Decomposition of multiplier operators on character amenable banach algebras

Abdol majid Fattahi

Deparment of Mathematics
Razi University
Kermanshah, Iran

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$$T(xy) = xT(y) = T(x)y \qquad \text{for all } x,\ y \in \mathcal{A}.$$

- Let me state the Notation and preliminaries

In [7], Lau, Pym and Kaniuth introduced and investigated a large class of Banach algebras which they called φ*-amenable* Banach algebras. Given $\varphi \in \Delta(\mathcal{A})$, a Banach algebra $\mathcal{A}$ is said to be φ-amenable if there exists $m \in \mathcal{A}^{**}$ such that $\langle m, \varphi \rangle = 1$ and $\langle m, f.a \rangle = \varphi(a)\langle m, f \rangle$ for all $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$. A commutative Banach algebra $\mathcal{A}$ is said to be *character amenable*, if $\mathcal{A}$ has a bounded approximate identity and for each $\varphi \in \Delta(\mathcal{A}) \cup \{0\}$, $\mathcal{A}$ is φ-amenable. Here we mention some of the well-known properties of these algebras that we shall need. By theorem 1.4. of [7], for $\varphi \in \Delta(\mathcal{A})$ the Banach algebra $\mathcal{A}$ is φ-amenable if and only if there exists a bounded net $(u_\alpha)_\alpha$ in $\mathcal{A}$ such that $\|au_\alpha - \varphi(a)u_\alpha\| \to 0$ for all $a \in \mathcal{A}$ and $\varphi(u_\alpha) = 1$ for all α. Also, a commutative Banach algebra is character amenable if and only if for each $\varphi \in \Delta(\mathcal{A})$, the ideal $\ker \varphi$ has a bounded approximate identity, see [7, corollary 2.3].

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Lemma

If $\mathcal{A}$ is a character amenable Banach algebra, and $T : \mathcal{A} \to \mathcal{A}$ is a multiplier with closed range, then for each $\varphi \in \Delta(T(\mathcal{A}))$ the Banach algebra $T(\mathcal{A})$ is φ-amenable.

Proof. For arbitrary $\varphi \in \Delta(T(\mathcal{A}))$ we can choose $b \in \mathcal{A}$ for which $\varphi(T(b)) = 1$. If now define the linear functional $\widetilde{\varphi}$ on $\mathcal{A}$ by $\widetilde{\varphi}(a) := \varphi(T(b)a)$ for $a \in \mathcal{A}$, then $\widetilde{\varphi}$ is multiplicative and non-zero, and the definition of $\widetilde{\varphi}$ is independent of the choice of b. Therefore $\widetilde{\varphi} \in \Delta(\mathcal{A})$. As we mentioned in preliminaries, by $\widetilde{\varphi}$-amenability of $\mathcal{A}$, there exist a net $(u_\alpha)_{\alpha \in I}$ in $\mathcal{A}$ such that $\widetilde{\varphi}(u_\alpha) = 1$ for all $\alpha \in I$, and $\|au_\alpha - \widetilde{\varphi}(a)u_\alpha\| \to 0$ for each $a \in \mathcal{A}$.

Now for each $\alpha \in I$, set $\nu_\alpha := T(b)u_\alpha$. So we have $\varphi(\nu_\alpha) = 1$ and for each $a \in \mathcal{A}$

$$\|T(a)\nu_\alpha - \varphi(T(a))\nu_\alpha\| \leq \|T(b)\|.\|T(a)u_\alpha - \widetilde{\varphi}(T(a))u_\alpha\| \to 0$$

and this complete the proof. $\square$

Now we are going to show that for a closed range multiplier T on a commutative character amenable Banach algebra $\mathcal{A}$, the Banach algebra $T(A)$ has a bounded approximate identity. For this end I must state some definitions. Given a Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule X, a continuous derivation of $\mathcal{A}$ to X, or X-derivation is a continuous linear mapping D from $\mathcal{A}$ into X such that $D(ab) = D(a).b + a.D(b)$ for all $a, b \in \mathcal{A}$. For each $x \in X$, the mapping $D_x : \mathcal{A} \to X$ defined by $D_x(a) = a.x - x.a$ is a bounded X-derivation, called the inner derivation associated with x. We denote the space of all continuous X-derivations by $Z^1(\mathcal{A}, X)$ and the subspace of all inner derivations in X by $N^1(\mathcal{A}, X)$.

The quotient space $H^1(\mathcal{A}, X) = Z^1(\mathcal{A}, X)/N^1(\mathcal{A}, X)$ is called the first continuous cohomology group of $\mathcal{A}$ with coefficients in X. Therefore if $H^1(\mathcal{A}, X) = \{0\}$, then every continuous X-derivation is inner.

