Quasianalytic contractions and function algebras

László Kérchy

$\mathcal{H}$ complex Hilbert space, $\dim \mathcal{H} = \aleph_0$

$\mathcal{L}(\mathcal{H})$ bounded linear operators on $\mathcal{H}$

$\mathcal{S} \subset \mathcal{L}(\mathcal{H})$, $\quad \text{Lat}\, \mathcal{S} := \{\mathcal{M} \text{ subspace} : C\mathcal{M} \subset \mathcal{M} \ \ \forall C \in \mathcal{S}\}$

$T \in \mathcal{L}(\mathcal{H})$

$$\mathcal{W}(T) := \{p(T) : p \text{ polynomial}\}^-_{\text{WOT}}$$

$$\mathcal{R}(T) := \{q(T) : q \text{ rational function}\}^-_{\text{WOT}}$$

$$\{T\}' := \{C \in \mathcal{L}(\mathcal{H}) : CT = TC\}$$

$$\mathcal{W}(T) \subset \mathcal{R}(T) \subset \{T\}'$$

$$\text{Lat}\, T \supset \text{Rlat}\, T \supset \text{Hlat}\, T$$

(HSP) *Is it true that* $\text{Hlat}\, T \neq \big\{\{0\}, \mathcal{H}\big\}$ *whenever* $T \neq cI$?

We may assume: $\|T\| \leq 1$

$$\mathcal{H}_0(T) := \{x \in \mathcal{H} : \lim_{n\to\infty} \|T^n x\| = 0\} \in \text{Hlat}\, T$$

Sz.-Nagy–Foias classification:

$$T \in C_{0\cdot} \;\text{ if }\; \mathcal{H}_0(T) = \mathcal{H}$$

$$T \in C_{1\cdot} \;\text{ if }\; \mathcal{H}_0(T) = \{0\}$$

$$T \in C_{\cdot 0} \;\text{ if }\; \mathcal{H}_0(T^*) = \mathcal{H}$$

$$T \in C_{\cdot 1} \;\text{ if }\; \mathcal{H}_0(T^*) = \{0\}$$

$$C_{ij} = C_{i\cdot} \cap C_{\cdot j}$$

(HSP) for C_{00} is equivalent to the general (HSP)

(HSP) for the complement of C_{00} can be reduced to C_{10}

(HSP)* *Is it true for every* $T \in C_{10}$ *that* $\text{Hlat}\, T \neq \big\{\{0\}, \mathcal{H}\big\}$?

Typical example for a C_{10}-contraction:

$S \in \mathcal{L}(H^2),\ Sf = \chi f$ unilateral shift $(\chi(\zeta) = \zeta)$

$H_{\mathrm{i}}^{\infty} = \{\vartheta \in H^{\infty} : |\vartheta(\zeta)| = 1$ for a.e. $\zeta \in \mathbb{T}\}$

inner functions

Beurling Theorem:

$$\operatorname{Lat} S = \left\{\vartheta H^2 : \vartheta \in H_{\mathrm{i}}^{\infty}\right\}$$

$$\vartheta_1 H^2 = \vartheta_2 H^2 \iff \exists \kappa \in \mathbb{T},\ \vartheta_2 = \kappa \vartheta_1$$

$T \in C_{10}$

$W \in \mathcal{L}(\mathcal{K})$ unitary operator

$X \in \mathcal{L}(\mathcal{H}, \mathcal{K}),\ \|X\| \leq 1$ and $XT = WX$

(X, W) is a *unitary asymptote* of T, if

$$\forall (X', W'),\ \exists! Y \in \mathcal{L}(\mathcal{K}, \mathcal{K}'),\ \|Y\| \leq 1,$$
$$X' = YX \text{ and } YW = W'Y$$

Exists uniquely up to isomorphism.

$\gamma \colon \{T\}' \to \{W\}',\ C \mapsto D$, where $XC = DX$

contractive algebra-homomorphism

W absolutely continuous unitary operator

$\delta\colon \mathbb{T} \to \mathbb{N} \cup \{0, \infty\}$ spectral-multiplicity function

$$\forall n \in \mathbb{N}, \quad \omega(T, n) := \big\{\zeta \in \mathbb{T} : \delta(\zeta) \geq n\big\}$$

