Group actions on topological graphs

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(work in progress)

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Outline

- We recall basic facts about topological graphs and their C^*-algebras, with examples.
- We define the action of a group G on a topological graph E. We give a structure theorem for free and proper actions, and define the quotient graph E/G.
- This action induces a natural action of G on the C^*-correspondence $\mathcal{H}(E)$ and on the graph C^*-algebra $C^*(E)$ such that $C^*(E) \rtimes_r G$ is strongly Morita equivalent to $C^*(E/G)$.
- We also introduce the fundamental group and the universal covering of a topological graph via a geometric realization. We give examples, one having the Baumslag-Solitar group as fundamental group.

Topological graphs and their C^*-algebras

- Let $E = (E^0, E^1, s, r)$ be a topological graph. Recall that E^0 (vertices) and E^1 (edges) are locally compact (Hausdorff) spaces, $s, r : E^1 \to E^0$ are continuous maps, and s is a local homeomorphism.
- The C*-algebra $C^*(E)$ is the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ of the C^*-correspondence $\mathcal{H} = \mathcal{H}(E)$ over $A = C_0(E^0)$, obtained as a completion of $C_c(E^1)$ using

$$\langle \xi, \eta \rangle(v) = \sum_{s(e)=v} \overline{\xi(e)}\eta(e),\ \xi, \eta \in C_c(E^1)$$

$$(\xi \cdot f)(e) = \xi(e)f(s(e)),\ (f \cdot \xi)(e) = f(r(e))\xi(e).$$

Examples

- Example 1. Let $E^0 = E^1 = \mathbb{T}, s(z) = z$, and $r(z) = e^{2\pi i\theta}z$ for $\theta \in [0, 1]$ irrational. Then $C^*(E) \cong A_\theta$, the irrational rotation algebra.
- Example 2. Let $E^0 = E^1 = X$, for X a locally compact metric space, let $s = id$ and let $r = h : X \to X$ be a homeomorphism.
  Then $C^*(E) \cong C_0(X) \rtimes \mathbb{Z}$, since $C^*(E)$ is the universal C^*-algebra generated by $C_0(X)$ and a unitary u satisfying $\hat{h}(f) = u^*fu$ for $f \in C_0(X)$, where $\hat{h}(f) = f \circ h$.
- Example 3. Let $n \in \mathbb{N} \setminus \{0\}$ and $m \in \mathbb{Z} \setminus \{0\}$. Take

$$E^0 = E^1 = \mathbb{T}, s(z) = z^n, r(z) = z^m.$$

  If $m \notin n\mathbb{Z}$, then $C^*(E)$ is simple and purely infinite.

Skew products

- Skew products of topological graphs. Let $E = (E^0, E^1, s, r)$ be a topological graph, let G be a locally compact group, and let $c : E^1 \to G$ be continuous. Define the skew product graph $E \times_c G = (E^0 \times G, E^1 \times G, \tilde{s}, \tilde{r})$, where

$$\tilde{s}(e, g) = (s(e), g), \;\; \tilde{r}(e, g) = (r(e), gc(e)).$$

- Then $E \times_c G$ becomes a topological graph using the product topology. If E has one vertex and n loops $\{e_1, ..., e_n\}$ and if G has a set of generators $S = \{h_1, ..., h_n\}$ such that $c(e_i) = h_i, i = 1, ..., n$ then we get the Cayley graph $E(G, S)$.

Graph morphisms

- Let E, F be two topological graphs. A graph morphism $\phi : E \to F$ is a pair of continuous maps $\phi = (\phi^0, \phi^1)$ such that the diagram

$$\begin{tikzcd} E^0 \arrow[d, "\phi^0"'] & E^1 \arrow[l, "s"'] \arrow[r, "r"] \arrow[d, "\phi^1"'] & E^0 \arrow[d, "\phi^0"'] \\ F^0 & F^1 \arrow[l, "s"'] \arrow[r, "r"] & F^0 \end{tikzcd}$$

  is commutative.
- A graph morphism ϕ is a graph covering if both ϕ^0, ϕ^1 are covering maps.
- An isomorphism is a graph morphism $\phi = (\phi^0, \phi^1)$ such that ϕ^i are homeomorphisms for $i = 0, 1$. It follows that $\phi^{-1} = ((\phi^0)^{-1}, (\phi^1)^{-1})$ is also a graph morphism.

