Linear independence of time frequency translates for special configurations

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We will be concerned with the so called HRT Conjecture

Conjecture (L^2 HRT conjecture: Heil, Ramanathan and Topiwala, 1996)

Let $(t_j, \xi_j)_{j=1}^n$ be $n \geq 2$ distinct points in the plane. Then there is no nontrivial L^2 function $f : \mathbb{R} \to \mathbb{C}$ satisfying a nontrivial linear dependence

$$\sum_{j=1}^{n} d_i f(x + t_j) e^{2\pi i \xi_j x} = 0,$$

for a.e. $x \in \mathbb{R}$.

A weaker version of this is

Conjecture (The Schwartz HRT Conjecture)

Let $(t_j, \xi_j)_{j=1}^n$ be $n \geq 2$ distinct points in the plane. Then the above equation has no **Schwartz** *solution.*

These conjectures are far from being resolved, but a few cases are known. It is easy to see that the L^2 HRT is true if the shifts $(\tilde{t}_j, \tilde{\xi}_j)_{j=1}^n$ are collinear. Indeed, via metaplectic transforms, we can assume they lie on the time axis, so $\xi_j = 0$. Then the equation becomes

$$\sum_j d_j f(x + t_j) = 0.$$

By taking the Fourier transform we get

$$\widehat{f}(\xi)(\sum_j d_j e^{2\pi i \xi t_j}) = 0.$$

The key is that the zeros of the trig polynomial $\sum_j d_j e^{2\pi i \xi t_j}$ are discrete (in particular, they have measure 0)

A nice result, of a reasonably general nature is

Theorem (Linnell, 1999)

The L^2 HRT Conjecture holds for any configuration of points that sit on a lattice (any discrete subgroup of $\mathbb{R}^2$).

Linnell's argument uses von Neumenn algebra techniques. Zubin Gautam and I have an alternative argument, inspired by the proof of a.e. invariance of spectra of random Schrödinger oparators. Our argument was motivated by Nazarov and Volberg's observation that the almost Mathieu operator provides a counterexample to the analogue of the HRT conjecture on the group $\mathbb{Z}$ (**The HRT Conjeture can be asked on any topological Abelian group**)

Let $e(x) := e^{2\pi i x}$.
Recall the almost Mathieu operator

$$H_{\omega,\lambda,\theta}\Psi(n) := \Psi(n+1)+\Psi(n-1)+\lambda[e(\omega n+\theta)+e(-\omega n-\theta)]\Psi(n).$$

The next result follows from a much stronger theorem due to Jitomirskaya, 1999.

Theorem

Let ω be Diophantine and let $\lambda > 1$. Then for a.e. $\theta \in [0,1]$, there exists at least one eigenfunction $\Psi_\theta \in l^2(\mathbb{Z})$. In other words

$$\Psi_\theta(n+1)+\Psi_\theta(n-1)+\lambda[e(\omega n+\theta)+e(-\omega n-\theta)]\Psi_\theta(n) = E_\theta \Psi_\theta(n)$$

Thus, for fixed λ, θ, ω, we get a linear dependence on $\mathbb{Z}$

$$f(n+1) + f(n-1) + Ae(\omega n)f(n) + Be(-\omega n)f(n) = Ef(n),$$

where $f(n) := \psi_\theta(n), A = \lambda e(\theta), B = \lambda e(-\theta)$
If E_θ did not depend on θ, say $E_\theta = E$, then we would obtain a linear dependence on $\mathbb{R}$ (by patching together solutions along individual orbits), using the following 5 shifts

$$(-1,0),(0,0),(0,\omega),(0,-\omega),(1,0).$$

(Incidentally, these points sit on the lattice $\mathbb{Z} \times \omega\mathbb{Z}$, so we know this is impossible, since it would contradict Linnell's result!)

How about HRT for non-lattice configurations?
Linnell's result implies that HRT holds true for arbitrary 3 shifts, since any 3 points sit on a lattice. However, the conjecture is open for arbitrary 4 shifts.

Definition (D. 2010)

We will call an (n, m) configuration, any collection of $n + m$ distinct points in the plane, such that there exist 2 distinct parallel lines such that one of them contains exactly n of the points, and the other one contains exactly m of the points.