$\omega(T) := \omega(T, 1)$ is the *residual set* of T

Theorem 1. *If* $\omega(T) = \mathbb{T}$, *then* $\forall \varepsilon > 0$, $\vee \mathrm{Lat}(T, \varepsilon) = \mathcal{H}$,

where

$$\mathrm{Lat}(T, \varepsilon) = \big\{\mathcal{M} \in \mathrm{Lat}\ T : \exists Q \in \mathcal{L}(\mathcal{M}, H^2) \ \textit{invertible}, \\ Q(T|\mathcal{M}) = SQ,\ \|Q\|\|Q^{-1}\| < 1 + \varepsilon\big\}.$$

Sz.-Nagy–Foias calculus:

$$\Phi_T\colon H^\infty \to \mathcal{L}(\mathcal{H}),\ \vartheta \mapsto \vartheta(T) \ \text{ algebra-hom.}$$

$\vartheta_2 \prec \vartheta_1$ if $|\vartheta_2(z)| \le |\vartheta_1(z)|\ \forall z \in \mathbb{D}$

$A_2 \prec A_1$ if $\|A_2 x\| \le \|A_1 x\|\ \forall x \in \mathcal{H}$

$$\vartheta_2 \prec \vartheta_1 \implies \vartheta_2(T) \prec \vartheta_1(T)$$

$\Theta = \{\vartheta_n\}_{n=1}^\infty$ decreasing, $G_\Theta(\zeta) := \lim_{n\to\infty} |\vartheta_n(\zeta)|\ (\zeta \in \mathbb{T})$

$\implies \{\vartheta_n(T)\}_{n=1}^\infty$ decreasing,

$$\mathcal{H}_0(T,\Theta) := \left\{x \in \mathcal{H} : \lim_{n\to\infty} \|\vartheta_n(T)x\| = 0\right\}$$

T is non-vanishing on the set $\alpha \subset \mathbb{T}$, if

$$\mathcal{H}_0(T,\Theta) = \{0\} \ \text{ whenever } \chi_\alpha G_\Theta \not\equiv 0$$

$\pi(T)$ is the largest Borel set where T is non-vanishing

is called the *quasianalytic spectral set* of T

Theorem 2. (a) $\pi(T) \subset \omega(T)$

(b) *If* $\pi(T) \neq \omega(T)$, *then* Hlat T *is non-trivial.*

We may assume that T is a *quasianalytic contraction*:

$$\pi(T) = \omega(T).$$

Let us assume: W is *cyclic*, that is $\omega(T, 2) = \emptyset$.

$$\mathcal{L}_0(\mathcal{H}) := \left\{ T \in C_{10}(\mathcal{H}) : \pi(T) = \omega(T) \text{ and } \omega(T, 2) = \emptyset \right\}$$

$$\mathcal{L}_1(\mathcal{H}) := \left\{ T \in \mathcal{L}_1(\mathcal{H}) : \omega(T) = \mathbb{T} \right\}$$

Theorem 3. *If* $T \in \mathcal{L}_0(\mathcal{H})$ *and* $\omega(T)$ *contains an arc,*

$$\textit{then } \exists\, \widetilde{T} \in \mathcal{L}_1(\mathcal{H}),\ \{T\}' = \{\widetilde{T}\}'.$$

Examples for operators in $\mathcal{L}_1(\mathcal{H})$:

(a) $S \in \mathcal{L}(H^2),\ Sf = \chi f$ unilateral shift

(b) $T \in C_{10}$ and $d_{T^*} = d_T + 1 < \infty$, where

$$d_T = \text{rank}(I - T^*T)$$

(c) $T \in C_{10},\ d_T = \infty,\ \dim\ker T^* < \infty$,

$$\exists \Psi\colon \mathbb{D} \to \mathcal{L}(\mathcal{D}_{T^*}, \mathcal{D}_T) \text{ bounded, analytic,}$$

$$\exists \delta \in H^\infty \setminus \{0\},\ \Psi\Theta_T = \delta I.$$

(d) Cyclic subspaces of orthogonal sums of operators in $\mathcal{L}_1$.

We assume: $T \in \mathcal{L}_1(\mathcal{H})$

$\phi: L^\infty(\mathbb{T}) \to \{W\}',\ f \mapsto f(W)$

$\varphi := \phi^{-1}: \{W\}' \to L^\infty$ isometric algebra-isomorphism

$\widehat{\gamma}_T := \varphi \circ \gamma: \{T\}' \to L^\infty$ contractive alg.-hom., injective

independent of the choice of (X, W)