Group actions

- A locally compact group G acts on E if there are continuous maps $\lambda^i : G \times E^i \to E^i$ for $i = 0, 1$ such that $g \mapsto \lambda_g$ is a homomorphism from G into $Aut(E)$.
- The action λ is called free if $\lambda^0_g(v) = v$ for some $v \in E^0$ implies $g = 1_G$. In this case the action of G is also free on E^1.
- The action is called proper if the maps $G \times E^0 \to E^0 \times E^0, (g, v) \mapsto (\lambda^0_g(v), v)$ and $G \times E^1 \to E^1 \times E^1, (g, e) \mapsto (\lambda^1_g(e), e)$ are proper. (It is sufficient to require properness of the first map).
- A group G acts freely and properly on a skew product $E \times_c G$ by $\lambda^0_g(v, h) = (v, gh)$ and $\lambda^1_g(e, h) = (e, gh)$.

Principal G-bundles and the quotient graph

A map $q : P \to X$ is called a principal G-bundle if there is a free and proper action of G on P such that P/G can be identified with X.

Theorem

Given $F = (F^0, F^1, s, r)$ a topological graph, a principal G-bundle $P \to F^0$ and an isomorphism of pull-backs $s^*(P) \cong r^*(P)$, there is a topological graph $E = (E^0, E^1, \tilde{s}, \tilde{r})$ with a free and proper action of G such that $E^0 = P, E^1 = s^*(P)$ and $F \cong E/G$. Moreover, every topological graph E on which G acts freely and properly arises this way.

Corollary

The topological graph E constructed above is G-equivariantly isomorphic to a skew product $F \times_c G$ iff the principal bundle $E^0 \to F^0$ is trivial.

Group actions on C^*-correspondences

- A group G acts on a C^*-correspondence $\mathcal{H}$ over A if there is a map $G \times \mathcal{H} \to \mathcal{H}, \quad (g, \xi) \mapsto g \cdot \xi$ such that $g \mapsto g \cdot \xi$ is continuous, $\xi \mapsto g \cdot \xi$ is linear, and if G acts on A by $*$-automorphisms such that $\langle g{\cdot}\xi, g{\cdot}\eta \rangle = g{\cdot}\langle \xi, \eta \rangle,\ g{\cdot}(\xi a) = (g{\cdot}\xi)(g{\cdot}a),\ g{\cdot}(\varphi(a)\xi) = \varphi(g{\cdot}a)(g{\cdot}\xi)$.
- An action of G on the C^*-correspondence $\mathcal{H}$ defines an action on the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ since all defining relations are equivariant.

Proposition.

If G acts on the topological graph $E = (E^0, E^1, s, r)$, then G acts on the C^*-correspondence $\mathcal{H} = \mathcal{H}(E)$ and hence on $C^*(E)$.

Proof.

Define $g \cdot \xi(e) = \xi(g^{-1}e)$ for $\xi \in C_c(E^1)$, $g \cdot f(v) = f(g^{-1}v)$ for $f \in C_0(E^0)$. Then this action is compatible with the bimodule structure since s and r are equivariant. $\square$

Proper actions on C^*-algebras

- The action α of a locally compact group G on a C^*-algebra A is *proper* if there is a dense α-invariant $*$-subalgebra A_0 of A such that
- for every $a, b \in A_0$ the functions

$$x \mapsto a\alpha_x(b) \text{ and } x \mapsto \Delta(x)^{-1/2}a\alpha_x(b)$$

  are integrable on G, and
- for all $a, b \in A_0$ there exists $\langle a, b\rangle_r \in M(A_0)$, where

$$M(A_0) := \{m \in M(A) : a \in A_0 \Rightarrow ma \in A_0\}$$

  such that

$$c\langle a, b\rangle_r = \int_G c\alpha_x(a^*b)dx \text{ for all } c \in A_0.$$

Proper action cont'd

- For such an action,

$$A^\alpha := \overline{span}\{\langle a, b\rangle_r : a, b \in A_0\} \subset M(A)$$

  is called the *generalized fixed-point algebra*.
- Define a (left) inner product on A_0 with values in $A \rtimes_{\alpha,r} G$ by

$${}_\ell\langle a, b\rangle(x) = \Delta(x)^{-1/2} a\alpha_x(b^*).$$

- The set

$$I := \overline{span}\{{}_\ell\langle a, b\rangle : a, b \in A_0\}$$

  is an ideal in $A \rtimes_{\alpha,r} G$, and the closure $\mathcal{Z}$ of A_0 in the norm $\|a\|^2 := \|\langle a, a\rangle_r\|$ is an $I - A^\alpha$ imprimitivity bimodule.
- The action is called *saturated* if $I = A \rtimes_{\alpha,r} G$.

