OPERATOR REPRESENTATIONS OF WEAK*- DIRICHLET ALGEBRAS

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Outline

Introduction and preliminaries

- Let X be a compact Hausdorff space and $C(X)$ the Banach algebra of all complex continuous functions on X;
- Denote by A a function algebra on X, $\mathcal{M}(A)$ stands for the set of all non zero complex homomorphisms of A;
- For $\gamma \in \mathcal{M}(A)$, A_γ means the kernel of γ, and M_γ designates the set of all representing measures m for γ, that is m is a probability Borel measure on X satisfying $\gamma(f) = \int f dm$, $f \in A$.

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- Let A be a function algebra on X which is *weak*-Dirichlet* in $L^\infty(m)$, that is $A + \overline{A}$ is weak* dense in $L^\infty(m)$, for some fixed $m \in M_\gamma$ and $\gamma \in \mathcal{M}(A)$.
- Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the Banach algebra of all bounded linear operators on $\mathcal{H}$.
- Any bounded linear and multiplicative map Φ of A in $\mathcal{B}(\mathcal{H})$ with $\Phi(1) = I$ (the identity operator on $\mathcal{H}$) is called a *representation* of A on $\mathcal{H}$. When $\|\Phi\| \leq 1$ one says that Φ is *contractive*. Here, we only consider representations Φ for which there exist a scalar $\rho > 0$ and a system $\{\mu_x\}_{x \in \mathcal{H}}$ of positive measures on X with $\|\mu_x\| = \|x\|^2$ such that

$$\langle \Phi(f)x, x\rangle = \int [\rho f + (1-\rho)\gamma(f)]d\mu_x$$

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- $$w(\Phi(f)) \leq \|\rho f + (1-\rho)\gamma(f)\| \qquad (f \in A) \qquad (1.1)$$
- If the representation Φ of A on $\mathcal{H}$ admits a system $\{\mu_x\}_{x\in\mathcal{H}}$ of weak $\rho-$spectral measures attached by γ such that μ_x is $m-a.c.$ for any $x \in \mathcal{H}$, then Φ has a $\gamma-$spectral $\rho-$dilation, that is there exists a contractive representations $\widetilde{\Phi}$ of $C(X)$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ satisfying the relation
$$\Phi(f) = \rho P_{\mathcal{H}}\widetilde{\Phi}(f)|\mathcal{H} \qquad (f \in A_\gamma), \qquad (1.2)$$
where $P_{\mathcal{H}}$ is the orthogonal projection on $\mathcal{H}$. Moreover, in this case there exists a unique semispectral measure $F_\Phi : Bor(X) \to \mathcal{B}(\mathcal{H})$ such that $\langle F_\Phi(\cdot)x, x\rangle = \mu_x$, or equivalently
$$\langle\Phi(f)x, y\rangle = \int [\rho f + (1-\rho)\gamma(f)]d\langle F_\Phi x, y\rangle \qquad (f \in A), \qquad (1.3)$$

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Extension of a representation to the space $L^p(m)$

We characterize below some representations Φ of A on $\mathcal{H}$ which can be linearly and boundedly extended to the space $L^p(m)$ for $1 \leq p \leq \infty$. Our characterization is given in the terms of Radon-Nikodym derivative with respect to m of the corresponding $\mathcal{B}(\mathcal{H})-$valued semispectral measure F_Φ. In the sequel we put $\varphi_{x,y} = \langle F_\Phi(\cdot)x, y\rangle \in L^1(m)$ for $x, y \in \mathcal{H}$.

Extension of a representation to the space $L^p(m)$

Theorem 2.1

Let Φ be a representation of A on $\mathcal{H}$ which admits a system of $m - a.c.$ weak $\rho-$spectral measures attached by γ. Then Φ has a bounded linear extension Φ_p from $L^p(m)$ into $\mathcal{B}(\mathcal{H})$ for $1 \leq p \leq \infty$, if and only if $\varphi_{x,y} \in L^q(m)$ and there exists a constant $c > 0$ such that

$$||\varphi_{x,y}||_q \leq c||x|| \, ||y|| \qquad (x, y \in \mathcal{H}), \tag{2.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. In this case, Φ_p is uniquely determined and it satisfies for $h \in L^p(m)$ and $x, y \in \mathcal{H}$ the relation

$$\langle \Phi_p(h)x, y\rangle = \int [\rho h + (1-\rho)\int hdm]\varphi_{x,y}dm. \tag{2.2}$$

Extension of a representation to the space $L^p(m)$

Theorem 2.1

Furthermore, for $h \in L^2(m)$ and $x \in \mathcal{H}$ we have the inequality

$$\|\Phi_2(h)x\|^2 \leq \int |\rho h + (1-\rho)\int h dm|^2 \varphi_{x,x} dm. \tag{2.3}$$

Hence, if $\{h_\alpha\} \subset L^\infty(m)$ is a bounded net such that $\{h_\alpha\}$ converges a.e.(m) to $h \in L^\infty(m)$, then $\{\Phi_p(h_\alpha)\}$ strongly converges to $\Phi_p(h)$ in $\mathcal{B}(\mathcal{H})$, for $p \geq 2$.

