Periodic points and hyperbolicity in higher dimensional complex dynamics Eugen Mihailescu Doctor of Philosophy Mathematics 1999 Professor John E. Fornaess, Chair
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To my parents

My first and foremost thanks are due to my advisor, John-Erik Fornaess. His constant encouragement and insights in both mathematical and human matters were very valuable in this exciting, although at times stressful period of graduate studies.

I am also grateful to the professors who agreed to be on my doctoral committee and especially to Berit Stensones who prepared a detailed report about my thesis.

Thanks are due to professors at the University of Michigan who helped me through discussions or advice during my graduate studies here: David Barrett, Peter Duren, Alejandro Uribe, Andreas Blass.

I wish to express my gratitude too for the travel support to attend conferences that I received from many sources, at UM, Stony Brook, Indiana U., U. of Washington, U. of Florence, etc. This proved to be very important.

Thanks also to my graduate student friends in the Department for making life merrier and for putting things into perspective.

Last, but certainly not least my family in Bucharest, my parents, Eugenia and Stelian Mihailescu and my brother Cristi were of great moral help during these years. Their letters, phone calls and my habitual summer stay there not only helped to concentrate on my research, but also made me very happy along the way.

Contents

Chapter 1

Introduction

The research in this thesis is concerned with the behavior of higher dimensional complex dynamical systems. This field is relatively recent and it combines ideas and methods from the theory of real dynamical systems, several complex variables and global analysis on manifolds.

Following the study of Fatou components from the one dimensional case (i.e., connected components of the largest open set where the iterates of a rational map form a normal family), we are interested in the higher dimensional case of a holomorphic endomorphism f of a Kobayashi hyperbolic Stein manifold. The passage from one complex variable to several complex variables provides numerous new phenomena. For example, in one complex variable, Sullivan's theorem ([8]) states that there can be only four types of Fatou components: attracting basins of periodic orbits, parabolic domains, Siegel disks and Herman rings. In the last two cases, the map is conjugated on those components to an irrational rotation, which has one, respectively no fixed point. There are also conditions that the multiplier of a fixed point has to satisfy in order for that point to belong to a Siegel disk; these conditions deal with number theoretical concepts, like Diophantine approximation and the Brjuno condition. Siegel disks have found applications in celestial mechanics, see for example [37].

In one variable there exists also a relationship between rotation domains and the postcritical set, in particular the boundary of rotation domains is delineated by forward iterates of critical points.

Sullivan's classification in one complex variable was proved using quasiconformal mappings.

In the case of recurrent Fatou components in $\mathbf{P}^2$, a theorem of Fornaess-Sibony provides a partial answer: we know that there can only be attracting basins, basins of attraction to an invariant Riemann surface Σ, or Siegel domains.

Siegel domains appear in the higher dimensional case as Fatou components on which there exists a sequence of iterates $(f^{k_n})_n$ converging to the identity.

On Siegel domains we will have a natural S^1 or a T^2 action with discrete sets of periodic points; especially in the case of an S^1-action, there is no classification and indeed the starting point of this problem would be first to see how many “rotation centers” can there be in the given Siegel domain.

This problem can be studied in the larger context of maps of Stein, Kobayashi hyperbolic manifolds Ω actions of tori on them, because any Fatou component is Stein and Kobayashi hyperbolic, according to theorems by Fornaess - Sibony and Ueda.

The first chapter is concerned especially with the number of fixed (periodic) points that can exist in such a manifold Ω for a given T^s-torus action which does not have whole lines of fixed points. In the case when fixed points do exist, we look also at the invariant manifolds containing the fixed points.

Here it should be said that, in the case when $dim\ \Omega = s$, a theorem of Barrett - Bedford - Dadok states that the action is globally conjugated to a linear action on a Reinhardt domain, therefore its dynamics is relatively simple.

In Chapter II, we show as an independent result that, for Ω of any dimension, a torus action which has a finite number of periodic points of period m, for all m, will have in fact a finite number of periodic points of all periods altoghether (i.e there is N and $x_1, ..., x_N$ such that these are the only periodic points of the action).

Then, in complex dimension 2 we give a reasonable cohomology condition that will imply existence of at most one fixed point in Ω for an S^1 - action (see also [26]).

The key point of the proofs is the use of multidimensional residue theory combined with methods from one variable complex dynamics and topology.

We prove also a global result pertaining to the existence of “global axes of coordinates” in Ω which are invariant under the action. Examples of manifolds with S^1 - actions which do not satisfy the cohomological condition are also given in Chapter II. One consequence of Theorem 2.4.13 is that any S^1 - action without whole lines of fixed points, on a bounded domain D in $\mathbb{C}^2$, where D is of the form $D_1 \times D_2$, with D_1 simply connected and D_2 any plane domain, will have at most one fixed point. Also, as a corollary of the same theorem we get that any Siegel domain for a skew product map of $\mathbb{P}^2$ has at most one periodic point.

More generally, bounding the number of fixed points of an S^1 action in the Stein manifold Ω (action without whole lines of fixed points), by the number of generators of the group $H^2(\Omega, \mathbb{Z})$ will be proved to hold in Chapter III. The general idea of this proof is to apply the construction of ”global axes of coordinates” from Theorem 2.4.13 to obtain submanifolds invariant to the action, and then to construct cocycles with their defining functions.

Let us notice that the more general case of holomorphic actions on Stein manifolds is also important from the point of view of vector field results. This problem has been studied by several researchers, including Forstneric [17], Palis, Suzuki, etc., in conjunction to singularities of vector fields. Indeed, if V is an $\mathbb{R}$ -

complete vector field,its flow ϕ_t defines an action of ($\mathbb{R}, +$) on Ω by holomorphic automorphisms. Conversely, one associates to every action $\phi : \mathbb{R} \times \Omega \to \Omega$ by holomorphic automorphisms of Ω the vector field $V(\phi)(x) = \frac{d}{dt}\phi(t,x)|_{t=0}$, called the infinitesimal generator of ϕ. For each nonfixed point of such an action, the maximal complex orbit C_x is generically biholomorphic to one of the following Riemann surfaces: a complex line $\mathbb{C}$, a punctured complex line $\mathbb{C}^\star$, the unit disk, the punctured unit disk, or an annulus.

In the last two cases, the action ϕ factors through an action of the circle group. This is making the connection with our study.

Chapter III will also study several examples of Siegel domains for holomorphic endomorphisms of $\mathbf{P}^2$; in particular we will introduce a new type of example, obtained by using a partial linearization result and a Brjuno arithmetic condition.

Chapter IV deals with the "opposite case", i.e., when the map f of $\mathbf{P}^2$ is hyperbolic on its nonwandering set. In this case it is not difficult to prove that there can be no Siegel domains. Indeed, a Siegel domain, if it would exist, would be a component of the non-wandering set; however, the fact that there exists a sequence of iterates on it converging to the identity, would contradict the hyperbolicity condition for f.

A slightly stronger notion of hyperbolicity has also been studied by Fornaess - Sibony in [15]. Here, also the No Cycle Condition ([35]) is playing an important role, making possible the use of methods from the diffeomorphism case, like the filtration method, in order to prove stability results.

We introduce the notion of ws - hyperbolicity and show in Theorem 4.3.1 that it is open in the set of all holomorphic mappings of $\mathbf{P}^2$. This will provide a large number of examples of ws - hyperbolic maps obtained from simpler ones.

An interesting set is also $K^-(f)$ which is the set of points whose preimages do not converge uniformly towards the support of the classical ergodic invariant measure of f, μ_f. This is similar to the set K^- of points with bounded backward iterates, from the case of Henon diffeomorphisms ([5]). We show that K^- is the union of unstable sets of nonwandering points of unstable index 1 and of the attracting periodic points (see also [15], where the stronger notion of s-hyperbolicity is not needed), and then, using auxiliary hyperbolicity results by Brody and Siu - Yeung, we prove that the interior of K^- contains only wandering points, except eventually for a finite number of attracting periodic points.

In many cases of interest, like product functions, functions constructed by Ueda's method, or perturbations of real solenoids, the set K^- is actually an algebraic variety so its interior is empty. This might hold for a larger class, if not for all ws-hyperbolic mappings.

In the case of ws-hyperbolic mappings on $\mathbf{P}^2$ having a totally invariant hyperplane Π, the repelling part of the non-wandering set, S_2, has interesting properties. Bedford-Jonsson [4] showed that it is actually a topological quotient of $J_\Pi \times S^1$, where J_Π is the Julia set of the restriction of f to the hyperplane

Π. This means that whenever J_Π is connected or locally connected, the same properties will be shared by S_2.

Finally we prove that any repellor with empty interior will have zero Lebesgue measure for a special class of functions. In this proof we will use estimates of the derivatives of inverse iterates and then the Lebesgue Differentiation Theorem.

Chapter 2

Periodic Points for Actions of Tori

2.1 Motivation

In one complex variable dynamics, Sullivan's theorem ([8]) gives a complete classification of the Fatou components that can appear for a rational map f.

Consequently we can have only periodic components U and U can be one of the following : 1. U attracting basin; 2. U parabolic domain; 3. U Siegel disk; 4. U Herman ring.

Cases 3. and 4. are called *rotation domains.* In these cases, the rational map f is conjugated on U to an irrational rotation, hence by taking all the iterates of f and their limits we obtain an S^1-action on U which has at most one fixed (periodic) point in U.

The goal of this chapter is to give conditions when the generalization of the one variable situation is true. In particular we consider actions of tori on Stein manifolds and study their periodic (fixed) points.

Outline of the chapter:

The main results of this chapter are contained in Theorems 2.3.1, 2.4.13 and Proposition 2.4.8.

In §1. we introduce notation and basic definitions; also some examples of Siegel domains are reviewed.

In §2. we will prove that the number of periodic points (of all periods) for a large class of actions of tori on Stein manifolds is finite.

