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On problems involving the weak Laplacian operator on the Sierpinski gasket

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Physical phenomena on fractal domains

- reaction - diffusion problems
- elastic properties of fractal media
- fluid flow through fractal regions

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Efforts to develop an appropriate framework to cope with PDEs on fractals

Appropriate framework to study PDEs on fractals

Classical domains (open subsets of $\mathbb{R}^{\mathbb{N}}$):

Sobolev spaces

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Question

How to define differential operators, for example the Laplacian, on fractals?

Defining the Laplacian on fractals

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- J. Kigami, *In quest of fractal analysis*, in *Mathematics of Fractals* (H. Yamaguti, M. Hata, J. Kigami, eds.), Providence, RI: American Mathematical Society, 1993, 53–73.
- J. Kigami, *Harmonic calculus on p.c.f. self-similar sets*, Trans. Am. Math. Soc., **335** (1993), 721–755.

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Kigami defined the Laplacian as the limit of discrete differences on graphs approximating the fractal (applicable to *post-critically finite fractals*).

Defining the Laplacian on fractals

- U. Mosco, *Dirichlet forms and self-similarity*, in *New directions in Dirichlet forms* (J. Josta et al., eds.), Cambridge, International Press, 1998.
- U. Mosco, *Lagrangian metrics on fractals*, Proc. Symp. Appl. Math. **54** (1998), 301–323.

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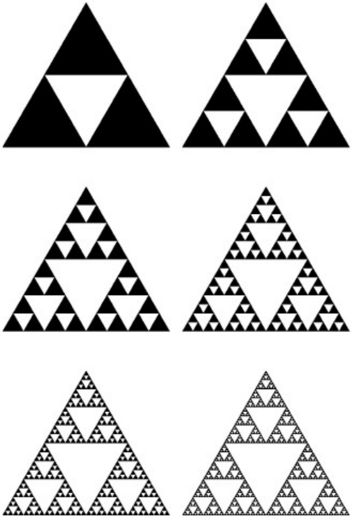
- U. Mosco, *Lagrangian metrics on fractals*, Proc. Symp. Appl. Math. **54** (1998), 301–323.

Mosco introduced a framework for the Laplacian by taking as a starting point a Dirichlet form that reflects the self-similarities of the underlying fractal (this leads to a very general theory of *variational fractals* that have been analysed by other means).

- M.T. Barlow, R.F. Bass, *Transition densities for Brownian motion on the Sierpinski carpet*, Probab. Theory Related Fields **91** (1992), 307–330.
- J. Kigami, *In quest of fractal analysis*, in *Mathematics of Fractals* (H. Yamaguti, M. Hata, J. Kigami, eds.), Providence, RI: American Mathematical Society, 1993, 53–73.
- J. Kigami, *Harmonic calculus on p.c.f. self-similar sets*, Trans. Am. Math. Soc., **335** (1993), 721–755.
- S. Kusuoka, Z.X. Yin, *Dirichlet forms on fractals: Poincaré constant and resistance*, Probab. Theory Related Fields **93** (1992), 169–196.

The Sierpinski gasket in the plane



Let $N \geq 2$ be a natural number and let $p_1, \dots, p_N \in \mathbb{R}^{N-1}$ be so that $|p_i - p_j| = 1$ for $i \neq j$.

The Sierpinski gasket in a general setting

Let $N \geq 2$ be a natural number and let $p_1, \ldots, p_N \in \mathbb{R}^{N-1}$ be so that $|p_i - p_j| = 1$ for $i \neq j$.

Define, for every $i \in \{1, \ldots, N\}$, the map $S_i \colon \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ by

$$S_i(x) = \frac{1}{2}(x + p_i).$$

Let $\mathcal{S} := \{S_1, \ldots, S_N\}$.

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Let $\mathcal{S} := \{S_1, \ldots, S_N\}$.

Denote by $F \colon \mathcal{P}(\mathbb{R}^{N-1}) \to \mathcal{P}(\mathbb{R}^{N-1})$ the map assigning to a subset A of $\mathbb{R}^{N-1}$ the set

$$F(A) = \bigcup_{i=1}^{N} S_i(A).$$

There is a unique nonempty compact subset V of $\mathbb{R}^{N-1}$, called the *attractor of the family* $\mathcal{S}$, such that $F(V) = V$ (that is, V is a fixed point of the map F).

