mathcal{N}(S_{T_0})}$.
Let $T = (T_0, T_1)$ be a bicontraction on $\mathcal{H}$ with T_i^* hyponormal, (that is $T_i T_i^* \leq T_i^* T_i$), $j = 0, 1$, such that either $T_1 S_{T_0} = S_{T_0} T_1$, or $T_0 S_{T_1} = S_{T_1} T_0$. It is known that $S_{T_i} = S_{T_i}^2$ for $i = 0, 1$, and by previous Theorem one has $S_T = S_T^2$. In particular, if T_i are quasinormal (that is $T_i T_i^* T_i = T_i^* T_i^2$) and $T_i S_{T_{1-i}} = S_{T_{1-i}} T_i$ and $T_i S_{T_{1-i}^*} = S_{T_{1-i}^*} T_i$ for either $i = 0$ or $i = 1$, then $S_T = S_T^2$ and $S_{T^*} = S_{T^*}^2$, because $S_{T_i^*} = S_{T_i^*}^2$, $i = 0, 1$.

Theorem

Let $T = (T_0, T_1)$ be a bicontraction on $\mathcal{H}$ such that $\{T_0^n\}_{n\in\mathbb{N}}$ strongly converges. Then $S_{T_0} = S_{T_0}^2$ and $T_1 S_{T_0} = S_{T_0} T_1$. Furthermore, if $(I - T_0)T_1^n h \to 0$ ($n \to \infty$) for $h \in \mathcal{H}$ then $S_T = S_{T_1}$.

Corollary

If $\{T_i^n\}_{n\in\mathbb{N}}$ strongly converges for $i = 0, 1$ then S_{T_0}, S_{T_1} and S_T are orthogonal projections, and $S_{T_i} T_{1-i} = T_{1-i} S_{T_i}$ for $i = 0, 1$.

Corollary

Suppose that $\{T_0^n\}_{n\in\mathbb{N}}$ strongly converges and that $(I - T_1)T_0^n h \to 0$ $(n \to \infty)$ for $h \in \mathcal{H}$. Then $S_T = S_{T_0}$ is an orthogonal projection, and $S_{T_1} = I \oplus S_{Q_1}$.

Theorem

Let $T = (T_0, T_1)$ be a bicontraction on $\mathcal{H}$ such that $\{T_0^n\}_{n\in\mathbb{N}}$ strongly converges. Then $S_{T_0} = S_{T_0}^2$ and $T_1 S_{T_0} = S_{T_0} T_1$. Furthermore, if $(I - T_0)T_1^n h \to 0$ $(n \to \infty)$ for $h \in \mathcal{H}$ then $S_T = S_{T_1}$.

Corollary

If $\{T_i^n\}_{n\in\mathbb{N}}$ strongly converges for $i = 0, 1$ then S_{T_0}, S_{T_1} and S_T are orthogonal projections, and $S_{T_i} T_{1-i} = T_{1-i} S_{T_i}$ for $i = 0, 1$.

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Suppose that $\{T_0^n\}_{n\in\mathbb{N}}$ strongly converges and that $(I - T_1)T_0^n h \to 0$ $(n \to \infty)$ for $h \in \mathcal{H}$. Then $S_T = S_{T_0}$ is an orthogonal projection, and $S_{T_1} = I \oplus S_{Q_1}$.

Theorem

Let $T = (T_0, T_1)$ *be a bicontraction on* $\mathcal{H}$ *such that* $\{T_0^n\}_{n\in\mathbb{N}}$ *strongly converges. Then* $S_{T_0} = S_{T_0}^2$ *and* $T_1 S_{T_0} = S_{T_0} T_1$*. Furthermore, if* $(I - T_0)T_1^n h \to 0$ *(*$n \to \infty$*) for* $h \in \mathcal{H}$ *then* $S_T = S_{T_1}$*.*

Corollary

If $\{T_i^n\}_{n\in\mathbb{N}}$ *strongly converges for* $i = 0, 1$ *then* S_{T_0}*,* S_{T_1} *and* S_T *are orthogonal projections, and* $S_{T_i} T_{1-i} = T_{1-i} S_{T_i}$ *for* $i = 0, 1$*.*

Corollary

Suppose that $\{T_0^n\}_{n\in\mathbb{N}}$ *strongly converges and that* $(I - T_1)T_0^n h \to 0$ $(n \to \infty)$ *for* $h \in \mathcal{H}$*. Then* $S_T = S_{T_0}$ *is an orthogonal projection, and* $S_{T_1} = I \oplus S_{Q_1}$*.*

Theorem

Let $T = (T_0, T_1)$ be a bicontraction on $\mathcal{H}$. Then the sequence $\{T_0^m T_1^n\}_{m,n\in\mathbb{N}}$ strongly converges as $m, n \to \infty$ if and only if $T_i = I \oplus S_i$ $(i = 0, 1)$ relative to an orthogonal decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, such that $S_0^m S_1^n h \to 0$ as $m, n \to \infty$ for any $h \in \mathcal{M}^\perp$. In this case we have $S_T = S_T^2$.

