Group actions on topological graphs

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(work in progress)

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We recall basic facts about topological graphs and their $C^*$-algebras, with examples.

We define the action of a group $G$ on a topological graph $E$. We give a structure theorem for free and proper actions, and define the quotient graph $E/G$.

This action induces a natural action of $G$ on the $C^*$-correspondence $\mathcal{H}(E)$ and on the graph $C^*$-algebra $C^*(E)$ such that $C^*(E) \rtimes_r G$ is strongly Morita equivalent to $C^*(E/G)$.

We also introduce the fundamental group and the universal covering of a topological graph via a geometric realization. We give examples, one having the Baumslag-Solitar group as fundamental group.
Let $E = (E^0, E^1, s, r)$ be a topological graph. Recall that $E^0$ (vertices) and $E^1$ (edges) are locally compact (Hausdorff) spaces, $s, r : E^1 \to E^0$ are continuous maps, and $s$ is a local homeomorphism.

The C*-algebra $C^*(E)$ is the Cuntz-Pimsner algebra $O_H$ of the C*-correspondence $H = H(E)$ over $A = C_0(E^0)$, obtained as a completion of $C_c(E^1)$ using

$$\langle \xi, \eta \rangle(v) = \sum_{s(e) = v} \overline{\xi(e)} \eta(e), \; \xi, \eta \in C_c(E^1)$$

$$(\xi \cdot f)(e) = \xi(e)f(s(e)), \; (f \cdot \xi)(e) = f(r(e))\xi(e).$$
Examples

- **Example 1.** Let $E^0 = E^1 = \mathbb{T}$, $s(z) = z$, and $r(z) = e^{2\pi i \theta} z$ for $\theta \in [0, 1]$ irrational. Then $C^*(E) \cong A_\theta$, the irrational rotation algebra.

- **Example 2.** Let $E^0 = E^1 = X$, for $X$ a locally compact metric space, let $s = id$ and let $r = h : X \to X$ be a homeomorphism. Then $C^*(E) \cong C_0(X) \rtimes \mathbb{Z}$, since $C^*(E)$ is the universal $C^*$-algebra generated by $C_0(X)$ and a unitary $u$ satisfying $\hat{h}(f) = u^*fu$ for $f \in C_0(X)$, where $\hat{h}(f) = f \circ h$.

- **Example 3.** Let $n \in \mathbb{N} \setminus \{0\}$ and $m \in \mathbb{Z} \setminus \{0\}$. Take

\[ E^0 = E^1 = \mathbb{T}, \ s(z) = z^n, \ r(z) = z^m. \]

If $m \not\in n\mathbb{Z}$, then $C^*(E)$ is simple and purely infinite.
Skew products

- **Skew products of topological graphs.** Let $E = (E^0, E^1, s, r)$ be a topological graph, let $G$ be a locally compact group, and let $c : E^1 \to G$ be continuous. Define the skew product graph $E \times_c G = (E^0 \times G, E^1 \times G, \tilde{s}, \tilde{r})$, where

\[
\tilde{s}(e, g) = (s(e), g), \quad \tilde{r}(e, g) = (r(e), gc(e)).
\]

- Then $E \times_c G$ becomes a topological graph using the product topology. If $E$ has one vertex and $n$ loops $\{e_1, \ldots, e_n\}$ and if $G$ has a set of generators $S = \{h_1, \ldots, h_n\}$ such that $c(e_i) = h_i, i = 1, \ldots, n$ then we get the Cayley graph $E(G, S)$. 
Graph morphisms

- Let $E, F$ be two topological graphs. A graph morphism $\phi : E \to F$ is a pair of continuous maps $\phi = (\phi^0, \phi^1)$ such that the diagram

\[
\begin{array}{ccc}
E^0 & \xleftarrow{s} & E^1 & \xrightarrow{r} & E^0 \\
\phi^0 \downarrow & & \phi^1 \downarrow & & \phi^0 \downarrow \\
F^0 & \xleftarrow{s} & F^1 & \xrightarrow{r} & F^0
\end{array}
\]

is commutative.

- A graph morphism $\phi$ is a graph covering if both $\phi^0, \phi^1$ are covering maps.