Extension of a representation to the space $L^p(m)$

Remark 2.2

The equivalent conditions of Theorem 2.1 imply

$$\|\Phi\|_p := \sup_{f\in A, \|f\|_p\leq 1} \|\Phi(f)\| < \infty. \tag{2.4}$$

It is easy to see that the condition (2.4) is equivalent to the existence of a bounded linear extension $\widehat{\Phi}_p$ of Φ to $H^p(m)$, where $H^p(m)$ is the closure of A into $L^p(m)$. In this case, $\widehat{\Phi}_p$ is the uniquely determined and it satisfies the relation (2.2) for $g \in H^p(m)$.

Extension of a representation to the space $L^p(m)$

Proposition 2.3

Let Φ be a representation of A on $\mathcal{H}$ as in Theorem 2.1 such that $\|\Phi\|_p < \infty$. Then

$$\widehat{\Phi}_p(fg) = \widehat{\Phi}_p(f)\widehat{\Phi}_p(g) \qquad (f \in H^\infty(m), \;\; g \in H^p(m)) \tag{2.5}$$

and, in particular, $\widehat{\Phi} := \widehat{\Phi}_p | H^\infty(m)$ is a representation of $H^\infty(m)$ on $\mathcal{H}$. Moreover, if $\{f_\alpha\} \subset H^\infty(m)$ is a bounded net which converges a.e.(m) to $f \in H^\infty(m)$, then $\{\widehat{\Phi}(f_\alpha)\}$ strongly converges to $\widehat{\Phi}(f)$ in $\mathcal{B}(\mathcal{H})$.

Extension of a representation to the space $L^p(m)$

Remark 2.4

If the representation Φ in Theorem 2.1 is contractive, that is $\rho = 1$ and $\|\Phi\| = 1$ (because $\Phi(1) = I$), then it extension Φ_p is also contractive, in the case when it exists. Indeed, if $\widetilde{\Phi}$ is as in the proof of Theorem 2.1, we have for $f \in A$, $g \in A_\gamma$ and $x, y \in \mathcal{H}$,

$$\begin{aligned} \left| \int (f + \overline{g})\varphi_{x,y} dm \right| &= |\langle (\Phi(f) + \Phi(g)^*)x, y\rangle| = |\langle P_{\mathcal{H}}\widetilde{\Phi}(f + \overline{g})x, y\rangle| \\ &\leq \|\widetilde{\Phi}(f + \overline{g})\|\|x\|\|y\| \leq \|f + \overline{g}\|\|x\|\|y\|, \end{aligned}$$

because $\widetilde{\Phi}$ is a contractive representation of $C(X)$.

Reduction to functional calculus

In the sequel we denote by $H_0^p(m)$ the closure (weak*, if $p = \infty$) of A_γ in $L^p(m)$, that is

$$H_0^p(m) = \left\{ f \in H^p(m) : \int fdm = 0 \right\}.$$

We say ([Na], [Sr]) that $H_0^p(m)$ is *simply invariant* if the closure of $A_\gamma H_0^p(m)$ in $L^p(m)$ is strictly contained into $H_0^p(m)$.
If $H_0^p(m)$ is simply invariant then there exists a function $Z \in H_0^\infty(m)$ with $|Z| = 1$ a.e.(m) such that $H_0^p(m) = ZH^p(m)$.
As in Theorem 3 [Lu] one can prove that, if m_0 is the normalized Lebesgue measure on $\mathbb{T}$, there exists an isometric $*-$isomorphism τ of $L^p(m_0)$ onto a closed subspace of $L^p(m)$, taking $H^p(m_0)$ onto a closed subspace of $H^p(m)$, for $1 \leq p \leq \infty$.

Reduction to functional calculus

The following main result shows that under the simple invariance of $H_0^p(m)$ with $1 \leq p \leq 2$, the representations from Theorem 2.1 and the extensions to $H^p(m)$ can be reduced to functional calculus. We define $S : H^p(m) \to L^p(m)$

$$Sg = \overline{Z}(g - \int gdm) \qquad (g \in H^p(m)). \tag{3.1}$$

Also, for $T \in \mathcal{B}(\mathcal{H})$ we denote by $r(T)$ the spectral radius of T.

Reduction to functional calculus

Theorem 3.1

Suppose that $H_0^p(m)$ is a simply invariant subspace for $1 \leq p < \infty$, and let Φ be a representation of A on $\mathcal{H}$ satisfying Theorem 2.1. Then $r(\widehat{\Phi}(Z)) < 1$, and if $1 \leq p \leq 2$ one has

$$\widehat{\Phi}_p(g) = \sum_{n=0}^{\infty} \widehat{g}(n)\widehat{\Phi}(Z)^n \qquad (g \in H^p(m)) \tag{3.2}$$

where $\widehat{g}(n) = \int \overline{Z}^n g dm$ for $n \in \mathbb{N}$, the series being absolutely convergent in $\mathcal{B}(\mathcal{H})$.
Moreover, the relation (3.2) is also true when $2 < p < \infty$, for $g \in H^p(m)$ such that $\{S^n g\}$ is a bounded sequence in $H^p(m)$, S being the operator from (3.1).