Theorem

Let $T = (T_0, T_1)$ be a bicontraction on $\mathcal{H}$. Then S_T is a compact operator if and only if $T_i = U_i \oplus S_i$ $(i = 0, 1)$ relative to an orthogonal decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ with $\mathcal{M}$ a finite dimensional subspace, such that U_i are unitary operators on $\mathcal{M}$ and $\{S_0^m S_1^n\}_{m,n\in\mathbb{N}}$ strongly converges to 0, (as $m, n \to \infty$) in $\mathcal{B}(\mathcal{M}^\perp)$. In this case, S_T is a finite dimensional orthogonal projection, which commutes with T_0 and T_1.

Theorem

Let $T = (T_0, T_1)$ *be a bicontraction on* $\mathcal{H}$*. Then the sequence* $\{T_0^m T_1^n\}_{m,n\in\mathbb{N}}$ *strongly converges as* $m, n \to \infty$ *if and only if* $T_i = I \oplus S_i$ *(*$i = 0, 1$*) relative to an orthogonal decomposition* $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$*, such that* $S_0^m S_1^n h \to 0$ *as* $m, n \to \infty$ *for any* $h \in \mathcal{M}^{\perp}$*. In this case we have* $S_T = S_T^2$*.*

Theorem

Let $T = (T_0, T_1)$ *be a bicontraction on* $\mathcal{H}$*. Then* S_T *is a compact operator if and only if* $T_i = U_i \oplus S_i$ *(*$i = 0, 1$*) relative to an orthogonal decomposition* $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$ *with* $\mathcal{M}$ *a finite dimensional subspace, such that* U_i *are unitary operators on* $\mathcal{M}$ *and* $\{S_0^m S_1^n\}_{m,n\in\mathbb{N}}$ *strongly converges to* 0*, (as* $m, n \to \infty$*) in* $\mathcal{B}(\mathcal{M}^{\perp})$*. In this case,* S_T *is a finite dimensional orthogonal projection, which commutes with* T_0 *and* T_1*.*

Theorem

Let $T = (T_0, T_1)$ be a bicontraction on $\mathcal{H}$. Then
(i) T is similar to a bi-isometry if and only if S_T is invertible. In this case S_{T_i} is invertible, too, for $i = 0, 1$.
(ii) $\mathcal{R}(S_T)$ is closed if and only if $T^0 = (T_{00}, T_{10})$ is similar to a isometry, where $T_{i0} = P_{\overline{\mathcal{R}(S_T)}} T_i|_{\overline{\mathcal{R}(S_T)}}$, $i = 0, 1$.

Theorem

Let $T = (T_0, T_1)$ and $T' = (T_0', T_1')$ be two bicontractions on $\mathcal{H}$ and $\mathcal{H}'$, respectively. Then an operator $A \in \mathcal{B}(\mathcal{H}', \mathcal{H})$ satisfies $A = T_i^* A T_i'$ for $i = 0, 1$ if and only if there exists an operator $B \in \mathcal{B}(\overline{\mathcal{R}(S_{T'})}, \overline{\mathcal{R}(S_T)})$ such that $A = S_T^{1/2} B S_{T'}^{1/2}$ and $B = V_i^* B V_i'$, where V_i and V_i' are the isometries on $\overline{\mathcal{R}(S_T)}$ and $\overline{\mathcal{R}(S_{T'})}$ respectively, which satisfy the relations $V_i S_T^{1/2} = S_T^{1/2} T_i$ and $V_i' S_{T'}^{1/2} = S_{T'}^{1/2} T_i'$, for $i = 0, 1$. In this case, B is uniquely determined and $||B|| = ||A||$.

Corollary

Under the hypotheses of previous Theorem, if either $S_T = 0$, or $S_{T'} = 0$, then the only operator $A \in \mathcal{B}(\mathcal{H}', \mathcal{H})$ satisfying $A = T_i^* A T_i'$ for $i = 0, 1$ is $A = 0$.