Then, §3. will give topological conditions on the manifold guaranteeing the existence of at most one fixed point. In particular, this will become true when the second cohomology group with integer coefficients vanishes.

Finally, §4. introduces an example of an S^1-action on a connected and Kobayashi hyperbolic Stein manifold having exactly two fixed points.

2.2 Basic facts and examples

First we will introduce the notions of pseudoconvex manifold, domain of holomorphy, Kobayashi distance and Kobayashi hyperbolicity, and also that of a Stein manifold.

The following definition of a pseudoconvex (or holomorphically convex) manifold extends the definition of a convex region in $\mathbb{C}^n$ as being equal to its envelope under linear functions.

Definition 2.2.1. *A complex manifold M of dimension n is called pseudoconvex (or holomorphically convex) if, for any compact set $K \subset M$, we have that the holomorphic envelope of K, $\hat{K} := \{z \in M, |f(z)| \leq sup_K|f|, \forall f \in O(M)\}$ is relatively compact in M.*

There are several other equivalent definitions of pseudoconvexity, one of them being for instance the existence of a globally plurisubharmonic function on M which converges to ∞ towards the boundary.

The importance of pseudoconvex manifolds resides in the fact that they are the natural domains of definitions for holomorphic functions. If a domain is not pseudoconvex, then all holomorphic functions on that domain extend to a fixed strictly larger domain.

Related to the notion of holomorphic convexity is that of Stein manifolds:

Definition 2.2.2. *A manifold Ω of complex dimension n is called Stein if it satisfies the following three conditions:*

i) holomorphic convexity.

ii) holomorphic separability, i.e for any two points p, q in Ω, there exists a global holomorphic function $f \in O(\Omega)$ such that $f(p) \neq f(q)$.

iii) $\forall p \in \Omega$, there are n global functions $f_\nu \in O(\Omega)$, forming a system of local coordinates in a neighbourhood of p.

Stein manifolds have many interesting properties given by their abundance of global holomorphic functions. For example any Stein manifold of dimension n can be embedded properly into $\mathbb{C}^{2n+1}$ (see [20], [10]).

Next we will define the notion of domain of holomorphy in $\mathbb{C}^n$ or $\mathbf{P}^n$; this corresponds to the idea of a domain U on which holomorphic functions do not extend to a fixed larger domain, even if the boundary of U is complicated.

Definition 2.2.3. *A domain $U \subset \mathbb{C}^n$ or $U \subset \mathbf{P}^n, U \neq \mathbf{P}^n$, is called a domain of holomorphy if there is no point $p \in \partial U$, no fixed connected neighbourhood V of p and no nonempty component W of $V \cap U$ so that whenever f is holomorphic on U, there exists a holomorphic function g on V so that $f = g$ on W.*

The following theorems of Takeuchi are given in [14].

Theorem 2.2.4. *If $U \subset \mathbb{C}^n$ or $\mathbf{P}^n, U \neq \mathbf{P}^n$ and U is pseudoconvex, then U is a domain of holomorphy.*

Theorem 2.2.5. *The complement of a domain of holomorphy in* $\mathbf{P}^n, n \geq 2$, *is connected.*

Here one can make the observation that in $\mathbb{C}$ or $\mathbf{P}^1$, any open domain is pseudoconvex. Also, the analogue of Theorem 2.2.5 is not true, there exist rational maps with totally disconnected Julia sets, see [8].

Theorem 2.2.6. *A domain of holomorphy in* $\mathbf{P}^n, n \geq 2$, *cannot contain a compact complex analytic set of positive dimension.*

We will see an interesting consequence of this theorem after we will define the Fatou set and the Julia set of a map on $\mathbf{P}^n$, in 2.2.13.

Now let us define the important concept of Kobayashi distance and Kobayashi hyperbolicity for a complex manifold M.

Denote by ρ the Poincare distance on the unit disk Δ, which, for two points $z, w \in \Delta$, is given by the formula

$$\rho(z, w) = ln\frac{|1-z\bar{w}|+|z-w|}{|1-z\bar{w}|-|z-w|}.$$

Denote by $O(\Delta, M)$ the set of holomorphic functions from the unit disk Δ to M. Let $p, q \in M$ two given points. We will denote by a *chain in M from p to q*, a collection σ of m functions $g_j \in O(\Delta, M)$ and m pairs of points $\xi_j', \xi_j'' \in \Delta, j = 1..m$, so that $g_1(\xi_1') = p, g_m(\xi_m'') = q$ and $g_j(\xi_j'') = g_{j+1}(\xi_{j+1}'), j = 1..m-1$.

Definition 2.2.7. *The Kobayashi pseudometric on M between p and q is defined to be equal to the quantity*

$$d_M(p, q) = inf_\sigma \sum_{j=1}^{m} \rho(\xi_j', \xi_j''),$$

where ρ is the Poincaré distance in the unit disk and where the infimum is taken with respect to all chains of arbitrary length in M, joining p, q.

The function $d_M(\cdot, \cdot)$ is positive, symmetric and satisfies the triangle inequality, so it is a pseudodistance. The Kobayashi pseudodistance is being contracted by holomorphic mappings.

Theorem 2.2.8. *If M and N are complex manifolds and $f : M \to N$ is holomorphic, then $d_N(f(p), f(q)) \leq d_M(p, q)$.*

Therefore, it is easy to see that the Kobayashi pseudodistance is preserved by biholomorphisms.

Definition 2.2.9. *The complex manifold M is called Kobayashi hyperbolic if $d_M(\cdot, \cdot)$ is actually a distance, i.e if $d_M(p, q) \neq 0$ for $p \neq q$.*

Clearly, since the Kobayashi distance on Δ coincides with the Poincaré distance, it is obvious that Δ is Kobayashi hyperbolic; similarly it can be shown that any ball in $\mathbb{C}^n$ is Kobayashi hyperbolic. In fact, from Theorem 2.2.8 we see that any submanifold of a hyperbolic manifold is going to be also hyperbolic.

In particular it means that it is impossible to embed a holomorphic copy of $\mathbb{C}$ or $\mathbb{C}^\star$ into a Kobayashi hyperbolic manifold.

In the following we will consider Ω a Stein manifold, which is also supposed to be Kobayashi hyperbolic. The following theorem of Kobayashi [25], which will be used in the sequel, shows that the group of biholomorphisms on a hyperbolic manifold has special properties:

Theorem 2.2.10. *If Ω is hyperbolic, then its group of holomorphic automorphisms, $Aut(\Omega)$, is a Lie group.*

The following lemma follows from the structure theorem of commutative Lie groups (for example Onishchik [30]); for the convenience of the reader, a proof of the lemma may be found in Ueda [39].

Lemma 2.2.11. *Let G be a Lie group and assume there exists an element f in G such that the subgroup generated by f is infinite cyclic and there exists a subsequence $(f^{j_n})_n$ converging to the identity element. Then the closure of $(f^n)_n$ is compact and it contains a torus $\boldsymbol{T}^s$, for some $s > 0$.*

Hence by taking $G = Aut(\Omega)$ with the compact-open topology, and $f \in Aut(\Omega)$ so that the hypothesis of the lemma is satisfied, we will obtain a $\boldsymbol{T^s}$-action on Ω .

For future reference, let us write down that in the notation of Ueda, H is the closure of $(f^j)_j$ in G and H_0 is the connected component of H containing the identity element.

If $s = \dim_{\mathbb{C}}\Omega$, then the action is described by a theorem of Barrett- Bedford-Dadok [3] :

Theorem 2.2.12. *In the above assumptions(so $G = Aut(\Omega)$), if $s = dim_{\mathbb{C}}\Omega$, there exists $\tilde{\Phi} : \Omega \to U$, a biholomorphism of Ω to a Reinhardt domain U, and an integer $l > 0$ so that $\tilde{\Phi} \circ f^l = R \circ \tilde{\Phi}$, with $R(z_1, ..., z_s) = (e^{i\alpha_1} z_1, ..., e^{i\alpha_s} z_s)$.*

Consequently we may have at most one fixed point of f in Ω in this case. However there is no classification as the one above for general actions of tori on Stein manifolds.

A natural example of a Stein manifold with a torus action is provided by a Siegel domain for a holomorphic map on a projective space.

In the following definition the notion of degree of a holomorphic mapping $f : \mathbf{P}^n \to \mathbf{P}^n$ will be in the sense of Fornaess - Sibony ([12], [14]), i.e as degree of a homogeneous polynomial lifting for f, $F : \mathbb{C}^{n+1} \setminus 0 \to \mathbb{C}^{n+1} \setminus 0$.

Definition 2.2.13. *Let $f : \mathbf{P}^n \to \mathbf{P}^n$ a holomorphic map of degree $d \geq 2$.*

The set $\mathcal{F} = \{z \in \mathbf{P}^n$, z has a neighbourhood V s.t $(f^j)_j$ forms a normal family in $V\}$ is called the Fatou set of f. Its complement in $\mathbf{P}^n$ is called the Julia set of f.

Let us review some of the properties of the Fatou and Julia sets. First, a Fatou component is a domain of holomorphy according to a theorem of Fornaess-Sibony [14], so we can apply in this case the theorems of Takeuchi given above.

Theorem 2.2.14. *Every Fatou component of a holomorphic map on $\mathbf{P}^n$ is a domain of holomorphy.*

This implies, according to 2.2.5 and 2.2.6 that the Julia set of a function in $\mathbf{P}^n, n \geq 2$ is always connected and that it intersects any analytic set of positive dimension in $\mathbf{P}^n$. In particular the Julia set intersects the critical set of f, which is an analytic variety. This is in contradiction to the one variable case, where there are rational functions whose Julia sets are totally disconnected; also another contradiction is that, if a rational function is expanding on its Julia set, then its entire critical set is contained in a union of Fatou components.