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The SG can be constructed inductively as follows: Put $V_0 := \{p_1, \ldots, p_N\}$, $V_m := F(V_{m-1})$, for $m \geq 1$, and $V_* := \cup_{m \geq 0} V_m$. It can be proved that $V = \overline{V_*}$.

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In the sequel V is considered to be endowed with the relative topology induced from the Euclidean topology on $\mathbb{R}^{N-1}$.
V_0 is the *intrinsic boundary* of V.

Properties of V:

- Hausdorff dimension $d = \frac{\ln N}{\ln 2}$;
- $0 < \mathcal{H}^d(V) < \infty$, where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure on $\mathbb{R}^{N-1}$; let μ be the normalized restriction of $\mathcal{H}^d$ to the subsets of V, so $\mu(V) = 1$;
- $\mathrm{supp}\mu = V$, i.e.,

$$\mu(B) > 0, \text{ for every nonempty open subset } B \text{ of } V.$$

Denote by $C(V)$ the space of real-valued continuous functions on V and by

$$C_0(V) := \{u \in C(V) \mid u|_{V_0} = 0\}.$$

The spaces $C(V)$ and $C_0(V)$ are endowed with the usual supremum norm $||\cdot||_{sup}$.

Defining the space $H_0^1(V)$

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For a function $u\colon V \to \mathbb{R}$ and for $m \in \mathbb{N}$ let

$$W_m(u) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y \in V_m \\ |x-y|=2^{-m}}} (u(x)-u(y))^2.$$

We have $W_m(u) \le W_{m+1}(u)$ for every natural m, so we can put

$$W(u) = \lim_{m\to\infty} W_m(u).$$

Definition of $H_0^1(V)$

Put

$$H_0^1(V) := \{u \in C_0(V) \mid W(u) < \infty\}$$

and endow $H_0^1(V)$ with the norm

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$$||u|| = \sqrt{W(u)}.$$

In fact, there is an inner product defining this norm: For $u, v \in H_0^1(V)$ and $m \in \mathbb{N}$ let

$$\mathcal{W}_m(u,v) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y \in V_m \\ |x-y|=2^{-m}}} (u(x)-u(y))(v(x)-v(y)).$$

Put $\mathcal{W}(u,v) = \lim_{m\to\infty} \mathcal{W}_m(u,v)$.

Properties of $H_0^1(V)$:

- $\mathcal{W}\colon H_0^1(V) \times H_0^1(V) \to \mathbb{R}$ is an inner product which induces the norm $||\cdot||$.
- $(H_0^1(V), ||\cdot||)$ becomes a separable real Hilbert space.
- $H_0^1(V)$ is a dense subset of $(L^2(V,\mu), ||\cdot||_2)$.
- $\mathcal{W}$ is a Dirichlet form on $L^2(V,\mu)$.
- The inequality
$$||u||_s \leq (2N+3)||u||$$
holds for every $u \in H_0^1(V)$.
- The embedding
$$(H_0^1(V), ||\cdot||) \hookrightarrow (C_0(V), ||\cdot||_s)$$
is compact.

- **Idea**: J. Kigami, *Harmonic metric and Dirichlet form on the Sierpinski gasket*, in *Asymptotic Problems in Probability Theory: Stochastic Models and Diffusions on Fractals* (K.D. Elworthy and N. Ikeda, eds.), Longman, Harlow/New York, 1993.
- **N=3**: S.M. Kozlov, Harmonization and homogenization on fractals, *Commun. Math. Phys.* **153** (1993), 339–357.
- **The general case**: K.J. Falconer and J. Hu, Non-linear elliptical equations on the Sierpinski gasket, *J. Math. Anal. Appl.* **240** (1999), 552–573.

For every $v \in L^2(V,\mu)$ there exists a unique $u_v \in H_0^1(V)$ such that

$$\langle v, u\rangle_2 = \mathcal{W}(u_v, u), \text{ for all } u \in H_0^1(V).$$

For every $v \in L^2(V, \mu)$ there exists a unique $u_v \in H_0^1(V)$ such that

$$\langle v, u \rangle_2 = \mathcal{W}(u_v, u), \text{ for all } u \in H_0^1(V).$$

The map $\varphi\colon L^2(V, \mu) \to L^2(V, \mu)$, defined by

$$\varphi(u) = u_v \in H_0^1(V) \hookrightarrow L^2(V, \mu),$$

is

- linear, continuous, injective,
- symmetric (thus self-adjoint),
- compact.