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The L^2*-HRT Conjecture holds for special* $(1,3)$ *configurations*

$$(0,0),(1,0),(1,\alpha),(1,\beta)$$

(a) if $\frac{\beta}{\alpha}$ *is sufficiently non-Diophantine, more exactly, if there exists* $\gamma > 1$ *such that*

$$\liminf_{n\to\infty} n^{\gamma}\|n\frac{\beta}{\alpha}\| < \infty$$

(b) if at least one of α, β *is rational*
In either case, no solution f *can exist satisfying minimal decay*

$$\lim_{|n|\to\infty} |f(x+n)| = 0, \quad a.e.\ x$$

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Theorem (D., Zaharescu, 2010)

The L^2*-HRT Conjecture holds for all* $(2,2)$ *configurations. Moreover, when the points sit in a special* $(2,2)$ *configuration* $(0,0),(1,0),(0,\alpha),(1,\beta)$*, no nontrivial solution* f *can exist satisfying minimal decay*

$$\lim_{\substack{|n|\to\infty \\ n\in\mathbb{Z}}} |f(x+n)| = 0, \quad a.e.\ x$$

The main ideas in the proof:
Via metaplectic transforms, one can easily reduce to standard (m, n) configurations, where m of the points sit on the y axis and n of the points sit on the line $x = 1$.
The crucial property of any such standard (m, n) configuration is that any hypothetical linear dependence between the shifts of f can be investigated via **scalar** recurrences. For example, the recurrences become

$$f(x+1) = f(x)(C_0 + C_1 e(\alpha x) + C_2 e(\beta x)).$$

for $(1, 3)$ configurations and

$$f(x+1)(A + Be(\alpha x)) = f(x)(C + De(\beta x))$$

for $(2, 2)$ configurations.

I will now show you the details for

Theorem (D. 2010)

The Schwartz-HRT Conjecture holds for all $(1,3)$ *configurations.*

Assume for contradiction that there exists a Schwartz solution f. Let (I, S, d) be a triple such that $d > 0$ and $I \subset [0,1)$ is an interval such that

$$\lim_{\substack{|n|\to\infty \\ n\in\mathbb{Z}}} |n|^C f(x+n) = 0, \text{ for each } x \in I \text{ and each } C > 0, \tag{1}$$

$$d < |f(x)|, \quad \text{for each } x \in I \cup (I+1), \tag{2}$$

$$f(x+1) = f(x)P(x), \quad \text{for each } x \in I + \mathbb{Z}. \tag{3}$$

where

$$P(x) = C_0 + C_1 e(\alpha x) + C_2 e(\beta x)$$

We will construct triples (x_k, x_k', P_k) with $x_k, x_k' \in I$, $P_k \to \infty$ and apply the recurrence for the $\mathbb{Z}$-orbits of x_k (forwards) and x_k' (backwards)

$$f(x_k + [P_k]) = f(x_k) \prod_{n=0}^{[P_k]-1} P(x_k + n) \tag{4}$$

$$f(x_k' - [P_k]) = f(x_k')[\prod_{n=1}^{[P_k]} P(x_k' - n)]^{-1}. \tag{5}$$

The selection is in such a way that

$$\frac{\prod_{n=0}^{[P_k]-1} |P(x_k + n)|}{\prod_{n=1}^{[P_k]} |P(x_k' - n)|} \gtrsim P_k^{-L},$$

for some universal L, independent of k. Using (3), and multiplying (4) and (5) we get (and this contradicts f Schwartz, as $k \to \infty$)

$$|f(x_k + [P_k])f(x_k' - [P_k])| \gtrsim d^2 P_k^{-L}$$

The difficult part is to construct triples (x_k, x_k', P_k) with $x_k, x_k' \in I$, $P_k \to \infty$ such that

$$\frac{\prod_{n=0}^{[P_k]-1} |P(x_k+n)|}{\prod_{n=1}^{[P_k]} |P(x_k'-n)|} \gtrsim P_k^{-L}.$$

It turns out that it's very difficult to control a product such as

$$\prod_{n=0}^{N} P(x+n) = \prod_{n=0}^{N} [C_0 + C_1 e(\alpha x + \alpha n) + C_2 e(\beta x + \beta n)]$$

along any given orbit.

Thus, since **estimating** products along one orbit is too difficult, the new idea is to **compare** products along 2 different orbits. Recall that our goal is to construct triples (x_k, x_k', P_k) with $x_k, x_k' \in I$, $P_k \to \infty$ such that

$$\frac{\prod_{n=0}^{[P_k]-1} |P(x_k+n)|}{\prod_{n=1}^{[P_k]} |P(x_k'-n)|} \gtrsim P_k^{-L}.$$

It will be clear later that we need to estimate averages along a fixed orbit

$$\frac{1}{P_k}\sum_{n=0}^{[P_k]-1} \frac{1}{|P(x+n)|},$$

where

$$P(x) = C_0 + C_1 e(\alpha x) + C_2 e(\beta x)$$

The proof will rely on the almost periodicity of P.