More generally it can be shown ([14]) that, if T is the invariant current whose support is the Julia set J, then T^l is well defined as a positive closed current on $\mathbf{P}^n$ and supp T^l is connected whenever $2l \leq n$. The definition of T is given in Section 4.2.

Ueda proved in [40] that:

Theorem 2.2.15. *Fatou components are Kobayashi hyperbolic.*

Definition 2.2.16. *A connected component U of $\mathcal{F}$ is called a Siegel domain if there exists a subsequence $(f^{j_\nu})_\nu$ converging uniformly on compacts to id_U.*

This notion has been introduced by Fornæss and Sibony ([11], [13]) as a generalization of the Siegel disks and Herman rings from one variable.

Directly from the definition it is clear that any Siegel domain is periodic, i.e there is $m > 0$ s.t f^m(U)=U. It follows easily that f:U $\rightarrow$ U is a biholomorphism (we may assume WLOG that $m = 1$, hence that f(U)=U) and, from degree considerations, also that there is no $q > 0$ s.t $f^q/_U = id_U$.

We notice also that, since according to Theorem 2.2.15, U is Kobayashi hyperbolic, $G = Aut(\Omega)$ will be a Lie group (from 2.2.10) and we are in the conditions of Lemma 2.2.11.

Remark 2.2.17.

Examples of Siegel domains in $\mathbf{P}^2$ can be constructed by the method of Ueda ([41], starting from rational functions on $\mathbf{P}^1$ that have Siegel disks or Herman rings.

The main idea is to double cover $\mathbf{P}^2$ with $\mathbf{P}^1 \times \mathbf{P}^1$ by means of a projection π such that the diagonal of $\mathbf{P}^1 \times \mathbf{P}^1$ is the branch locus and $\pi([\xi : \eta], [\xi\prime : \eta\prime]) = \pi([\xi\prime : \eta\prime], [\xi : a])$. Then, if $f : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ is rational, there will exist a holomorphic mapping $F : \mathbf{P}^2 \rightarrow \mathbf{P}^2$ s.t $F \circ \pi = \pi \circ \hat{f}$, where $\hat{f} = (f, f)$.

Hence if U is a Siegel disk for f, then since f has no periodic points of period two in U (other than the fixed point), the associated map F has only one fixed point in π(U$\times$ U) . Similarly, if U is a Herman ring for f, we obtain that π(U$\times$ U) is a Siegel domain for F with no fixed points.

Taking the iterates of F on U and their limits in $Aut(\Omega)$ will provide us with an example of an action on a Siegel domain which has no fixed points. We will give more details about this method in Chapter III.

2.3 Periodic points

Theorem 2.3.1. *Let Ω be a Stein manifold of dimension n and assume that Ω is Kobayashi hyperbolic. Assume also that we are in the conditions from the hypothesis of Lemma 2.2.11 with $G = Aut(\Omega)$ and that the holomorphic automorphism f satisfies the following condition :*

($\star$)

for all $m > 0$, m integer, there exists only at most a finite number of solutions of the equation f^m $(x) = x$ (we call these solutions periodic points of period m).

Then the number of periodic points of f of all periods, belonging to Ω is finite.

Proof

As in the proof of Lemma 2.2.11 (see Ueda [39]), if H_0 is the connected component of $(f^j)_j$ in $Aut(\Omega)$ containing the identity, then $\exists p \in \mathbb{Z}$ such that H_0 is the closure of $(f^{pj})_j$ and there is a Lie group isomorphism $\Phi : H_0 \to \mathrm{T}^s$, for some $s > 0$.

Then $\Phi(f^p) = (e^{i\alpha_1} z_1, ..., e^{i\alpha_s} z_s)$.

Assume that one of the $\alpha_1, ..., \alpha_s$ is rational, for instance $\alpha_1 = \frac{m_1}{n_1}$.

Then $\Phi(f^{pm}) = (e^{2\pi i m \alpha_1}, ..., e^{2\pi i m \alpha_s})$ and obviously $e^{2\pi i m \alpha_1}$ takes only a finite number of values. Therefore , $(f^{pm})_m$ would not be dense in H_0, contradiction. So all α_j's are irrational.

Suppose now that we can find an infinite number of periodic points in Ω. Because of the condition ($\star$), we should have a sequence of integers $m_j \to \infty$ and points p_j s.t p_j is a periodic point of period m_j, for all j.

Let $j > 0$ be fixed and consider m arbitrary s.t $f^m \in H_0$. Since all α_k's are irrational, it is clear that the closed subgroup generated by f^{pm_j} is still dense in H_0 , for all j. Consequently there is a sequence k_q s.t $f^{pm_j k_q} \to f^m$ in H_0, when $q \to \infty$.

So $f^{pm_j k_q}(p_j) = \underbrace{f^{m_j} \circ ... \circ f^{m_j}}_{p \cdot k_q}(p_j) = p_j \to f^m(p_j)$, hence p_j is a periodic point of period m. But j was chosen arbitrarily, so all points p_j are periodic of period m, which would contradict ($\star$).

$\diamond$

Remark 2.3.2. *Let $f : \mathbf{P}^n \to \mathbf{P}^n$ a holomorphic mapping of degree $d \geq 2$ and Ω a Siegel domain of f. Now Ω is Stein and Kobayashi hyperbolic and f satisfies condition ($\star$) , (Fornæss - Sibony [9]).*

Hence we get that there exist at most a finite number of periodic points of f (of all periods) belonging to Ω.

2.4 Fixed points for holomorphic S^1 - actions in Stein manifolds

In this paragraph we shall specialize our results to the case of Stein manifolds of dimension 2. We consider $f \in Aut(\Omega)$, a holomorphic automorphism of Ω , s.t the closure of $(f^n)_n$ in $Aut(\Omega)$ (with the compact-open topology) is isomorphic as a Lie group to S^1.

Assume that 0 is a fixed point of f in Ω and also that f satisfies $(\star)$.

Ω is not supposed for the moment to be Kobayashi hyperbolic. First let us state a local linearization result (see also [1] and [21]).

Lemma 2.4.1. *If Ω is a complex manifold of dimension 2 and $f \in Aut(\Omega)$ is generating an S^1 - action as above, then there will exist a local biholomorphism ϕ from a neighbourhood U of 0 in Ω to a neighbourhoood W of 0 in $\mathbb{C}^2$, so that*

$$\phi \circ f = A \cdot \phi \text{ on } U,$$

where $A := Df(0)$.

Note : we denote by 0 both the point in Ω and the point in $\mathbb{C}^2$; which one is meant will become clear from the context.

Proof : Let us fix a small coordinate chart V near 0 in Ω.

Since f generates a compact group action on Ω, we can find a neighbourhood U of 0 in Ω s.t f^n (U) $\subset$ V, for all integers n.

By taking $\tilde{U} = \cap_{n\in\mathbb{Z}} f^n(U)$, we can show that $\tilde{U}$ is a neighbourhood of 0 and that $f(\tilde{U}) = \tilde{U}$.

Indeed if $\tilde{U}$ would not be a neighbourhood of 0, then we could find a sequence of points p_{n_k} converging to 0, but so that $f^{n_k}(p_{n_k}) \notin$ U, for all k.

But since we can extract a subsequence of f^{n_k} that converges uniformly on compacts to an automorphism g such that $g(0) = 0$, we would get that $f^{n_k}(p_{n_k}) \in$ U for large k, hence contradiction.

So if f preserves a neighbourhood of 0 in Ω, it is enough to solve the problem locally in $\mathbb{C}^2$.

Hence let denote by A the derivative of f at 0 and define the following functions near o:

$$\varphi_n(z) = \frac{1}{n} \cdot \sum_{j=1}^{n-1} A^{-j} f^j(z)$$

Let us consider now the matrix A . We will show that A = $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $|\lambda_1| = |\lambda_2| = 1$.

By a suitable change of coordinates one can assume that A is in its Jordan canonical form, so

$A = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$ or A = $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

But in the first case $A^n = \begin{pmatrix} \lambda^n & 0 \\ n\lambda^{n-1} & \lambda^n \end{pmatrix}$ and since there exists a sequence of iterates of f converging to the identity, we would get a contradiction.

So $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and using the same argument as above, $|\lambda_1|$ and$|\lambda_2|$ must be 1.

Now, since f preserves the neighbourhood $\tilde{U}$ of 0, there exists a positive constant M s.t $|f^j| \leq M, \forall j$ and $\forall z \in \tilde{U}$.

However, as A is unitary, it is obvious that $|\varphi_n(z)| \leq M, \forall n, \forall z$.

Now the ball B(0,2M) in $\mathbb{C}^2$ is a convex bounded domain and therefore it is taut.

Considering φ_n as holomorphic maps from $\tilde{U}$ to this ball, we obtain a normal family, which does not converge towards the boundary.

Let φ be a limit of a subsequence $(\varphi_{k_n})_n$.

But $\varphi_{k_n} \circ f = A \circ \varphi_{k_n} + O(\frac{1}{k_n})$, which implies that $\varphi \circ f = A \circ \varphi$ near 0.

$f(0)$= 0, hence it follows that $\varphi(0) = 0$ in $\mathbb{C}^2$. Also $D\varphi(0) = Id$, directly from the formula for φ_n.

So φ is the local biholomorphism near 0 which will conjugate f to its derivative.

Knowing that f generates an S^1 action we can say more about the matrix A from the previous Lemma.

Lemma 2.4.2. *In the same setting as in Lemma 2.4.1, there will exist* $q_1, q_2 \in \mathbb{Z}$ *and*

λ *with* $|\lambda| = 1$ *and* λ *is not a root of unity, such that* $A = \begin{pmatrix} \lambda^{q_1} & 0 \\ 0 & \lambda^{q_2} \end{pmatrix}$.

Proof In the proof of Lemma 3.1 we showed that $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $|\lambda_1| = |\lambda_2| = 1$.

Assume one of λ_1 or λ_2 is a root of unity, for instance $\lambda_1^m = 1$.