$\forall \vartheta \in H^\infty,\ \widehat{\gamma}_T(\vartheta(T)) = \vartheta$

$\mathcal{F}(T) :=$ ran $\widehat{\gamma}_T$ is a subalgebra of L^∞,

$\mathcal{F}(T) \supset H^\infty$ *functional commutant* of T

$\widehat{\Phi}_T: \mathcal{F}(T) \to \{T\}',\ f \mapsto f(T)$, where $\widehat{\gamma}_T(f(T)) = f$

algebra-isomorphism, extension of Φ_T

Theorem 4. $\mathcal{F}(T)$ *is a quasianalytic function algebra*:

$$\forall f \in \mathcal{F}(T) \setminus \{0\},\ f(\zeta) \neq 0 \ \text{for a.e. } \zeta \in \mathbb{T}.$$

Proof. $f \in \mathcal{F}(T) \setminus \{0\}$

$\exists C \in \{T\}',\ \widehat{\gamma}_T(C) = f$

$0 \neq v \in \mathcal{H}$

T quasianalytic $\implies Xv$ is cyclic for $\{W\}'$

W cyclic $\implies Xv$ is separating for $\{W\}'$

$XCv = f(W)Xv \neq 0 \implies Cv \neq 0$

$\implies f(W)Xv = XCv$ is cyclic for $\{W\}$;

$\implies f(\zeta) \neq 0$ for a.e. $\zeta \in \mathbb{T}$

F. & M. Riesz Theorem $\Longrightarrow$ H^∞ is quasianalytic

$\mathcal{B} \subset H^\infty_{\mathrm{i}}$, $\big[\overline{\mathcal{B}}, H^\infty\big]_0 :=$ is the algebra, generated by $\overline{\mathcal{B}} \cup H^\infty$

quasianalytic, *pre-Douglas algebra*

$\big[\overline{\mathcal{B}}, H^\infty\big] :=$ closure of $\big[\overline{\mathcal{B}}, H^\infty\big]_0$, *Douglas algebra*

Chang–Marshall Theorem.

Every closed algebra $H^\infty \subset \mathcal{A} \subset L^\infty$ *is a Douglas algebra.*

$\mathcal{A} \neq H^\infty$ Douglas algebra $\Longrightarrow$ $\mathcal{A} \supset [\overline{\chi}, H^\infty] = C(\mathbb{T}) + H^\infty$

$\Longrightarrow$ $\mathcal{A}$ is not quasianalytic

$A_T := X^*X = \lim_{n\to\infty} T^{*n}T^n$

$L_T\colon \{T\}' \to \mathcal{L}(\mathcal{H}),\ C \mapsto A_T C$

Theorem 5. *TFAE*:

(a) $\mathcal{F}(T)$ *is a Douglas algebra*

(b) $\widehat{\Phi}_T$ *is bounded* (c) $\widehat{\Phi}_T$ *is an isometry*

(d) L_T *is bounded from below* (e) L_T *is an isometry*

(f) $\mathcal{F}(T) = H^\infty$

(g) $\{T\}' = H^\infty(T)$

Then $\mathrm{Hlat}\ T = \mathrm{Lat}\ T$ *is rich.*

$T \in \mathcal{L}_1(\mathcal{H}),\ T \prec S \implies \mathcal{F}(T)$ Douglas algebra

Proposition 6. $\sigma(T) = \mathbb{D}^- \iff \mathcal{F}(T) \cap \overline{H^\infty_{\mathrm{i}}} = \mathbb{T} \cdot 1$

Theorem 7. (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c), *where*

(a) $\mathcal{F}(T)$ *is a pre-Douglas algebra*

(b) $\{T\}' = \mathcal{R}(T)$

(c) Hlat $T =$ Rlat T

Sketch of the proof.

Suppose $\mathcal{F}(T) \neq H^\infty$ is a pre-Douglas algebra

$\mathcal{B} := \{\eta \in H^\infty_{\mathrm{i}} : \overline{\eta} \in \mathcal{F}(T)\}$, $[\overline{\mathcal{B}}, H^\infty]_0 = \mathcal{F}(T)$

Question: $\eta \in \mathcal{B} \Longrightarrow \overline{\eta}(T) \in \mathcal{R}(T)$?

Easily verified if $\eta = \prod_{n=1}^{N} b_{a_n}$ is a finite Blaschke product.

$$\left(b_0(z) = z, \quad b_a(z) = -\tfrac{\overline{a}}{|a|}\tfrac{z-a}{1-\overline{a}z} \text{ for } a \in \mathbb{D} \setminus \{0\} \right)$$

Assume $\eta = \prod_{n=1}^{\infty} b_{a_n}$ infinite Blaschke product.