Theorem

Let $T = (T_0, T_1)$ and $T' = (T'_0, T'_1)$ be two bicontractions on $\mathcal{H}$ and $\mathcal{H}'$, respectively. Then an operator $A \in \mathcal{B}(\mathcal{H}', \mathcal{H})$ satisfies $A = T_i^* A T'_i$ for $i = 0, 1$ if and only if there exists an operator $B \in \mathcal{B}(\overline{\mathcal{R}(S_{T'})}, \overline{\mathcal{R}(S_T)})$ such that $A = S_T^{1/2} B S_{T'}^{1/2}$ and $B = V_i^* B V'_i$, where V_i and V'_i are the isometries on $\overline{\mathcal{R}(S_T)}$ and $\overline{\mathcal{R}(S_{T'})}$ respectively, which satisfy the relations $V_i S_T^{1/2} = S_T^{1/2} T_i$ and $V'_i S_{T'}^{1/2} = S_{T'}^{1/2} T'_i$, for $i = 0, 1$. In this case, B is uniquely determined and $||B|| = ||A||$.

Corollary

Under the hypotheses of previous Theorem, if either $S_T = 0$, or $S_{T'} = 0$, then the only operator $A \in \mathcal{B}(\mathcal{H}', \mathcal{H})$ satisfying $A = T_i^* A T'_i$ for $i = 0, 1$ is $A = 0$.

Let $\mathcal{K} \supset \mathcal{H}$ be a Hilbert space. An **isometric dilation** on $\mathcal{K} \supset \mathcal{H}$ of the bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$ is a bi-isometry $V = (V_0, V_1)$ on $\mathcal{K}$ satisfying

$$T_0^m T_1^n = P_{\mathcal{H}} V_0^m V_1^n|_{\mathcal{H}} \quad (m, n \in \mathbb{N}). \tag{9}$$

The dilation V of T is **minimal**, and we denote it by $[\mathcal{K}, V]$, if

$$\mathcal{K} = \bigvee_{m,n \geq 0} V_0^m V_1^n \mathcal{H}. \tag{10}$$

The existence of such a dilation was firstly proved by Ando, but it also follows from the commutant dilation Nagy-Foiaş's theorem.

An isometric dilation $V = (V_0, V_1)$ of $T = (T_0, T_1)$ is **regular** if

$$T_0^{*m} T_1^n = P_{\mathcal{H}} V_0^{*m} V_1^n|_{\mathcal{H}} \quad (m, n \in \mathbb{N}). \tag{11}$$

The minimal regular isometric dilation of T is uniquely determined up to a unitary equivalence.

We can use the operators S_{T_0}, S_{T_1} and S_T in order to obtain an isometric dilation of a bicontraction $T = (T_0, T_1)$ satisfying the condition

$$\Delta_T^2 := I - T_0^* T_0 - T_1^* T_1 + T_1^* T_0^* T_0 T_1 \geq 0,$$

which means T has a regular dilation.
We remark that $\Delta_T^2 = D_{T_0}^2 - T_1^* D_{T_0}^2 T_1$, where
$D_{T_i} = (I - T_i^* T_i)^{1/2}$ is the defect operator of T_i, $i = 0, 1$.

$$||h||^2 = \sum_{m,n=0}^{\infty} ||\Delta_T T_0^m T_1^n h||^2 + ||S_{T_0}^{1/2} h||^2 + ||S_{T_1}^{1/2} h||^2 - ||S_T^{1/2} h||^2 \tag{12}$$

$$= \sum_{m,n=0}^{\infty} ||\Delta_T T_0^m T_1^n h||^2 + ||(S_{T_0} - \frac{1}{2} S_T)^{1/2} h||^2 + ||(S_{T_1} - \frac{1}{2} S_T)^{1/2} h||^2.$$

Denote $\mathcal{D}_T = \overline{\Delta_T \mathcal{H}}$ and let $\mathcal{H}_T = \bigoplus_{m,n\in\mathbb{Z}} \mathcal{D}_T^{(m,n)}$ be the Hilbert space of all sequences $\{h_{m,n}\}_{m,n\in\mathbb{Z}}$ with $h_{m,n} \in \mathcal{D}_T^{(m,n)}$ and

$$\sum_{m,n\in\mathbb{Z}} ||h_{m,n}||^2 < \infty.$$

The space $\mathcal{H}$ can be isometrically embedded in the space

$$\mathcal{G} = \mathcal{H}_T \oplus \overline{\mathcal{R}(S_{T_0})} \oplus \overline{\mathcal{R}(S_{T_1})}$$

by identifying the element h of $\mathcal{H}$ with the element $\widetilde{h} \oplus (S_{T_0} - \frac{1}{2}S_T)^{1/2}h \oplus (S_{T_1} - \frac{1}{2}S_T)^{1/2}h$ of $\mathcal{G}$, where $\widetilde{h} = \{\widetilde{h}_{m,n}\}_{m,n\in\mathbb{Z}}$ such that

$$\widetilde{h}_{m,n} = \begin{cases} \Delta_T T_0^m T_1^n h, \text{ if } m,n \geq 0 \\ 0, \text{ if } m < 0, \text{ or } n < 0. \end{cases}$$