Then, since A generates an S^1 - action as well, λ_2 cannot be also a root of unity.

But because the closure of the sequence of iterates must be only one copy of S^1, m must be 1, and in this case A would have the form from the statement.

Suppose now that neither λ_1 nor λ_2 are roots of unity.

Let H denote the closed group generated by A in GL(2,$\mathbb{C}$).

Let $\Phi : S^1 \to H$ the Lie group isomorphism and $\Phi_1 = p_1 \circ \Phi, \Phi_2 = p_2 \circ \Phi$, where p_1, p_2 are the projections on the first, respectively second factor of the diagonal of a matrix of the above type.

So Φ_1, Φ_2 are group endomorphisms of S^1.

But any endomorphism of S^1 looks like $z \to z^q$, for some $q \in \mathbb{Z}$.

So there exists q_1, q_2 s.t $\Phi_1(z) = z^{q_1}, \Phi_2 = z^{q_2}$, hence $\lambda_1^{q_2} = \lambda_2^{q_1}$.

If $q_1 = q_2$, then $\lambda_2 = \varepsilon\lambda_1$, with ε a root of unity , and then $A^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \varepsilon^n \lambda_1^n \end{pmatrix}$, so the closure of $(A^n)_n$ would contain several copies of S^1, unless $\varepsilon = 1$.

In this case $A = \lambda_1 \cdot Id$.

Assume now that $q_2 > q_1$, but q_1 does not divide q_2.

Then $\lambda_2 = \lambda_1^{\frac{q_2}{q_1}}$, where we fixed a value of the fractional power $\lambda_1^{\frac{1}{q_1}}$.

Obviously $\lambda_1 = (\lambda_1^{\frac{1}{q_1}})_1^q$, so by denoting $\lambda =$ the fixed value of $\lambda_1^{\frac{1}{q_1}}$, we get $\lambda_1 = \lambda^{q_1}, \lambda_2 = \lambda^{q_2}$; this implies that A has the claimed form in this case as well.

Among matrices that have the above form, we can differentiate between those for which q_1, q_2 have the same sign and the ones with q_1, q_2 of opposite signs. This classification does not depend on which λ we choose above and it will prove to be essential in the study of S^1 -actions.

Case 1

Let us consider first the case when q_1, q_2 have the same sign.WLOG we may suppose that q_1, q_2 are both positive, otherwise we just take f^{-1} instead of f and clearly f^{-1} would have the same fixed points as f. Although versions of the next theorem are more or less known ([7]), we will give nevertheless a complete proof for future reference and notation.

Theorem 2.4.3. *Let Ω be a Stein manifold of dimension 2 with a holomorphic S^1 -action generated by an automorphism f as above.*

If the eigenvalues of the derivative of f at 0 look like $\lambda_1 = \lambda^{q_1}, \lambda_2 = \lambda^{q_2}$, $q_1, q_2 \in \mathbb{Z}$ and λ not a root of unity, and if q_1, q_2 have the same sign, then the action will have exactly one fixed point in Ω.

Proof Assume as above that 0 is a fixed point for f.

First let us notice that there is a neighbourhood U of 0 s.t we can continue the orbits of points in U to analytic disks in U going thru 0.

Indeed, by Lemma 2.4.1 one can assume that $f = A = \begin{pmatrix} \lambda^{q_1} & 0 \\ 0 & \lambda^{q_2} \end{pmatrix}$; choose $z_0 = (z_{0,1}, z_{0,2}) \in U$.

Then the orbit of z_0 under the S^1 - action generated by f looks like $(\mu^{q_1} \cdot z_{0,1}, \mu^{q_2} \cdot z_{0,2})$, $\mu \in S^1, q_1, q_2 > 0$.

Hence $\Delta \ni \tilde{z} \to (\tilde{z}^{q_1} \cdot z_{0,1}, \tilde{z}^{q_2} \cdot z_{0,2})$ would be a parametrization for an analytic disk passing through 0, such that if $\tilde{z} \in S^1$, then we obtain the orbit of z_0.

Suppose now there exists another fixed point $p \in \Omega$.

Since Ω is Stein, we can find g holomorphic on Ω s.t $g(0) = 0, g(p) = 1$.

Define $\tilde{g}(z) = \int_{S^1} g(\theta \cdot z) d\mu(\theta)$, where $\theta \cdot z$ denotes the action of θ on z and μ is the normalized Haar measure on S^1.

Then $\tilde{g}$ is well defined and since the action is holomorphic, it gives a holomorphic function on Ω. However $\tilde{g}$ must be constant on orbits and since both 0 and p are fixed points, $\tilde{g}(0) = 0, \tilde{g}(p) = 1$.

We know that near 0 the orbits of the action are boundaries of analytic disks passing through 0, therefore considering the restriction of $\tilde{g}$ to one of these analytic disks, we get that $\tilde{g}$ is constant also in the interior.

But since $\tilde{g}(0) = 0$, it follows that $\tilde{g} \equiv 0$ near 0, hence in the entire Ω.

On the other hand $\tilde{g}(p) = 1$, which leads to a contradiction.

Case 2

The case when q_1, q_2 have different signs is more complicated, mainly because we do not have analytic disks passing thru 0 and whose boundaries are orbits of the action. In this situation the orbits can only be "complexified" up to analytic annuli in Ω. In what follows we will also need the condition Ω Kobayashi hyperbolic.

In the sequel we will obtain a foliation of Ω with 1-dimensional complex varieties. Let us summarize how to obtain this foliation.

As in the proof of Theorem 2.4.3, we can consider the average over orbits of a fixed arbitrary non-constant holomorphic function $g \in O(\Omega)$ (assuming such a function g exists s.t $\tilde{g}$ is not constant).

Notation 2.4.4. *If $f \in Aut(\Omega)$ generates an S^1 - action and if $g \in O(\Omega)$, then define its averaged function $\tilde{g}(z) = \int_{S^1} g(\theta \cdot z) d\mu(\theta)$, where μ is the normalized Haar measure on S^1 and $\theta \cdot z$ denotes the action of θ on z through the S^1 - action generated by f.*

Like in the proof of Theorem 2.4.3, one can easily check that $\tilde{g}$ is holomorphic on Ω and it is constant on the orbits of the S^1 - action.

If we take the level sets $L_C(\tilde{g})$ of this averaged function, we will obtain complex 1-dimensional varieties in Ω. Since $\tilde{g}$ is constant on orbits one can see that the action preserves the sets $L_C(\tilde{g})$.

Also, if $z_0 \in \Omega$ and $O(z_0)$ denotes its orbit, $\exists C$ complex number, s.t $O(z_0) \subset L_C(\tilde{g})$. Then, we will consider the global irreducible component of $L_0(\tilde{g})$ containing the local "z-axis" near 0.

One can show that this global irreducible component is an analytic submanifold of dimension 1, invariant to the action of f, playing the role of a "global z-axis" in Ω. This will be used in the end for the topological condition in Theorem 3.9, see also [26].

Let us make this submanifold construction more precise in the next lemmas and in Proposition 2.4.8.

Firstly one has to make sure there is a non-constant averaged function $\tilde{g}$ as above.

Lemma 2.4.5. *In the notation from 2.4.4, if Ω is a Stein manifold, there exists a holomorphic function $g \in O(\Omega)$ such that $g(0) = 0$, but $\tilde{g} \not\equiv 0$.*

Proof First we will recall briefly the definition of an Oka-Weil domain (see [20])

Definition : An *Oka - Weil domain* U in a holomorphic variety V is an open subset of V with the property that there are finitely many global holomorphic functions $f_1, ..., f_N \in O(V)$ s.t the restrictions of these functions to U are the coordinate fu nctions of a biholomorphic mapping between U and a holomorphic subvariety of an open polydisk in $\mathbb{C}^N$.

$\diamond$

Hence, since Ω is a Stein manifold, we can find a neighbourhood of 0, denoted by U, which is an Oka - Weil domain.

We can also assume that U is invariant to the action and that f is linearizable in U. Suppose now that for all $g \in O(\Omega)$ with $g(0) = 0$, we have $\tilde{g} \equiv 0$. Consider the function $h \in O(U)$ which in the local linearizing coordinates (z, w) has the formula $(z, w) \to z^{|q_2|} \cdot w^{q_1}$, if $q_1 > 0, q_2 < 0$. Then, since U is an Oka -Weil domain one can approximate h uniformly on compacts of U by global holomorphic functions g_n (Gunning [20]).

By taking $g_n - g_n(0)$, we can suppose that $g_n(0) = 0$.

Then $\tilde{g_n}(z, w) = \int_{S^1} g_n(\lambda^{q_1} \cdot z, \frac{1}{\lambda^{|q_2|}} \cdot w) d\mu(\lambda)$, for $(z, w) \in U$, and clearly we get $\tilde{g_n} \to \tilde{h}$ on U. But if $\tilde{g_n} \equiv 0, \forall n$, then also $\tilde{h} \equiv 0$. On the other hand by the formula for $\tilde{h}$, $\tilde{h}(z, w) = \int_{S^1} \lambda^{q_1 |q_2|} \cdot z^{|q_2|} \cdot \frac{1}{\lambda^{|q_2| \cdot q_1}} \cdot w^{q_1} d\mu(\lambda) = z^{|q_2|} \cdot w^{q_1} \not\equiv 0$, contradiction. Hence we proved that there is a function $g \in O(\Omega)$, $g(0) = 0$ s.t $\tilde{g} \not\equiv 0$.

$\diamond$

In the following we will only consider functions g of the type reffered in Lemma 2.4.5.

Lemma 2.4.6. *In the above setting, the analytic set $L_0(\tilde{g})$ has only two local irreducible components near 0.*

Proof Since $\tilde{g}(0) = 0$, but $\tilde{g} \not\equiv 0$, we can decompose $L_0(\tilde{g})$ near 0 as a union of a finite number of irreducible analytic sets: $L_0(\tilde{g}) \cap U = S_1 \cup ... \cup S_l$.