The main results

Theorem

If G acts freely and properly on the topological graph E, then G acts properly on $C^*(E)$ and the action is saturated. Moreover, $C^*(E) \rtimes_r G$ and $C^*(E/G)$ are strongly Morita equivalent.

Sketch of proof. Since G acts freely and properly on E^0 and there is an equivariant map $C_0(E^0) \to M(C^*(E))$, it follows that the action of G on $C^*(E)$ is proper and saturated with respect to the $*$-subalgebra

$$A_0 = C_c(E^0)C^*(E)C_c(E^0).$$

To prove that the generalized fixed point algebra is isomorphic to $C^*(E/G)$, we construct an injective homomorphism from $C^*(E/G)$ into $M(C^*(E))$ whose image is $C^*(E)^\alpha$. This is done in several steps, using multipliers of C^*-correspondences.

The main results cont'd

Recall that G acts freely and properly on $E \times_c G$ and that $(E \times_c G)/G = E$. We have

Corollary

The C^*-algebras $C^*(E \times_c G) \rtimes_r G$ and $C^*(E)$ are strongly Morita equivalent. In particular, for a finitely generated locally compact group G with generators $S = \{h_1, h_2, ..., h_n\}$ and Cayley graph $E(G, S)$, we get that $C^*(E(G, S)) \rtimes_r G$ is strongly Morita equivalent to the Cuntz algebra $\mathcal{O}_n$.

Corollary

If G is abelian, $c : E^1 \to G$ induces an action α^c of $\hat{G}$ on $C^*(E)$ such that $(\alpha^c_\chi \xi)(e) = \langle \chi, c(e) \rangle \xi(e)$ for $\xi \in C_c(E^1)$ and $\chi \in \hat{G}$. Then

$$C^*(E) \rtimes_{\alpha^c} \hat{G} \cong C^*(E \times_c G).$$

The fundamental group

- The geometric realization of a topological graph E is

$$R(E) := E^1 \times [0,1] \sqcup E^0 / \sim,$$

  where $(e,0) \sim s(e)$ and $(e,1) \sim r(e)$ (a kind of double mapping torus).
- If the group G acts on the topological graph E, then G acts on $R(E)$ by

$$g \cdot (e,t) = (\lambda_g^1(e), t), e \in E^1, t \in [0,1], \quad g \cdot v = \lambda_g^0(v), v \in E^0.$$

- The fundamental group $\pi_1(E)$ is by definition $\pi_1(R(E))$. The universal covering $\tilde{E}$ of E is a simply connected graph which covers E.
- The group $\pi_1(E)$ acts freely on $\tilde{E}$, and the orbit space $\tilde{E}/G$ is isomorphic to E.
- If E is discrete, then $\pi_1(E)$ is free, and the universal covering is a tree T.

Examples of coverings

- Example 1. Let E with $E^0 = E^1 = \mathbb{T}$ and with $s(z) = z, r(z) = e^{2\pi i\theta}z$ for θ irrational. The geometric realization is homeomorphic to $\mathbb{T}^2$, hence $\pi_1(E) \cong \mathbb{Z}^2$.
- The universal covering $\tilde{E}$ has $\tilde{E}^0 = \tilde{E}^1 = \mathbb{R} \times \mathbb{Z}$, and $s(y,k) = (y,k), r(y,k) = (y+\theta, k+1)$. Here $\mathbb{Z}^2$ acts on $\tilde{E}$ by $(j,m)\cdot(y,k) = (y + m\theta + j, k+m)$, and $\tilde{E}/\mathbb{Z}^2 \cong E$.
- Example 2. Let $h : X \to X$ be a homeomorphism, and let E with $E^0 = E^1 = X$, $s = id$ and $r = h$. The geometric realization of E is homeomorphic to the mapping torus of h.
- The universal covering $\tilde{E}$ has $\tilde{E}^0 = \tilde{E}^1 = \tilde{X} \times \mathbb{Z}$, where $\tilde{X}$ is the universal covering of X. The source and range maps are $s(y,k) = (y,k)$, $r(y,k) = (\tilde{h}(y), k+1)$, where $\tilde{h} : \tilde{X} \to \tilde{X}$ is a lifting of h.
- Then $\pi_1(E) \cong \pi_1(X) \rtimes \mathbb{Z}$, and the action of $\pi_1(X) \rtimes \mathbb{Z}$ on $\tilde{X} \times \mathbb{Z}$ is given by $(g,m)\cdot(y,k) = (g\cdot\tilde{h}^m(y), k+m)$.