Now we can define an isometry W_i on $\overline{\mathcal{R}(S_{T_i})}$ by

$$W_i(S_{T_i} - \frac{1}{2}S_T)^{1/2}h = (S_{T_i} - \frac{1}{2}S_T)^{1/2}T_i h, \quad h \in \mathcal{H},$$

because T_i is a S_{T_i}-isometry and also, a S_T-isometry. Similarly, since T_{1-i} is a S_{T_i}-contraction, we can define a contraction $\widetilde{T}_{1-i}$ on $\overline{\mathcal{R}(S_{T_i})}$ by

$$\widetilde{T}_{1-i}(S_{T_i} - \frac{1}{2}S_T)^{1/2}h = (S_{T_i} - \frac{1}{2}S_T)^{1/2}T_{1-i}h, \quad h \in \mathcal{H}.$$

In addition, we have

$$W_i\widetilde{T}_{1-i} = \widetilde{T}_{1-i}W_i$$

because $T_iT_{1-i} = T_{1-i}T_i$, for $i = 0, 1$.

Let $[\mathcal{K}_i, \widetilde{V}_i]$ be the minimal isometric dilation of $\widetilde{T}_i$ and $\widetilde{W}_{1-i}$ be the isometric extension of W_{1-i} on $\mathcal{K}_i$ such that

$$\widetilde{W}_{1-i}\widetilde{V}_i = \widetilde{V}_i\widetilde{W}_{1-i},$$

for $i = 0, 1$.
Let $S_i \in \mathcal{B}(\mathcal{H}_T)$ be the bilateral shift defined by

$$S_0\{h_{m,n}\} = \{h_{m+1,n}\}, \quad S_1\{h_{m,n}\} = \{h_{m,n+1}\} \tag{13}$$

if $\{h_{m,n}\} \in \mathcal{H}_T$. Clearly, S_i is unitary and $S_0S_1 = S_1S_0$.
Consider the isometries V_0 and V_1 on the Hilbert space

$$\mathcal{K} = \mathcal{H}_T \oplus \mathcal{K}_1 \oplus \mathcal{K}_0$$

given by

$$V_0 = S_0 \oplus \widetilde{W}_0 \oplus \widetilde{V}_0, \quad V_1 = S_1 \oplus \widetilde{V}_1 \oplus \widetilde{W}_1. \tag{14}$$

Theorem

If $T = (T_0, T_1)$ is a bicontraction on $\mathcal{H}$ such that $\Delta_T^2 \geq 0$, then the bi-isometry $V = (V_0, V_1)$ on $\mathcal{K}$ given by (14) is an isometric dilation of T.

Theorem

Let $T = (T_0, T_1)$ be a bicontraction on $\mathcal{H}$ such that T_0 is normal and $S_{T_1} = S_{T_1}^2$. Then the isometric dilation of T given by (14) is regular.

Theorem

If $T = (T_0, T_1)$ is a bicontraction on $\mathcal{H}$ such that $\Delta_T^2 \geq 0$, then the bi-isometry $V = (V_0, V_1)$ on $\mathcal{K}$ given by (14) is an isometric dilation of T.

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Let $T = (T_0, T_1)$ be a bicontraction on $\mathcal{H}$ such that T_0 is normal and $S_{T_1} = S_{T_1}^2$. Then the isometric dilation of T given by (14) is regular.

Theorem

Let $T = (T_0, T_1)$ be a bidisk isometry on $\mathcal{H}$. Then
(i) $[\mathcal{K}, V]$ is a minimal isometric dilation of T, where $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_0$ and $V = (V_0, V_1)$ is given by (14).
(ii) If $S_{T_0} = S^2_{T_0}$ then we have $S_T = S^2_T$ if and only if $S_{T_1} = S^2_{T_1}$.
(iii) If $S_{T_i} = S^2_{T_i}$ for $i = 0, 1$ then $[\mathcal{K}, V]$ is the minimal regular isometric dilation of T.