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The following theorem is known; $(a) \Leftrightarrow (b)$ and $(b) \Leftrightarrow (c)$ were proved in [7] and [5] respectively.

Theorem

Let $\mathcal{A}$ be a commutative Banach algebra and $T : \mathcal{A} \to \mathcal{A}$ be a multiplier with closed range. Then the following assertion are equivalent.
(a) For each $\varphi \in \Delta(T(\mathcal{A})) \cup \{0\}$ the Banach algebra $T(\mathcal{A})$ is φ-amenable.
(b) For each $\varphi \in \Delta(T(\mathcal{A})) \cup \{0\}$, if X is a Banach $T()$-bimodule such that $T(a).x = \varphi(T(a)).x$ for all $x \in X$ and $a \in \mathcal{A}$, then $H^1(T(\mathcal{A}), X^*) = \{0\}$;
(c) $T(\mathcal{A})$ has a bounded approximate identity.

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Theorem

Let $T : \mathcal{A} \to \mathcal{A}$ be a multiplier on a commutative character amenable Banach algebra $\mathcal{A}$. Then the following statements are equivalent:
(a) T has closed range.
(b) $T(\mathcal{A})$ has a bounded approximate identity.
(c) $T^2(\mathcal{A}) = T(\mathcal{A})$
(d) $\mathcal{A} = T(\mathcal{A}) \oplus Ker(T)$
(e) $T = BP = PB$, where $B \in M(A)$ is invertible and $p \in M(A)$ is idempotent.

Proof.

(a) implies (b) by theorem 3.3. Suppose that (b) holds. Then by Cohen's factorization theorem we have $T^2(\mathcal{A}) = T^2(\mathcal{A}\mathcal{A}) = T(\mathcal{A})T(\mathcal{A}) = T(\mathcal{A})$. So (b)⇒(c). Now it is easy to see that the hypothesis (c) implies that $\mathcal{A} = T(\mathcal{A}) + Ker(T)$. Therefore the implication $(c) \Rightarrow (d)$ follows if we show that $T(\mathcal{A}) \cap Ker(T) = \{0\}$. For this end, if $T(z) = x \in Ker(T)$, then $xT(\mathcal{A}) = xKer(T) = \{0\}$. Thus, since $\mathcal{A}$ has a bounded approximate identity we have $x = 0$.

Now, suppose that $\mathcal{A} = T(\mathcal{A}) \oplus Ker(T)$. Since in this case $T^2(\mathcal{A}) = T(\mathcal{A})$ and $Ker(T) = Ker(T^2)$, T is a bijection on $T(\mathcal{A})$. Therefore, the linear operator B on $\mathcal{A}$, defined by $B(a + b) := T(a) + b$ for all $a \in T(\mathcal{A})$ and $b \in Ker(T)$, is obviously bijective. Moreover, let P be the linear projection on $\mathcal{A}$ defined by $P(a + b) = a$ for all $a \in T(\mathcal{A})$ and $b \in Ker(T)$, it is straightforward to see that $T = PB = BP$. Thus (d) implies (e). That (e) implies (a) is trivial. $\square$

- Remark: It should be noted that, the class of the Banach algebras satisfying the hypothesis of the theorem 3.4 is quite rich. It contains for instance all the C^*-algebras, the commutative semisimple amenable Banach algebras and the most of Banach algebras which come from harmonic analysis, such as the Herz-Figa-Talamanca algebra $A_p(G)$ of a locally compact amenable group G. Also, there is no well-known Banach algebra which is not character amenable, but each multiplier T on it factors as a product of an idempotent multiplier P and an invertible multiplier B. i.e. $T = BP = PB$.

Theorem 3.4 gives us a necessary condition for character amenability of Banach algebras.

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Example

Let $\mathcal{A}$ be the classical disk algebra and $T : \mathcal{A} \rightarrow \mathcal{A}$ be the multiplier defined by $T(f)(z) = z.f(z)$. Then $T(\mathcal{A}) = \{f \in \mathcal{A} : \; f(0) = 0\}$ and $T^2(\mathcal{A}) = \{f \in \mathcal{A} : \; f(0) = f'(0) = 0\}$. The ideal $T(\mathcal{A})$ is closed in $\mathcal{A}$, but $T^2(\mathcal{A})$ is not dense in $T(\mathcal{A})$, and the closed ideal $T(\mathcal{A})$ does not have a bounded approximate identity. Therefore, by 3.4, the classical disk algebra is not character amenable.

Example

Since every closed ideal of a C^*-algebra has a bounded approximate identity, a multiplier T on a commutative C^*-algebra has a closed range if and only if T is the product of an idempotent multiplier P and an invertible multiplier B (i.e. $T = P \circ B$).

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Thank you very much for your attention