For $N \in \mathbb{N}$, $B_N = \prod_{n=1}^{N} b_{a_n}$.

$$\overline{B_N} = \overline{\eta} \prod_{n=N+1}^{\infty} b_{a_n} \in \mathcal{F}(T) \implies \overline{B_N}(T) \in \mathcal{R}(T)$$

$B_N \to \eta$ weak-$*$ $\implies B_N(T) \to \eta(T)$ WOT $(N \to \infty)$

$$\implies \|B_N(T)^{-1}\| \leq \|\eta(T)^{-1}\| = \|\overline{\eta}(T)\| \ \forall N \in \mathbb{N}$$

$\exists \{B_{N_j}\}_{j=1}^{\infty}$ subsequence:

$B_{N_j}(T)^{-1} \to C$ WOT, and

$B_{N_j}(\zeta) \to \eta(\zeta)$ for a.e. $\zeta \in \mathbb{T}$.

$$\implies B_{N_j}(T) \to \eta(T) \text{ SOT}$$

$\implies I = B_{N_j}(T)^{-1} B_{N_j}(T) \to C\eta(T)$ WOT

$$\implies \overline{\eta}(T) = \eta(T)^{-1} = C \in \mathcal{R}(T).$$

Suppose $\eta \in \mathcal{B}$ arbitrary non-constant.

Frostman Theorem $\implies \exists a \in \mathbb{D}, 0 < |a| < 4^{-1}\|\overline{\eta}(T)\|^{-1}$,

$b_a \circ \eta = b$ is a Blaschke product.

$H^\infty \subset \mathcal{A} \subset L^\infty$ is a *generalized Douglas algebra*, if

$$f \in \mathcal{A},\ \lambda \in \mathbb{C},\ |\lambda| > \|f\|_\infty \implies \tfrac{1}{f-\lambda} \in \mathcal{A}$$

Tolokonnikov's Theorem.

If $\mathcal{A} \neq H^\infty$ is a generalized Douglas algebra, then $\overline{\chi} \in \mathcal{A}$.

Theorem 8.

(a) $\mathcal{F}(T)$ *is a generalized Douglas algebra*

$$\iff \ \forall f \in \mathcal{F}(T),\ r(f(T)) = \|f\|_\infty$$

(b) $\mathcal{F}(T) \neq H^\infty$ *is a generalized Douglas alg.* $\implies \sigma(T) = \mathbb{T}$.

Questions.

(a) *What are the quasianalytic function algebras in* L^∞*?*

(b) *Which quasianalytic function algebras can be attained as a*

functional commutant $\mathcal{F}(T)$ *of some* $T \in \mathcal{L}_1(\mathcal{H})$*?*

(c) *Which pre-Douglas algebras are attainable?*

(d) *Which generalized Douglas algebras are attainable?*

(e) *What is the spectral behaviour of the contractions in* $\mathcal{L}_1(\mathcal{H})$*?*

$A \subset \mathbb{D}, \quad \mathcal{B}_A = \{b_a : a \in A\}$

$$\forall a \in \mathbb{D},\ \overline{b_a} \in \overline{\mathcal{B}_A}, H^\infty]_0 \iff a \in A$$

If $[\overline{\mathcal{B}_A}, H^\infty]_0 = \mathcal{F}(T)$ for some $T \in \mathcal{L}_1(\mathcal{H})$, then

$\sigma(T) \cap \mathbb{D} = \mathbb{D} \setminus A$, and so A must be open.

Example

$$G = \left\{ re^{it} : \sqrt{\delta} < r < 1,\ 0 < t < \pi \right\} \quad (0 < \delta < 1)$$

Riemann & Charathéodory Theorems $\Longrightarrow$

$$\exists \vartheta_0 \colon \mathbb{D}^- \to G^- \ \text{ homeomoprhism,}$$

$$\vartheta_0 | \mathbb{D} \ \text{ analytic, } \ \vartheta_0(\zeta) = \zeta \ \text{ for } \ \zeta = 1, i, -1.$$

$T = \vartheta(S) \in \mathcal{L}(H^2)$, where $\vartheta = \vartheta_0^2$

Spectral mapping theorems $\Longrightarrow$

$$T \in \mathcal{L}_1(H^2),$$

$$\sigma(T) = \vartheta(\mathbb{D})^- = \left\{ re^{it} : \delta \le r \le 1, 0 \le t \le 2\pi \right\}$$

$$\mathcal{F}(T) = \left\{ g \in L^\infty : g \circ (\vartheta | \mathbb{T}_+) = h | \mathbb{T}_+ \ \text{ for some } \ h \in H^\infty \right\}.$$