Also, $\forall k$, $f^k(S^1) \subset S_1 \cup ... \cup S_l$. But $f^k(S_1)$ is an irreducible analytic set, so $f^k(S_1) \subset \{S_1, .., S_l\}$, hence since f is a biholomorphism of U, there will exist k_1 s.t $f^{k_1}(S_1) = S_1$. Similarly we can prove that $\exists k$ s.t $f^k(S_i) = S_i, \forall i = 1..l$.

Because the problem is local, f can be assumed to be equal to $A = \begin{pmatrix} \lambda^{q_1} & 0 \\ 0 & \lambda^{q_2} \end{pmatrix}$, q_1, q_2 of different signs, for example $q_1 > 0, q_2 < 0$.

Since f^k has the same form as f, we may assume WLOG that $f(S_i) = S_i, i = 1..l$. The question is reduced now to studying for which germs of analytic sets S_i, at 0, we get invariance by f . An easy exercise in one complex variable will prove that the only analytic germs at 0, which are invariated by the above matrix are the coordinate axes. This implies that $l = 2$, so $L_0(\tilde{g})$ as only two irreducible local components near 0.

$\diamond$

In the sequel let us fix a holomorphic function g_0 on Ω s.t $g_0(0) = 0$ and $\tilde{g_0} \not\equiv 0$ in Ω. Denote by S the global irreducible component of $L_0(\tilde{g_0})$ which contains the local "z-axis" near 0. Here and in the sequel, by local z - axis near 0 in Ω we mean the preimage of the z-axis near $0 \in \mathbb{C}^2$ by the local biholomorphism Φ of Lemma 2.4.1.

Remark 2.4.7. *In fact, since any two irreducible analytic subsets of Ω of dimension 1 coinciding in an open set are actually identical globally (this follows from the identity principle), one can see that the set S does not depend on the defining function $\tilde{g_0}$.*

We will show that this set S plays the role of a "global z-axis" passing through 0 in Ω.

Proposition 2.4.8. *Let g_0 be one of the functions whose existence is proved in Lemma 2.4.5, i.e so that $g_0 \in O(\Omega)$, $g_0(0) = 0$, but the averaged function $\tilde{g_0} \not\equiv 0$. If S is the global irreducible component of $\tilde{g_0}^{-1}(0)$ containing the local z-axis near 0, then $Sing(S) = \emptyset$, so S is a complex submanifold of dimension 1 in Ω.*

Proof $Sing(S)$ is an analytic set of dimension 0, so it is a discrete subset in Ω, without accumulation points in the interior. We will show first that 0 is not a singular point of S; the proof in general will follow then easily from this.

Lemma 2.4.9. *In the above notation, the point 0 (fixed by the action) is not a singular point for S.*

Proof Assume 0 is a singular point of S; this will be shown to give a contradiction. $\exists$ a small neighbourhood in which S is the union of the two local coordinate axes (one can actually identify the z- and w- axes near 0 in $\mathbb{C}^2$ with their preimages in Ω). We can also assume that in this neighbourhood, 0 is the only singular point of S. Let now $\tilde{S} = S \setminus Sing(S)$; $\tilde{S}$ is a Riemann surface and it is hyperbolic (since it is contained in Ω which was supposed to be Kobayashi hyperbolic). Therefore $f|\tilde{S} : \tilde{S} \to \tilde{S}$ is a biholomorphism.

According to Milnor's classification theorem ([28]), any holomorphic map f of a hyperbolic Riemann surface $\tilde{S}$ falls into one of the following four mutually disjoint cases:

1. f has a unique attracting fixed point in $\tilde{S}$, or

2. every orbit diverges to infinity w.r.t the Kobayashi distance $d_{\tilde{S}}$, or

3. f is an automorphism of finite order, or

4. $\tilde{S}$ is isomorphic to the unit disk Δ, the punctured unit disk $\Delta^\star$ or an annulus, and f corresponds to an irrational rotation.

In our case, since f has a sequence of iterates converging to identity, we cannot have 1. or 2. Also case 3. cannot appear because of the local form of f. Here and in the rest of the paper we are still in the case when the exponents q_1, q_2 of the derivative of f at 0 have different signs. Then f is isomorphic to $\Delta, \Delta^\star$ or an annulus by means of a map β, and f is conjugated to an irrational rotation.

So $\beta : \tilde{S} \to \Delta, \Delta^\star$, or an annulus A is a biholomorphism and let an invariant neighbourhood U of 0 in which f can be linearized and s.t according to the previous lemma 2.4.6, $\tilde{S} \cap U$ is the union of the two local coordinate axes, minus the point 0. Denote the two connected components of $\tilde{S} \cap U$ by S_1, S_2; obviously $f(S_1) = S_1, f(S_2) = S_2$. Consider now $\beta(S_1)$ and $\beta(S_2)$ open connected subsets in one of $\Delta, \Delta^\star$ or an annulus, in which f corresponds to an irrational rotation.

If we take for example $\beta(S_1)\cap$ segment on the positive axis belonging to $\Delta, \Delta^\star$ or an annulus centered at 0, we will obtain an open subset of this segment. By rotating this open subset, the original set $\beta(S_1)$ will be reobtained; hence this open subset of the interval must be connected. But the only connected open subsets of a real interval are real open subintervals. Hence $\beta(S_1)$ must be actually a set of the form $\{0 \leq |z| < r\}$, or of the form $\{0 < |z| < r$, for some $r < 1$. The same can be said about $\beta(S_2)$.

Since $S_1 \cap S_2 = \emptyset \Longrightarrow \beta(S_1) \cap \beta(S_2) = \emptyset$.

According to the previous discussion, $\beta(S_1)$ and $\beta(S_2)$ are both annuli or disks or punctured disks centered at 0 in $\mathbb{C}$. Hence if they are disjoint, they cannot be both punctured disks.

On the other hand S_1, S_2 were defined as the local coordinate axes minus 0, hence they should be both biholomorphic to punctured disks.

But a disk, a punctured disk and an annulus cannot be biholomorphic to each other, therefore a contradiction . This shows that our assumption was wrong, so 0 is a regular point of S.

$\diamond$

Returning to the proof of Proposition 2.4.8, it is clear that $f(Sing(S)) = \underline{Sing(S)}$; let now $z_0 \in Sing(S)$ and consider its orbit $O(z_0)$. Since $\overline{O(z_0)} = \overline{(f^n(z_0))_n}$ is compact and $f^n(z_0) \in Sing(S), \forall n$, it follows that $O(z_0)$ is a finite set of points. WLOG one can assume that z_0 is a fixed point for f.

If $\mathrm{D}f(z_0)$ has eigenvalues of the type $\lambda^{q_1}, \lambda^{q_2}$, with q_1, q_2 of opposite signs, then $f|S$ is conjugated to an irrational rotation and we get a contradiction as above. Hence $Sing(S) = \emptyset$.

$\diamond$

Remark The above proposition has an interest in itself. Indeed , consider S_1 the global irreducible component of $L_0(\tilde{g}_0)$ containing the local "w-axis" near 0. Since S cannot contain both the "z-axis" and the "w-axis" near 0 (Lemma 2.4.9), it follows that $S_1 \neq S$. Also $f(S_1) = S_1$.

In fact the following result is true :

Corollary 2.4.10. $S \cap S_1 = \{0\}$.

This can be proved in a similar way, by looking at the orbit of a point from $S \cap S_1$ and at the action of f on S.

Remark(continued) S and S_1 are complex submanifolds of dimension 1 in Ω, they are invariated by the action and they extend the local "coordinate axes" near 0 . One can interpret them as "global axes of coordinates" in Ω. Next we want to prove that the action, under certain topological restrictions, has only one fixed point in Ω. First we need to introduce some notions and facts from multidimensional residue theory. Assume for the moment that X is a complex manifold of dimension n and S is an analytic submanifold of codimension 1 in X.

The following definition can be found in [2]:

Definition 2.4.11. *Let ω a closed regular form on $X \setminus S$.*

ω is said to have a polar singularity of first order if, $\forall a \in S$, if $\{s = 0\}$ is a minimal defining equation for S, $s \cdot \omega$ can be locally extended as a regular form across S.

Suppose now that ω is a form like in the definition. Then by a theorem of Leray ([2]), if $a \in S$, there exist U_a a neighbourhood of a, and regular forms ψ and θ defined in U_a, s.t

$$(1)\ \omega = \tfrac{ds}{s} \wedge \psi + \theta,$$

where $S \cap U_a = \{s = 0\}$, grad s $\neq 0$.

Moreover the form $\psi|_{S\cap U_a}$ can be extended globally to the entire S and it is closed. This restricted global form on S is called the *residue form of ω.*

Also if ω is holomorphic in $X \setminus S$, then the form $\psi|_S$ is holomorphic on S. The holomorphic forms on $X \setminus S$ which have polar singularities of first order play a special role in the case when $dim_X = 2$ and X is Stein. Denote by $\Omega^p(X, S)$ the sheaf of germs of p-forms χ that are holomorphic in $X \setminus S$ and s.t χ and $d\chi$ have polar singularities of first order on S.

In 1986, G.Raby ([32]) proved the following theorem relating the forms in $\Omega^p(X, S)$ to $H^1(X \setminus S, \mathbb{C})$:

Theorem 2.4.12. *If X is a Stein manifold of dimension 2 and S is a submanifold of X of dimension 1, then*

$$H^1(X \setminus S, \mathbb{C}) = H^1(\Gamma(X, \Omega^\bullet(X, S)))$$

So each cohomology class from $H^1(X \setminus S, \mathbb{C})$ contains a closed holomorphic form on $X \setminus S$ with polar singularities of first order on S.