$D := \varphi(L^2(V,\mu))$ is a dense subset both of $(H_0^1(V), ||\cdot||)$ and of $(L^2(V,\mu), ||\cdot||_2)$.

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$\Delta := -\varphi^{-1}$; $\Delta\colon D \to L^2(V,\mu)$ is a bijective, linear, and self-adjoint operator such that

$$-\mathcal{W}(u,v) = \int_V \Delta u \cdot v d\mu, \text{ for every } (u,v) \in D \times H_0^1(V).$$

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The operator Δ is called the *weak Laplacian on* V.

Let $h\colon V \times \mathbb{R} \to \mathbb{R}$ be given and possessing suitable properties. We can formulate now the following *Dirichlet problem on the SG*: Find appropriate functions $u \in H_0^1(V)$ (in fact, $u \in D$) such that

$$(P)\begin{cases} -\Delta u(x) = h(x, u(x)),\ \forall x \in V \setminus V_0, \\ \\ u|_{V_0} = 0. \end{cases}$$

A function $u \in H_0^1(V)$ is called a *weak solution of* (P) if

$$\mathcal{W}(u, v) - \int_V h(x, u(x))v(x)d\mu = 0,\ \forall v \in H_0^1(V).$$

Was investigated, among others, by:

- K.J. Falconer and J. Hu, Non-linear elliptical equations on the Sierpinski gasket, *J. Math. Anal. Appl.* **240** (1999), 552–573.
- J. Hu, Multiple solutions for a class of nonlinear elliptic equations on the Sierpinski gasket, *Sci. China Ser. A* **47** (2004), no. 5, 772–786.
- C. Hua and H. Zhenya, Semilinear elliptic equations on fractal sets, *Acta Math. Sci. Ser. B Engl. Ed.* **29 B** (2) (2009), 232–242.
- R.S. Strichartz, Solvability for differential equations on fractals, *J. Anal. Math.* **96** (2005), 247–267.
- R.S. Strichartz, *Differential Equations on Fractals. A Tutorial*, Princeton University Press, Princeton, NJ, 2006.

An important fact:

One attaches to problem (P) an energy functional $I\colon H_0^1(V) \to \mathbb{R}$ which is the key tool in the study of (P). In general:

- I is Fréchet differentiable on $H_0^1(V)$.
- $u \in H_0^1(V)$ is a weak solution of (P) $\Leftrightarrow$ u is a critical point of I.

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Methods used to investigate (P):

On can prove the existence of at least one nontrivial solution or of multiple solutions, using

- certain minimax results: mountain pass theorems, saddle-point theorems,
- minimization procedures.

The first method to investigate the Dirichlet problem on the SG (B.E. Breckner, V. Rădulescu, Cs. Varga)

Let $a\colon V \to \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}$, and $g\colon V \to \mathbb{R}$ be given. We consider the following Dirichlet problem on the SG:

$$(DP) \begin{cases} \Delta u(x) + a(x)u(x) = g(x)f(u(x)), \ \forall x \in V \setminus V_0, \\ \\ u|_{V_0} = 0. \end{cases}$$

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To apply this method we have to impose on the nonlinear term of the elliptic equation other conditions as those used in the papers mentioned before. Instead of requiring that the nonlinear term should satisfy certain symmetry properties, this term has to have an oscillating behavior.

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To apply this method we have to impose on the nonlinear term of the elliptic equation other conditions as those used in the papers mentioned before. Instead of requiring that the nonlinear term should satisfy certain symmetry properties, this term has to have an oscillating behavior.
$\hookrightarrow$ We can prove the existence of infinitely many weak solutions of (DP).

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- J. Saint Raymond, *On the multiplicity of solutions of the equation* $-\triangle u = \lambda \cdot f(u)$, J. Differential Equations **180** (2002), 65–88.

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- F. Faraci and A. Kristály, *One-dimensional scalar field equations involving an oscillatory nonlinear term*, Discrete Contin. Dyn. Syst. **18** (2007), no. 1, 107–120.
- A. Kristály and D. Motreanu, *Nonsmooth Neumann-type problems involving the p-Laplacian*, Numer. Funct. Anal. Optim. **28** (2007), no. 11-12, 1309–1326.