- An isomorphism is a graph morphism $\phi = (\phi^0, \phi^1)$ such that $\phi^i$ are homeomorphisms for $i = 0, 1$. It follows that $\phi^{-1} = ((\phi^0)^{-1}, (\phi^1)^{-1})$ is also a graph morphism.
Group actions

- A locally compact group $G$ acts on $E$ if there are continuous maps $\lambda^i : G \times E^i \rightarrow E^i$ for $i = 0, 1$ such that $g \mapsto \lambda_g$ is a homomorphism from $G$ into $\text{Aut}(E)$.

- The action $\lambda$ is called free if $\lambda^0_g(v) = v$ for some $v \in E^0$ implies $g = 1_G$. In this case the action of $G$ is also free on $E^1$.

- The action is called proper if the maps $G \times E^0 \rightarrow E^0 \times E^0$, $(g, v) \mapsto (\lambda^0_g(v), v)$ and $G \times E^1 \rightarrow E^1 \times E^1$, $(g, e) \mapsto (\lambda^1_g(e), e)$ are proper. (It is sufficient to require properness of the first map).

- A group $G$ acts freely and properly on a skew product $E \times_c G$ by $\lambda^0_g(v, h) = (v, gh)$ and $\lambda^1_g(e, h) = (e, gh)$.
Principal $G$-bundles and the quotient graph

A map $q : P \to X$ is called a **principal $G$-bundle** if there is a free and proper action of $G$ on $P$ such that $P/G$ can be identified with $X$.

**Theorem**

Given $F = (F^0, F^1, s, r)$ a topological graph, a principal $G$-bundle $P \to F^0$ and an isomorphism of pull-backs $s^*(P) \cong r^*(P)$, there is a topological graph $E = (E^0, E^1, \tilde{s}, \tilde{r})$ with a free and proper action of $G$ such that $E^0 = P$, $E^1 = s^*(P)$ and $F \cong E/G$. Moreover, every topological graph $E$ on which $G$ acts freely and properly arises this way.

**Corollary**

The topological graph $E$ constructed above is $G$-equivariantly isomorphic to a skew product $F \times_c G$ iff the principal bundle $E^0 \to F^0$ is trivial.
Group actions on \( C^* \)-correspondences

- A group \( G \) acts on a \( C^* \)-correspondence \( \mathcal{H} \) over \( A \) if there is a map \( G \times \mathcal{H} \to \mathcal{H}, \ (g, \xi) \mapsto g \cdot \xi \) such that \( g \mapsto g \cdot \xi \) is continuous, \( \xi \mapsto g \cdot \xi \) is linear, and if \( G \) acts on \( A \) by \(*\)-automorphisms such that \( \langle g \cdot \xi, g \cdot \eta \rangle = g \cdot \langle \xi, \eta \rangle \), \( g \cdot (\xi a) = (g \cdot \xi)(g \cdot a) \), \( g \cdot (\varphi(a) \xi) = \varphi(g \cdot a)(g \cdot \xi) \).

- An action of \( G \) on the \( C^* \)-correspondence \( \mathcal{H} \) defines an action on the Cuntz-Pimsner algebra \( O_{\mathcal{H}} \) since all defining relations are equivariant.

Proposition.

If \( G \) acts on the topological graph \( E = (E_0, E_1, s, r) \), then \( G \) acts on the \( C^* \)-correspondence \( \mathcal{H} = \mathcal{H}(E) \) and hence on \( C^*(E) \).

Proof.

Define \( g \cdot \xi(e) = \xi(g^{-1}e) \) for \( \xi \in C_c(E_1) \), \( g \cdot f(v) = f(g^{-1}v) \) for \( f \in C_0(E_0) \). Then this action is compatible with the bimodule structure since \( s \) and \( r \) are equivariant. □
Proper actions on $C^*$-algebras

The action $\alpha$ of a locally compact group $G$ on a $C^*$-algebra $A$ is proper if there is a dense $\alpha$-invariant $*$-subalgebra $A_0$ of $A$ such that for every $a, b \in A_0$ the functions

$$x \mapsto a\alpha_x(b) \quad \text{and} \quad x \mapsto \Delta(x)^{-1/2}a\alpha_x(b)$$

are integrable on $G$, and for all $a, b \in A_0$ there exists $\langle a, b \rangle_r \in M(A_0)$, where

$$M(A_0) := \{ m \in M(A) : a \in A_0 \Rightarrow ma \in A_0 \}$$

such that

$$c\langle a, b \rangle_r = \int_G c\alpha_x(a^*b)dx \quad \text{for all} \quad c \in A_0.$$
Proper action cont’d

- For such an action,

\[ A^\alpha := \overline{\text{span}}\{\langle a, b \rangle_r : a, b \in A_0\} \subset M(A) \]

is called the \textit{generalized fixed-point algebra}.