Examples cont'd

- Example 3. Let again $E^0 = E^1 = \mathbb{T}$ with $s(z) = z^p, r(z) = z^q$ for p, q positive integers. Then $R(E)$ is obtained from a cylinder, where the two boundary circles are identified using the maps s and r.

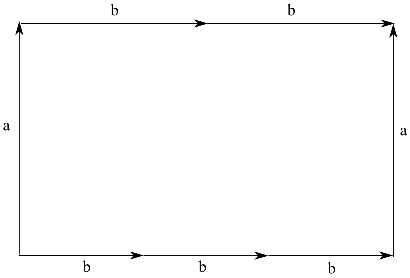

Figure: The case $p = 2, q = 3$.

Examples cont'd

- Then $\pi_1(E)$ is isomorphic to the Baumslag-Solitar group $B(p,q) = \langle a, b \mid ab^pa^{-1} = b^q \rangle$.
- For $p = 1$ or $q = 1$, this group is a semi-direct product and it is amenable. For $p \neq 1, q \neq 1$ and $(p,q) = 1$, it is not amenable.
- The universal covering space of $R(E)$ is obtained from the Cayley graph of $B(p,q)$ by filling out the squares. It is the cartesian product $T \times \mathbb{R}$, where T is the Bass-Serre tree of $B(p,q)$, viewed as an HNN-extension of $\pi_1(\mathbb{T})$.
- Recall that $B(p,q)$ is the quotient of the free product $\pi_1(\mathbb{T}) * \mathbb{Z}$ by the relation $as_*(b)a^{-1} = r_*(b)$, where a is the generator of $\mathbb{Z}$, $b \in \pi_1(\mathbb{T})$, and $s_*, r_* : \pi_1(\mathbb{T}) \hookrightarrow \pi_1(\mathbb{T})$.

Examples cont'd

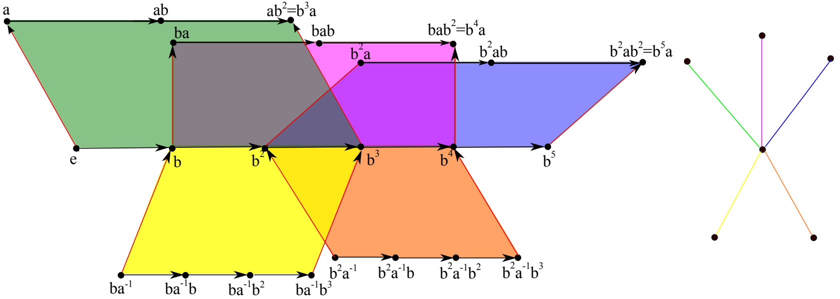

Figure: Cayley complex for $B(2,3)$.

Examples cont'd

- The 1-skeleton is the directed Cayley graph of $B(2,3)$, where the generators a, b multiply on the right. The group action is given by left multiplication.
- In the corresponding tree T, each vertex has 5 edges. The vertex set T^0 is identified with the left cosets $g\langle b\rangle \in B(2,3)/\langle b\rangle$, and the edge set T^1 with the left cosets $g\langle b^2\rangle \in B(2,3)/\langle b^2\rangle$.
- The source and range maps are given by $s(g\langle b^2\rangle) = g\langle b\rangle, r(g\langle b^2\rangle) = ga^{-1}\langle b\rangle$ for $g \in B(2,3)$.
- We have $\tilde{E}^0 \cong T^0 \times \mathbb{R}$, $\tilde{E}^1 \cong T^1 \times \mathbb{R}$ with $\tilde{s}(t,y) = (s(t), py), \tilde{r}(t,y) = (r(t), qy)$ for $t \in T^1$ and $y \in \mathbb{R}$.
- The group $B(p,q)$ acts freely and properly on $\tilde{E}$, and the quotient graph $\tilde{E}/B(p,q)$ is isomorphic to E.
- In particular, $\tilde{E}$ is not a skew product. We have $C^*(\tilde{E}) \rtimes_r B(p,q)$ strongly Morita equivalent to $C^*(E)$.

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