Theorem 2.4.13. *Let Ω be a Stein and Kobayashi hyperbolic manifold of complex dimension 2 with an S^1 - action as above (i.e generated by an automorphism f satisfying ($\star$), $f(0) = 0$ and with exponents q_1, q_2 with opposite signs). Assume the restriction morphism $\rho : H^1(\Omega) \to H^1(\Omega \setminus S)$ is not surjective, where S is the global analytic set of Proposition 2.4.8 . Then the S^1 - action has exactly one fixed point in Ω, namely 0.*

Proof By hypothesis, there exists a regular 1-form ω on $\Omega \setminus S$ which is not cohomologous to the restriction of a regular form on Ω. From Raby's Theorem 2.4.12, we can find $\tilde{\omega}$ holomorphic form on $\Omega \setminus S$, with polar singularities of first order on S, s.t $\tilde{\omega} \sim \omega$. Then the Leray expression (1) gives that locally

$$\tilde{\omega} = \frac{ds}{s} \wedge \Phi + \psi,$$

where s is chosen to be irreducible.

Also, $\Phi|_S$ is a holomomorphic function on S which, as a differential form is closed, so Φ is holomorphic near S and it is a constant C when restricted to S. Let a point Q $= (z_0, w_0), w_0 \neq 0$ close to 0. We can also assume that in the neighbourhood U, $s(z, w) = w$. It is easily seen that the orbit of Q, denoted now by $\gamma(Q)$, is homologous in $U \setminus S$ to a circle of the form $(z_0, \lambda \cdot w_0)$,$\lambda \in S^1$.

Then by Leray's Residue Theorem ([2]),

$$\frac{1}{2\pi i} \int_{\gamma(Q)} \tilde{\omega} = \frac{1}{2\pi i} \int_{\{(z_0, \lambda w_0), \lambda \in S^1\}} \tilde{\omega} = \Phi(z_0, 0) = C,$$

since Φ was constant on S.

Now, assume we would have another fixed point for the action, denoted by P. From the description of S given in the proof of Proposition 2.4.8, $P \notin S$. Since S is an analytic set in Ω, $\Omega \setminus S$ is connected; join Q to P by a simple path $\eta|[0, 1]$ contained in $\Omega \setminus S$.

"Rotating" η will give us that $\gamma(Q)$ and P are homologous in $\Omega \setminus S$.

Let us make this a little more precise. Indeed if we take the orbits of all points of η by the S^1 - action (we identify the parametrization of η with its image), we will get a homology between the orbit of $\eta(0)$ and that of $\eta(1)$

But the orbit of Q is $\gamma(Q)$ and the orbit of P is P itself since this point was assumed to be fixed for the S^1 - action. Also, because $f(\Omega \setminus S) = \Omega \setminus S$ and f is the biholomorphism of Ω generating our S^1- action, none of the orbits of points from $\eta([0, 1])$ will intersect S .

This proves that $\gamma(Q)$ and P are homologous as paths in $\Omega \setminus S$.

Nevertheless, $\tilde{\omega}$ is a closed form on $\Omega \setminus S$ and $\gamma(Q)$ is homologous to P in $\Omega \setminus S$.

Thus, $\frac{1}{2\pi i} \int_{\gamma(Q)} \tilde{\omega} = \frac{1}{2\pi i} \int_{\gamma(P)} \tilde{\omega} = 0 \Longrightarrow C = 0$.

Therefore Φ is holomorphic on a neighbourhood of S and vanishes on S.

So, if s is a local irreducible defining function for S , s will divide Φ.

Hence $\tilde{\omega} = \frac{ds}{s} \cdot \Phi + \Psi$ has no singularities on S, which means that it can be extended across S. So $\tilde{\omega}$ is the restriction to $\Omega \setminus S$ of a differential form on Ω, hence contradiction. In conclusion 0 is the only fixed point of the action.

◇

Corollary 2.4.14. *Any holomorphic S^1 - action with a discrete set of fixed points, on a 2 - dimensional Stein manifold Ω which is Kobayashi hyperbolic and has $H^2(\Omega, \mathbb{Z}) = 0$, has in fact at most one fixed point.*

Proof We will use the fact that, if Ω is Stein and $H^2(\Omega, \mathbb{Z}) = 0$, then the second Cousin problem can always be solved. By taking the local minimal defining functions for the "global z-axis" S, and constants equal to 1 outside S , we obtain a multiplicative Cousin data. Solving it will provide us with a global defining function, g_0 for S.

For more information about the second Cousin problem on Stein manifolds, we refer to any text on several complex variables, for example Chabat [10].

Now we can take the 1-form $\omega = \frac{dg_0}{g_0}$, which is closed on $\Omega \setminus S$ and has polar singularities of the first order on S.

Hence the hypothesis of Theorem 2.4.13 is fulfilled and there is at most one fixed point of the action in Ω.

◇

An application of this corollary will be for the case of a large class of product domains in $\mathbb{C}^2$ endowed with S^1-actions.

Corollary 2.4.15. *Any S^1-action with a discrete set of fixed points on a bounded domain D of $\mathbb{C}^2$ of the form $D = D_1 \times D_2$, where D_1 is a plane domain and D_2 is a simply connected plane domain, will have in fact at most one fixed point.*

Proof This is true because we can apply Kunneth formula to calculate the cohomology of D.

Indeed, because of the one variable theorems of Weierstrass and Mittag - Lefler, we know that $H^1(D_1, O^\star) = 0$ and this implies that actually $H^2(D_1, \mathbb{Z}) = 0$. Then obviously $H^1(D_2, \mathbb{Z}) = 0$ and $H^2(D_2, \mathbb{Z}) = 0$ since D_2 is simply connected. Therefore $H^2(D, \mathbb{Z}) = 0$ and then we are able to use the previous corollary.

◇

Definition 2.4.16. *A map f of $\mathbb{C}^2$ of the form $f(z, w) = (P(z), Q(z, w))$, with P a polynomial in z and Q a polynomial in z, w, s.t f can be extended holomorphically to a map of $\mathbf{P}^2$, will be called a skew - product.*

Corollary 2.4.17. *Any Siegel domain for a skew - product map f of $\mathbf{P}^2$ contains at most one fixed point of f.*

Proof First of all, since the map f on $\mathbf{P}^2$ is attracting near the line at infinity, it follows that if Siegel domains exist, then they have to be contained in $(t = 1)$ (this set is identified to $\mathbb{C}^2$).

If $P_0 = (z_0, w_0)$ is a fixed point for f, then z_0 is fixed for the polynomial P and the map f will preserve the line $z = z_0$ in $\mathbb{C}^2$. On this line, also $Q(z_0, w)$ will be defined as a polynomial.

If Ω is a Siegel domain for f and P_0 is a fixed point for f in Ω, then after a change of variables we can make the set S from Theorem 2.4.13 to be equal to the invariant line $z = z_0$. In this case the closed global holomorphic form $\frac{d(z-z_0)}{z-z_0}$ will make the condition from Theorem 2.4.13 to be satisfied, proving that there is no other fixed point in Ω.

2.5 Some examples

Let Ω a 2-dimensional Stein and Kobayashi hyperbolic manifold as before. We also have an S^1 - action on Ω by holomorphic automorphisms which have discrete sets of fixed points. A natural question would be to generalize the existence of only at most one fixed point to the case when the eigenvalues of the derivative A are of the form $\lambda^{q_1}, \lambda^{q_2}$, with q_1, q_2 of opposite signs. The following example shows this is not possible unless suplementary conditions are imposed.

Example

Let X = connected component of the set $\{(z,w,t) \in \mathbb{C}^3, z^m - 1 + wt = 0, |z| < 3, |w| < 3, |t| < 3\}$ containing the point (1,0,0).

The S^1 - action on X is given by $\lambda \cdot (z,w,t) \to (z, \lambda w, \frac{1}{\lambda} t), \lambda \in S^1$.

First we notice that X is a manifold of dimension 2, i.e it has no singular points as a subvariety in $\mathbb{C}^3$. Indeed $d(z^m - 1 + wt)(z,w,t) = (mz^{m-1}, t, w) = (0,0,0)$ iff $(z,w,t) = (0,0,0)$. But $(0,0,0) \notin$ X; also it is easy to see that the action is well -defined in the sense that it preserves the analytic set, the respective connected component, and $|\lambda w| = |w|, |\frac{1}{\lambda} t| = |t|, \forall \lambda \in S^1$. Next X is Stein and, since it is bounded, it is also Kobayashi hyperbolic.

We have that $(\epsilon_m, 0, 0) \in$ X, where ϵ_m is an m - root of unity; indeed we can join $(\epsilon_m, 0, 0)$ to $(\epsilon_m, 0, 1)$ in X by a path $u \to (\epsilon_m, 0, u), u \in [0,1]$, and then $(\epsilon_m, 0, 1)$ to (0,1,1) by a path $u \to ((1-u)\epsilon_m, 1-(1-u)^m, 1), u \in [0,1]$, which is contained in X as well.

We can do the same thing for a path going from (1,0,0) to (1,0,1) and then on to $(0,1,1)$. In fact the m points of the form $(\epsilon_m, 0, 0)$ are the only fixed points of this action on X. Indeed, if $\lambda \cdot (z,w,t) = (z,w,t), \forall \lambda \in S^1 \Longrightarrow w = t = 0 \Longrightarrow z$ is an m - root of unity, if $(z,w,t) \in$ X.

In conclusion we obtained a Stein, Kobayashi hyperbolic manifold of dimension 2 with an S^1 - action having exactly m fixed points.

$\diamond$

By a suitable modification of the above we can give an example showing that Corollary 2.4.14 does not cover all cases of Theorem 2.4.13.

Let X = connected component of $\{(z,w,t) \in \mathbb{C}^3, z + z^2 - 1 + wt = 0, |z| < 1, |w| < 1, |t| < 1\}$ containing the point $(\frac{-1+\sqrt{5}}{2}, 0, 0)$, with the action $\lambda \cdot (z,w,t) \to (z, \lambda w, \frac{1}{\lambda} t)$.