- Define a (left) inner product on \( A_0 \) with values in \( A \rtimes_{\alpha, r} G \) by

\[ \ell\langle a, b \rangle(x) = \Delta(x)^{-1/2} a\alpha_x(b^*) . \]

- The set

\[ I := \overline{\text{span}}\{\ell \langle a, b \rangle : a, b \in A_0\} \]

is an ideal in \( A \rtimes_{\alpha, r} G \), and the closure \( \mathcal{Z} \) of \( A_0 \) in the norm \( \|a\|^2 := \|\langle a, a \rangle_r\| \) is an \( I - A^\alpha \) imprimitivity bimodule.

- The action is called \textit{saturated} if \( I = A \rtimes_{\alpha, r} G \).
The main results

**Theorem**

*If G acts freely and properly on the topological graph E, then G acts properly on C*(E) and the action is saturated. Moreover, C*(E) RTimes, G and C*(E/G) are strongly Morita equivalent.*

**Sketch of proof.** Since G acts freely and properly on E₀ and there is an equivariant map \( C_0(E^0) \to M(C^*(E)) \), it follows that the action of G on C*(E) is proper and saturated with respect to the \(*\)-subalgebra

\[
A_0 = C_c(E^0)C^*(E)C_c(E^0).
\]

To prove that the generalized fixed point algebra is isomorphic to C*(E/G), we construct an injective homomorphism from C*(E/G) into M(C*(E)) whose image is C*(E)\(^\alpha\). This is done in several steps, using multipliers of C*-correspondences.
Recall that $G$ acts freely and properly on $E \times_c G$ and that $(E \times_c G)/G = E$. We have

**Corollary**

The $C^*$-algebras $C^*(E \times_c G) \rtimes_r G$ and $C^*(E)$ are strongly Morita equivalent. In particular, for a finitely generated locally compact group $G$ with generators $S = \{h_1, h_2, ..., h_n\}$ and Cayley graph $E(G,S)$, we get that $C^*(E(G,S)) \rtimes_r G$ is strongly Morita equivalent to the Cuntz algebra $\mathcal{O}_n$.

**Corollary**

If $G$ is abelian, $c : E^1 \to G$ induces an action $\alpha^c$ of $\hat{G}$ on $C^*(E)$ such that $(\alpha^c_\chi \xi)(e) = \langle \chi, c(e) \rangle \xi(e)$ for $\xi \in C_c(E^1)$ and $\chi \in \hat{G}$. Then

$$C^*(E) \rtimes_\alpha^c \hat{G} \cong C^*(E \times_c G).$$
The geometric realization of a topological graph $E$ is

$$R(E) := E^1 \times [0, 1] \sqcup E^0 / \sim,$$

where $(e, 0) \sim s(e)$ and $(e, 1) \sim r(e)$ (a kind of double mapping torus).

If the group $G$ acts on the topological graph $E$, then $G$ acts on $R(E)$ by

$$g \cdot (e, t) = (\lambda_g^1(e), t), e \in E^1, t \in [0, 1], \quad g \cdot v = \lambda_g^0(v), v \in E^0.$$

The fundamental group $\pi_1(E)$ is by definition $\pi_1(R(E))$. The universal covering $\tilde{E}$ of $E$ is a simply connected graph which covers $E$.

The group $\pi_1(E)$ acts freely on $\tilde{E}$, and the orbit space $\tilde{E}/G$ is isomorphic to $E$.

If $E$ is discrete, then $\pi_1(E)$ is free, and the universal covering is a tree $T$. 
Examples of coverings

- **Example 1.** Let $E$ with $E^0 = E^1 = \mathbb{T}$ and with $s(z) = z$, $r(z) = e^{2\pi i \theta} z$ for $\theta$ irrational. The geometric realization is homeomorphic to $\mathbb{T}^2$, hence $\pi_1(E) \cong \mathbb{Z}^2$.