The main technical lemma is the following

Lemma

Let (N_k) be a sequence of integers such that
(ii) $N_k \| N_k \frac{\alpha}{\beta} \| \lesssim \min_{1 \leq n \leq N_k} n \| n \frac{\alpha}{\beta} \|$,
(iii) $N_k \| N_k \frac{\alpha}{\beta} \| \lesssim 1$ *(i.e.* $N_k | \frac{\alpha}{\beta} - \frac{p_k}{N_k} | \lesssim 1$ *for some* $p_k \in \mathbb{Z}$*)*
Define $\frac{1}{M_k} := N_k \| N_k \frac{\alpha}{\beta} \| \lesssim 1$*, and let* $P_k := \frac{N_k}{\beta}$*. Then for each* k *and each* $\delta > 0$*, there exists an exceptional set* $E_{k,\delta} \subset [0,1]$ *such that*

$$|E_{k,\delta}| < \delta$$

and

$$\frac{1}{M_k P_k} \sum_{n=0}^{[P_k]-1} \frac{1}{|P(x+n)|} \lesssim_{\delta, C_0, C_1, C_2, \alpha, \beta} \log P_k,$$

for each $x \in [0,1] \setminus E_{k,\delta}$.

I will not prove this lemma, but let's first see how it implies (together with almost periodicity) the following comparison between 2 orbits.

Proposition

Assume

$$\frac{1}{M_k P_k} \sum_{n=0}^{[P_k]-1} \frac{1}{|P(x+n)|} \lesssim_{\delta, C_0, C_1, C_2, \alpha, \beta} \log P_k,$$

for some x, and let $x - P_k = y$. Then we have

$$|\prod_{n=0}^{[P_k]-1} P(y+n)| \leq P_k^L |\prod_{n=0}^{[P_k]-1} P(x+n)|.$$

Proof.

Let as before $\frac{1}{M_k} := P_k \| P_k \frac{\alpha}{\beta} \| \lesssim 1$. Then, we have $(P_k := \frac{N_k}{\beta})$

$$|e(\alpha x) - e(\alpha y)| = |e(P_k \alpha) - 1| \lesssim \frac{1}{P_k M_k}$$

$$|e(\beta x) - e(\beta y)| = |e(P_k \beta) - 1| \lesssim \frac{1}{P_k M_k}.$$

Thus, for each $n \in \mathbb{N}$, $|P(y+n)| \leq |P(x+n)| + \frac{C}{M_k P_k}$

Use the fact $a + b \leq a e^{\frac{b}{a}}$ for each $a, b > 0$ to get

$$|P(y+n)| \leq |P(x+n)| e^{\frac{C}{M_k P_k |P(x+n)|}},$$

and thus

$$|\prod_{n=0}^{[P_k]-1} P(y+n)| \leq |\prod_{n=0}^{[P_k]-1} P(x+n)| e^{\frac{C}{M_k P_k} \sum_{n=0}^{[P_k]-1} \frac{1}{|P(x+n)|}}.$$

Two things play a critical role in the estimate

$$\frac{1}{M_k P_k} \sum_{n=0}^{[P_k]-1} \frac{1}{|P(x+n)|} \lesssim_{\delta, C_0, C_1, C_2, \alpha, \beta} \log P_k .$$

1. The fact that $p(x,y) = C_0 + C_1 e(x) + C_2 e(y)$ has at most two real zeros $(\gamma_1^{(j)}, \gamma_2^{(j)}) \in [0,1)^2$, $j \in \{1,2\}$, and for each $x, y \in \mathbb{R}$,

$$|p(x,y)| \gtrsim_{C_0, C_1, C_2} \min_{j=1}^{2} (\|x - \gamma_1^j\|^2 + \|y - \gamma_2^j\|^2).$$

2. The second important fact is the geometry of the points $(n\alpha, n\beta)$, since

$$P(x+n) = C_0 + C_1 e(\alpha x + \alpha n) + C_2 e(\beta x + \beta n) = p(\alpha x + \alpha n, \beta x + \beta n).$$

The heuristic behind the proof is that if α/β is less Diophantine, then the estimate

$$\frac{1}{M_k P_k} \sum_{n=0}^{[P_k]-1} \frac{1}{\|\alpha x - \gamma_1^j + \alpha n\|^2 + \|\beta x - \gamma_2^j + \beta n\|^2} \lesssim \log P_k.$$

holds (for each j) because M_k is large, while if α/β is Diophantine, it holds because of the extra regularity the points $(n\alpha, n\beta)$.

Questions:

1. This approach fails for $(1, 4)$ configurations like $(0, 0), (1, 0), (1, \alpha), (1, \beta), (1, \gamma)$. This is because the best one can guarantee in general is the existence of arbitrarily large P such that $\max\{\|P\alpha\|, \|P\beta\|, \|P\gamma\|\} \lesssim \frac{1}{\sqrt{P}}$. It is not clear whether working with 3 or more orbits would have more to say about this case.

2. How to deal with configurations sitting on 3, rather than 2 lines? Here, matrix (rather than scalar) recurrences occur.