As before, X is Stein, Kobayashi hyperbolic manifold and it can be checked that the action has only one fixed point.

We will show that the "global z-axis S" is defined as the set of zeros of a global holomorphic function, but $H^2(X, \mathbb{Z}) \neq 0$. So there are cases when $\rho : H^1(X) \to H^1(X \setminus S)$ is not surjective and $H^2(X, \mathbb{Z}) \neq 0$.

The set S is given as $\{(z,w,t) \in X, t = 0\}$, so $\frac{dt}{t} \in H^1(X \setminus S)$, and $\frac{dt}{t} \notin \rho(H^1(X))$. We will prove first that the homology group $H_2(X \setminus \{P\}, \mathbb{Z})$ has a nonzero free part, where $P = (\frac{-1+\sqrt{5}}{2}, 0, 0)$.

Then, since the real dimension of X is 4, if $H_2(X \setminus \{P\}, \mathbb{Z})$ has nonzero free part, the same is true for $H_2(X, \mathbb{Z})$.

Now, if $(z, w, t) \in X \setminus \{P\}$, then $w \neq 0, t \neq 0$, so can write $t = \frac{1-z-z^2}{w}$. This means that $X \setminus \{P\} \cong D :=$ respective connected component of $\{(z, w) \in \mathbb{C}^2, |z| < 1, |w| < 1, z \neq \frac{-1+\sqrt{5}}{2}, |\frac{1-z-z^2}{w}| < 1\}$.

Denote by D_1 the first projection of D as a subset of $\Delta \times \Delta$. Now $\frac{-1+\sqrt{5}}{2} \notin D_1$, but $\exists V = V(\frac{-1+\sqrt{5}}{2}) \subset \Delta$ s.t $\forall z \in V \setminus \{\frac{-1+\sqrt{5}}{2}\}, |1-z-z^2| < |w|$ if $|w| > \varepsilon_0$, for some $\varepsilon_0 > 0$. So D_1 is open, connected, and contains $V \setminus \{\frac{-1+\sqrt{5}}{2}\}$.

From the definition of the set D, the second projection of D on the unit disk, D_2, is contained in $\Delta^\star$. Also, since $|\lambda \cdot w| = |w|, \forall \lambda \in S^1$, we get that D_2 is invariant to rotations. So D_2 is open, connected, invariant to rotations and contained in $\Delta^\star \Longrightarrow D_2$ is an annulus or a punctured disk.

Take next the product of a small closed loop around $\frac{-1+\sqrt{5}}{2}$ in D_1 with a larger circle around 0 in D_2; we can choose the loops so that to obtain a 2-simplex in D. By using Kunneth formula (Bredon [6]), it is easy to see that this 2-simplex will span a non-zero free subgroup in $H_2(D, \mathbb{Z})$.

Hence $H_2(X, \mathbb{Z})$ will have a non-zero free part. From the Universal Coefficient Theorem ([6]), there exists an epimorphism:

$$H^2(X, \mathbb{Z}) \rightarrow Hom(H_2(X, \mathbb{Z}), \mathbb{Z}) \rightarrow 0.$$

Since $H_2(X, \mathbb{Z})$ has a non-zero free subgroup, $Hom(H_2(X, \mathbb{Z}), \mathbb{Z}) \neq 0 \Longrightarrow H^2(X, \mathbb{Z}) \neq 0$.

Finally let us notice also that, with regard to the first example of this Section, where we discussed the case of a manifold X with an S^1-action having exactly m fixed points, we will be able to say something about the group $H^2(X, \mathbb{Z})$ after Theorem 3.1.1 will be proved in the next chapter.

Indeed it will follow that the group $H^2(X, \mathbb{Z})$ has at least $m+1$ generators.

Chapter 3

Generalizations and Applications

In this chapter we will give first a generalization of Corollary 2.4.14 and then we will study several applications of the theorems in Chapter II to particular cases of importance, like Siegel domains for holomorphic endomorphisms of $\mathbf{P}^2$.

Several examples of Siegel domains will be constructed in Section 3.4 after a preliminary introduction to rotation domains in one complex variable.

We will also use number theoretic results, like those related to diophantine numbers and to the Brjuno condition in order to guarantee the existence of certain Fatou components.

3.1 Generalization to the case of finitely many generators for the second cohomology

The following theorem gives an upper estimate for the number of fixed points of an S^1 action on a Stein, Kobayashi hyperbolic manifold when the second cohomology group is not zero, but it is still finitely generated.

Theorem 3.1.1. *Let Ω be a connected manifold of complex dimension 2, which is also Stein and Kobayashi hyperbolic. Consider a holomorphic action on Ω given by automorphisms without whole lines of fixed points as above. Assume that the cohomology group $H^2(\Omega, \mathbb{Z})$ has k-1 generators. Then the action will have at most k fixed points in Ω.*

Proof

Assume that the holomorphic action has more than k fixed points in Ω and let k+1 of them be denoted by $O, P_1, ..., P_k$ Recalling the construction from Proposition 2.4.8, we are able to construct for each of the fixed points above, global axes of coordinates. It does not matter if we take the global "z-axis" or the global "w-axis" below.

Making a choice, let $A_0, A_1, ..., A_k$ be such irreducible global axes of coordinates passing thru the fixed points respectively.

Each of these submanifolds is invariated by the action and the given action is conjugated to an irrational rotation on a disk or a punctured disk on each one of them.

We claim that $A_0, A_1, , , , , A_k$ are mutually disjoint. If not, suppose that A_0 and A_1 have a point in common. Clearly we can take a function $g \in Hol(\Omega)$ so that $g(0) = 0$ and $g(P_1) = 1$; then the averaged function $\tilde{g}$ will take the values 0 and 1 at $0, P_1$ respectively.

Therefore P_1 cannot belong to the set A_0, so A_0 and A_1 are distinct sets.

Also since $\widetilde{g-1} = \tilde{g} - 1$, it follows from the definition of the sets A_0, A_1 that $\widetilde{g-1}$ is equal to zero on A_1, while it is equal to 1 on A_0,implying that these two sets have no point in common.

Similarly one can show that all the analytic sets $A_0, ..., A_k$ are mutually disjoint.

Let now a simple covering of Ω with connected, simply connected open sets which do not intersect more than one of the analytic sets $A_0, ..., A_k$ and which is indexed by A, $(U_\alpha)_{\alpha \in A}$.

Such a covering is always possible and it has special properties; the cohomology of Ω relative to this covering, with coefficients in a sheaf F, will give the actual cohomology of Ω with coefficients in F. See for example [10].

Decompose A as $A_0 \bigcup ... \bigcup A_k \bigcup A_{rest}$ where A_i indexes the sets of the covering interstecting A_i and A_{rest} indexes the open sets of the covering that do not intesect any of the above axes.

Assume also that the covering is chosen so that if $\alpha \in A_i$, then there is a determining function ϕ^i_α for A_i on U_α. The function ϕ^i_α is holomorphic, irreducible and vanishes locally on A_i. If $\alpha \notin A_i$, then we take $\phi^i_\alpha = 1$.

Now, it is well-known that for a Stein manifold Ω, there is a group isomorphism between $H^2(\Omega, \mathbb{Z})$ and $H^1(\Omega, O^\star)$, where $O^\star$ is the multiplicative sheaf of nonvanishing holomorphic sections. Indeed there is an exact sequence of sheaves,

$$0 \to \mathbb{Z} \xrightarrow{i} O \xrightarrow{e} O^\star \to 1,$$

where i is the injection, $e : f \to e^{2\pi i f}$ a homomorphism of the sheaf O of additive groups to the sheaf $O^\star$ of multiplicative groups. Then the following sequence

$$H^1(\Omega, O) \to H^1(\Omega, O^\star) \to H^2(\Omega, \mathbb{Z}) \to H^2(\Omega, O)$$

will also be exact and since Ω is a Stein manifold, the groups at the ends of the above sequence are trivial, hence the groups in the middle are isomorphic.

Consider now $\phi^i_{\alpha\beta} = \frac{\phi^i_\alpha}{\phi^i_\beta}$ which is an element of $O^\star(U_\alpha \bigcap U_\beta)$ since any two adjoining determining functions for A_i have common simple zeros. This $(\phi^i_{\alpha\beta})_{\alpha\beta}$ will define a cocycle in $Z_1(U, O^\star)$ for any i.

Denote the generators of $H^1(\Omega, O^\star)$ by $(f_{1,\alpha\beta})_{\alpha\beta}, ..., (f_{k-1,\alpha\beta})_{\alpha\beta}$. There will exist positive integers $m_{i1}, ..., m_{ik-1}$ such that $\frac{\phi^i_{\alpha\beta}}{f^{m_{i1}}_{1,\alpha\beta} \cdots f^{m_{ik-1}}_{k-1,\alpha\beta}} \in B^1(U, O^\star)$, i.e the above fraction becomes a coboundary, $i \in \{0, ..., k\}$

Therefore we will find local nonvanishing functions $h^i_\alpha, \alpha \in A$ so that: $\frac{\phi^i_{\alpha\beta}}{f^{m_{i1}}_{1,\alpha\beta} \cdots f^{m_{ik-1}}_{k-1,\alpha\beta}} = \frac{h^i_\beta}{h^i_\alpha}$ on $U_\alpha \bigcap U_\beta$, for all i, α, β.