  The universal covering $\tilde{E}$ has $\tilde{E}^0 = \tilde{E}^1 = \mathbb{R} \times \mathbb{Z}$, and $s(y, k) = (y, k)$, $r(y, k) = (y + \theta, k + 1)$. Here $\mathbb{Z}^2$ acts on $\tilde{E}$ by $(j, m) \cdot (y, k) = (y + m\theta + j, k + m)$, and $\tilde{E}/\mathbb{Z}^2 \cong E$.

- **Example 2.** Let $h : X \to X$ be a homeomorphism, and let $E$ with $E^0 = E^1 = X$, $s = id$ and $r = h$. The geometric realization of $E$ is homeomorphic to the mapping torus of $h$.

  The universal covering $\tilde{E}$ has $\tilde{E}^0 = \tilde{E}^1 = \tilde{X} \times \mathbb{Z}$, where $\tilde{X}$ is the universal covering of $X$. The source and range maps are $s(y, k) = (y, k)$, $r(y, k) = (\tilde{h}(y), k + 1)$, where $\tilde{h} : \tilde{X} \to \tilde{X}$ is a lifting of $h$.

  Then $\pi_1(E) \cong \pi_1(X) \rtimes \mathbb{Z}$, and the action of $\pi_1(X) \rtimes \mathbb{Z}$ on $\tilde{X} \times \mathbb{Z}$ is given by $(g, m) \cdot (y, k) = (g \cdot \tilde{h}^m(y), k + m)$. 
**Example 3.** Let again $E^0 = E^1 = \mathbb{T}$ with $s(z) = z^p$, $r(z) = z^q$ for $p, q$ positive integers. Then $R(E)$ is obtained from a cylinder, where the two boundary circles are identified using the maps $s$ and $r$.

**Figure:** The case $p = 2, q = 3$. 
Examples cont’d

- Then $\pi_1(E)$ is isomorphic to the Baumslag-Solitar group $B(p, q) = \langle a, b \mid ab^p a^{-1} = b^q \rangle$.
- For $p = 1$ or $q = 1$, this group is a semi-direct product and it is amenable. For $p \neq 1, q \neq 1$ and $(p, q) = 1$, it is not amenable.
- The universal covering space of $R(E)$ is obtained from the Cayley graph of $B(p, q)$ by filling out the squares. It is the cartesian product $T \times \mathbb{R}$, where $T$ is the Bass-Serre tree of $B(p, q)$, viewed as an HNN-extension of $\pi_1(\mathbb{T})$.
- Recall that $B(p, q)$ is the quotient of the free product $\pi_1(\mathbb{T}) \star \mathbb{Z}$ by the relation $as_*(b)a^{-1} = r_*(b)$, where $a$ is the generator of $\mathbb{Z}$, $b \in \pi_1(\mathbb{T})$, and $s_*, r_* : \pi_1(\mathbb{T}) \hookrightarrow \pi_1(\mathbb{T})$. 
Examples cont’d

Figure: Cayley complex for $B(2, 3)$. 
The 1-skeleton is the directed Cayley graph of $B(2, 3)$, where the generators $a, b$ multiply on the right. The group action is given by left multiplication.

In the corresponding tree $T$, each vertex has 5 edges. The vertex set $T^0$ is identified with the left cosets $g\langle b \rangle \in B(2, 3)/\langle b \rangle$, and the edge set $T^1$ with the left cosets $g\langle b^2 \rangle \in B(2, 3)/\langle b^2 \rangle$.

The source and range maps are given by $s(g\langle b^2 \rangle) = g\langle b \rangle$, $r(g\langle b^2 \rangle) = ga^{-1}\langle b \rangle$ for $g \in B(2, 3)$.

We have $\tilde{E}^0 \cong T^0 \times \mathbb{R}$, $\tilde{E}^1 \cong T^1 \times \mathbb{R}$ with $\tilde{s}(t, y) = (s(t), py)$, $\tilde{r}(t, y) = (r(t), qy)$ for $t \in T^1$ and $y \in \mathbb{R}$.

The group $B(p, q)$ acts freely and properly on $\tilde{E}$, and the quotient graph $\tilde{E}/B(p, q)$ is isomorphic to $E$.

In particular, $\tilde{E}$ is not a skew product. We have $C^*(\tilde{E}) \rtimes_r B(p, q)$ strongly Morita equivalent to $C^*(E)$. 

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