Reduction to functional calculus

Remark 3.2

By

$$g = \sum_{j=0}^{n} \widehat{g}(j)Z^j + Z^{n+1}(S^{n+1}g) \qquad (g \in H^p(m)). \tag{3.3}$$

we have that the sequence $\{S^n g\}_n$ is bounded if and only if the sequence $\{\sum_{j=0}^{n} \widehat{g}(j)Z^j\}_n$ is bounded in $H^p(m)$, and in particular, this happens if S is a power bounded operator in $\mathcal{B}(H^p(m))$. But, even if the second sequence before converges, it limit is not necessary the function g. In fact, one has (by (3.3)) $g = \sum_{j=0}^{\infty} \widehat{g}(j)Z^j$ in $H^p(m)$ if and only if $S^n g \to 0$ $(n \to \infty)$.

Reduction to functional calculus

Theorem 3.4

Suppose $1 \leq p \leq 2$ and that $H_0^p(m)$ is a simply invariant subspace in $H^p(m)$. Let Φ be a representation of A on $\mathcal{H}$ satisfying Theorem 2.1. Then the semispectral measure F_Φ has the form $F_\Phi = \theta(\cdot)m$ where the function $\theta : X \to \mathcal{B}(\mathcal{H})$ is given by

$$\theta(s) = \sum_{n=-\infty}^{\infty} \overline{Z}^n(s)\widehat{\Phi}(Z)_\rho^{(n)}, \tag{3.6}$$

while the series converges absolutely and uniformly a.e.(m) for $s \in X$. Moreover, θ is a bounded function $a.e.(m)$ on X.

Reduction to functional calculus

From this theorem it follows that, for Φ as in Theorem 2.1, the $L^q(m)-$ boundedness of $\varphi_{x,y}$ in the sense of (2.1) for any $x, y \in \mathcal{H}$ and some q in the range $2 \leq q \leq \infty$, is equivalent to the fact that the Radon-Nikodym derivative of F_Φ is a bounded function a.e.(m) on X, if $H_0^p(m)$ is simply invariant. In this last case, Φ can be extended to whole $L^1(m)$ as in Theorem 2.1 and one has $\Phi_p = \Phi_1|L^p(m)$ for $1 < p \leq \infty$. Moreover, if $1 \leq p \leq r \leq \infty$ then $\widehat{\Phi}_r = \widehat{\Phi}_p|H^r(m)$.

Application to the scalar case

In this section we consider the case when Φ is an homomorphism of A, this is the one-dimensional case $\mathcal{H} = \mathbb{C}$. In this context, we generalize to weak* Dirichlet algebra some classical results concerning the function algebra with the uniqueness property for representing measures ([Gam], [SI])

Theorem 4.1

Suppose that $H_0^p(m)$ is a simply invariant subspace for some $p \in [1,2]$. Then for any homomorphism $\varphi \in \mathcal{M}(A)$ with $\|\varphi\|_p < \infty$ we have $|\widehat{\varphi}(Z)| < 1$ and

$$\varphi_p(g) = \sum_{n=0}^{\infty} \widehat{g}(n)\widehat{\varphi}(Z)^n \qquad (g \in H^p(m)) \tag{4.1}$$

where φ_p respectively $(\widehat{\varphi})$ is the bounded linear extension of φ to $H^p(m)$ (respectively, to $H^\infty(m)$), the series being absolutely convergent. Moreover, the measure

$$\mu = \frac{1 - |\varphi(Z)|^2}{|Z - \varphi(Z)|^2} m \tag{4.2}$$

is a representing measure for φ.

Application to the scalar case

Remark that only boundedness of φ on $H^p(m)$ assures that φ is $m-a.c.$ that is φ has a $m-a.c.$ representing measure, if $H^p_0(m)$ is simply invariant. In the general setting of Theorem 3.1, we cannot prove $r(\widehat{\Phi}(Z)) < 1$ without to suppose that Φ is $m-a.c.$ Concerning the existence of homomorphism of A which are bounded on $H^p(m)$, we give the following result which generalize Theorem 6.4 [S] (or Theorem V 7.1, and Theorem VI 7.2 of [CS]) in the context of weak* Dirichlet algebras.

Application to the scalar case

Theorem 4.2

Suppose that $H_0^p(m)$ is a simple invariant subspace for some $p \in [1,2]$. Then the set $\Delta_p(m)$ of all homomorphisms of A which are bounded on $H^p(m)$ is not reduced to $\{\gamma\}$, and $\Delta_p(m)$ is contained in the Gleason part of A which contains γ.
Moreover, there exists an one to one continuous map Γ from $\mathbb{D}$ into $\mathcal{M}(A)$ such that
(i) $\Gamma(\mathbb{D}) = \Delta_p(m)$, $\Gamma(0) = \gamma$,
(ii) For any $f \in A$, the function $\widehat{f} \circ \Gamma$ is analytic on $\mathbb{D}$, where $\widehat{f}$ is the Gelfand transform of f.

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