Assume that the following conditions hold:

(C1) $a \in L^1(V, \mu)$ and $a \leq 0$ a.e. in V.

(C2) $f : \mathbb{R} \to \mathbb{R}$ is continuous such that

- (1) There exist two sequences (a_k) and (b_k) in $]0, \infty[$ with $b_{k+1} < a_k < b_k$, $\lim_{k\to\infty} b_k = 0$ and such that $f(s) \leq 0$ for every $s \in [a_k, b_k]$.
- (2) Either $\sup\{s < 0 \mid f(s) > 0\} = 0$, or there is a $\delta > 0$ with $f|_{[-\delta,0]} = 0$.

(C3) $F : \mathbb{R} \to \mathbb{R}$, defined by $F(s) = \int_0^s f(t)dt$, is such that

- (3) $-\infty < \liminf_{s\to 0^+} \dfrac{F(s)}{s^2}$ and $\limsup_{s\to 0^+} \dfrac{F(s)}{s^2} = \infty$.

(C4) $g \in C(V)$ with $g \leq 0$ and such that the restriction of g to every open subset of V is not identically 0.

Then there is a sequence (u_k) of pairwise distinct weak solutions of problem (DP) such that $\lim_{k\to\infty} \|u_k\| = 0$. Thus $\lim_{k\to\infty} \|u_k\|_s = 0$.

The second method to investigate the Dirichlet problem on the SG (B.E. Breckner, D. Repovš, Cs. Varga)

Let $f, g\colon V \times \mathbb{R} \to \mathbb{R}$ be given. We consider the following Dirichlet problem on the SG depending on two parameters $\lambda, \eta \geq 0$

$$(DP_{\lambda,\eta}) \begin{cases} -\Delta u(x) = \lambda f(x, u(x)) + \eta g(x, u(x)), \ \forall x \in V \setminus V_0, \\ \\ u|_{V_0} = 0. \end{cases}$$

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B. Ricceri, *A further three critical points theorem*, Nonlinear Anal. **71** (2009), 4151–4157.

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B. Ricceri, *A further three critical points theorem*, Nonlinear Anal. **71** (2009), 4151–4157.
$\hookrightarrow$ We can prove the existence of at least three weak solutions of $(DP_{\lambda,\eta})$.

Hypotheses:

(C1) The function $f\colon V \times \mathbb{R} \to \mathbb{R}$ is continuous.

(C2) The function $F\colon V \times \mathbb{R} \to \mathbb{R}$, where $F\colon V \times \mathbb{R} \to \mathbb{R}$ is defined by

$$F(x,t) = \int_0^t f(x,\xi)d\xi,$$

satisfies the following conditions:

(1) There exist $\alpha \in [0,2[$, $a \in L^1(V,\mu)$, and $m \geq 0$ such that

$$F(x,t) \leq m(a(x) + |t|^\alpha), \text{ for all } (x,t) \in V \times \mathbb{R}.$$

(2) There exist $t_0 > 0$, $M \geq 0$, and $\beta > 2$ such that

$$F(x,t) \leq M|t|^\beta, \text{ for all } (x,t) \in V \times [-t_0, t_0].$$

(3) There exists $t_1 \in \mathbb{R} \setminus \{0\}$ such that for all $x \in V$ and for all t between 0 and t_1 we have

$$F(x,t_1) > 0 \text{ and } F(x,t) \geq 0.$$

Theorem (B.E. Breckner, D. Repovš, Cs. Varga)

Conclusion:
Then there exists a real number $\Lambda \geq 0$ such that, for each compact interval $[\lambda_1, \lambda_2] \subset]\Lambda, \infty[$, there exists a positive real number r with the following property: For every $\lambda \in [\lambda_1, \lambda_2]$ and every continuous function $g \colon V \times \mathbb{R} \to \mathbb{R}$ there exists $\delta > 0$ such that, for each $\eta \in [0, \delta]$, the problem $(DP_{\lambda,\eta})$ has at least three solutions whose norms are less than r.

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- Breckner, B.E., V. Rădulescu, and Cs. Varga, *Infinitely many solutions for the Dirichlet problem on the Sierpinski gasket*, to appear in Analysis and Applications.
- Breckner, B.E, D. Repovš, and Cs. Varga, *On the existence of three solutions for the Dirichlet problem on the Sierpinski gasket*, to appear in Nonlinear Analysis.