Recalling the definition of $\phi^i_{\alpha\beta}$, the following system of equations will be true:

$$\phi^0_\alpha \cdot h^0_\alpha = f^{m_{01}}_{1,\alpha\beta} ... f^{m_{0k-1}}_{k-1,\alpha\beta} \cdot \phi^0_\beta \cdot h^0_\beta$$
$$\phi^1_\alpha \cdot h^1_\alpha = f^{m_{11}}_{1,\alpha\beta} ... f^{m_{1k-1}}_{k-1,\alpha\beta} \cdot \phi^1_\beta \cdot h^1_\beta$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\phi^{k-1}_\alpha \cdot h^{k-1}_\alpha = f^{m_{k-11}}_{k-1,\alpha\beta} ... f^{m_{k-1k-1}}_{k-1,\alpha\beta} \cdot \phi^{k-1}_\beta \cdot h^{k-1}_\beta,$$

on the intersection $U_\alpha \bigcap U_\beta$.

Now the idea is to eliminate the group generators $f_{r,\alpha\beta}$ one by one from the above system so to obtain a gobally defined meromorphic function on Ω.

Indeed, for example in order to eliminate $f_{1,\alpha\beta}$ we can raise $\phi^0_\alpha \cdot h^0_\alpha$ to the power m_{11} and divide it by $\phi^1_\alpha \cdot h^1_\alpha$ raised to the power m_{01}; this will give us an equation containing no $f_{1,\alpha\beta}$.

Similarly, by division of powers of these fractions we will be able to get rid of $f_{2,\alpha\beta}$ and so on. In the end there will be a superposed fraction containing only local holomorphic functions indexed by α which is equal on $U_\alpha \bigcap U_\beta$ with a similar superposed fraction corresponding to functions indexed by β.

This will enable us to define a global meromorphic function on Ω, denoted by μ. The function μ will have its zeros and poles included in the union of analytic sets $A_0 \bigcup ... \bigcup A_{k-1}$ and will be holomorphic and nonvanishing on $\Omega \setminus (A_0 \bigcup ... \bigcup A_{k-1})$.

Obviously, since any proper analytic set does not disconnect a connected complex manifold, the set $\Omega \setminus (A_0 \bigcup ... \bigcup A_{k-1})$ is connected. Also it remains invariant to the S^1 - action from the construction of the analytic axes.

Consider now the form $\omega := \frac{d\mu}{\mu}$ which is a holomorphic closed form on $\Omega \setminus (A_0 \bigcup ... \bigcup A_{k-1})$ with polar singularities on $A_0 \bigcup ... \bigcup A_{k-1}$. Take now a path ν joining a point Q close to 0 to P_k inside $\Omega \setminus (A_0 \bigcup ... \bigcup A_{k-1})$; since this open set is invariant to the action, we can rotate ν inside it and obtain a "cone" whose base is the orbit γ of Q and whose tip is the fixed point P_k.

Indeed by rotating ν we mean that we take the S^1 orbits of all the points in the path; since each of the sets A_i is totally invariant for the action, these orbits will never intersect any of the A_i's, $i = 0..k-1$. This will provide a homology between γ and P_k.

But this means that by integrating the closed form ω on γ we should get the same result as by integrating it along P_k.

However in one case the result is $2\pi i$ and in the other 0, hence a contradiction. Therefore the action cannot have more than k fixed points in Ω.

$\diamond$

Remark:

In the above theorem we are talking about conditions when an S^1 action on a Stein, Kobayashi hyperbolic manifold has a given finite number of fixed points.

Actually there are Stein, Kobayashi hyperbolic manifolds with S^1 actions which have an infinite number of fixed points.

Here is an example:

Take $\Omega :=$ the connected component of the set $\{(z, w, t) \in \mathbb{C}^3, |z| < 3, |w| < 3, |t| < 3$ such that $e^{2\pi i/z-3} - 1 + wt = 0\}$ containing the point $(3 - \frac{1}{10}, 0, 0)$.

The S^1 action on Ω will be $\lambda \cdot (z, w, t) := (z, \lambda w, \frac{1}{\lambda} t), \forall \lambda \in S^1$.

This action preserves the analytic set given above and also the respective connected component. One can show that all the points of the form $(3 - \frac{1}{n}, 0, 0), n \in \mathbb{N}, n \geq 10$ are in fact in Ω.

For this we will connect first $3 - \frac{1}{10}$ with $3 - \frac{1}{11}$ by a path within Ω, then $3 - \frac{1}{11}$ with $3 - \frac{1}{12}$, and so on.

Indeed, if $s \to \eta(s)$ is the usual parametrization for the real line segment $[3-\frac{1}{10}..3-\frac{1}{11}]$, then we can find paths $s \to w(s)$ and $s \to t(s)$ such that $w(0) = 0$, $t(0) = 0$ and such that $|w(s)| < 3, |t(s)| < 3$ and $w(s) \cdot t(s) = 1 - e^{\frac{2\pi i}{\eta(s)-3}}$.

Hence it is possible to join any two points of the form $(3 - \frac{1}{n}, 0, 0)$ and $(3 - \frac{1}{m}, 0, 0)$ within Ω.

Therefore the S^1 action on Ω has an infinite and discrete number of fixed points which accumulate on the boundary of Ω in $\mathbb{C}^3$.

Indirectly, it follows from the previous theorem that the cohomology group $H^2(\Omega, \mathbb{Z})$ is infinitely generated.

3.2 Domains of rotation; the one complex variable case

Whenever talking about Siegel domains for holomorphic mappings on $\mathbf{P}^2$, one is inexorably drawn to consider the one complex variable case and the examples available there.

In the sequel we will first give several theorems describing the conditions that the derivative of a rational function has to satisfy in order to guarantee the existence of a Siegel disk or a Herman ring and also some of the properties of these Fatou components.

Then, in the next section, we will concentrate on Siegel domains for holomorphic mappings of $\mathbf{P}^2$ and several classes of examples. These Siegel domains are important to us because they provide interesting examples of actions of tori on Stein, Kobayashi hyperbolic manifolds in connection to Chapter II.

According to Sullivan's Classification Theorem, [8] there are only four types of Fatou components for a rational map f:

1. Attracting basins for periodic points

2. Parabolic domains (which turn out to have periodic points on their boundary)

3. Siegel disks (on which the map f is conjugated to an irrational rotation around the center of the unit disk)

4. Herman rings (on which the map is conjugated to an irrational rotation on an annulus or to an irrational rotation followed by an inversion)

In this thesis we discuss only types 3. and 4. since only they provide examples of S^1-actions.

The first example of a Siegel disk has been produced by C. L. Siegel in 1942, however the first examples of Herman rings had to wait until 1984, when M. Herman constructed them using a theorem of Arnold about homeomorphisms of the circle.

We will collect below several theorems, first about Siegel disks.

Let f a rational function so that $f(0) = 0$ and $f'(0) = \lambda = e^{2\pi i\theta}$, where θ is irrational. In order to linearize f near 0, we must find a solution ϕ of the *Schroder equation* $\phi(f(z)) = \lambda\phi(z)$, normalized by $\phi'(0) = 1$

The following theorems appear in Carleson - Gamelin [8].

Theorem 3.2.1. *A solution ϕ to the Schroder equation $\phi(f(z)) = \lambda\phi(z)$, with $\phi'(0) = 1$, is univalent in any disk $\{|\zeta| < r\}$ where it is well defined.*

Theorem 3.2.2. *A solution ϕ to the Schroder equation exists if and only if the sequence of iterates $\{f^n\}$ is uniformly bounded in some neighbourhood of the origin.*

Since uniform boundedness is a topological condition, the previous theorem implies that, if f is topologically conjugate to λz near 0, then it is also conformally conjugate.

Theorem 3.2.3. *There exists a $\lambda = e^{2\pi i\theta}$ so that the Schroder equation has no solution for any polynomial f.*

It is clear that in this case 0 is a point in the Julia set. An actual example as above was given in 1917 by Pfeifer.

Also, in 1938, H.Cremer showed that if λ is "almost" a root of unity, more precisely if $|\lambda| = 1$ and liminf $|\lambda^n - 1|^{1/n} = 0$, then there is a holomorphic function g, $g(0) = 0$ and $g'(0) = \lambda$, for which the Schroder equation has no solution.

Definition 3.2.4. *A real number θ is called Diophantine if it is badly approximable by rational numbers, i.e there is some $c > 0$ and $\mu < \infty$ so that $|\theta - \frac{p}{q}| \geq \frac{c}{q^\mu}$, for all integers $p, q \neq 0$.*

For a fixed $\mu > 2$, it is not difficult to show that the condition in the above definition is satisfied for almost every real number θ.

We have now a fundamental theorem of Siegel:

Theorem 3.2.5. *If θ is Diophantine, and if f has a fixed point at 0 with multiplier $e^{2\pi i\theta}$, then there exists a solution to the Schroder equation, that is f can be conjugated near 0 to multiplication by $e^{2\pi i\theta}$.*

We will be concerned at times with quadratic polynomials of the form $P(z) = e^{2\pi i\theta} \cdot z + z^2$. For these polynomials there is a necessary and sufficient condition for the existence of a local linearization near 0 (see [8]):

$$\sum_{n=1}^{\infty} \frac{log q_{n+1}}{q_n} < \infty,$$

where $\{\frac{p_n}{q_n}\}$ is a sequence of rational approximations to θ coming from its continued fraction expansion.

A first such example is the polynomial $P(z) = \lambda(z - z^2/2)$ where $\lambda = exp(2\pi i(\sqrt{5} - 1)/2)$. Indeed the continued fraction expansion of the golden mean $(\sqrt{5} - 1)/2$ is $\frac{1}{1+\frac{1}{1+...}}$ which satisfies the sufficient condition above; for this polynomial there will exist a Siegel disk at the origin.

An interesting relationship exists between critical points and Siegel disks or Herman rings.

Denote the critical set of f by C and the closure of the union of the forward orbits of points in C by CL.

The following theorem appears as well in Carleson-Gamelin [8].

Theorem 3.2.6. *If U is a Siegel disk or a Herman ring, then the boundary of U is contained in the closure of the postcritical set, CL.*

In fact the above theorem provides a practical way to delineate the boundary of a rotation domain U; it is enough to find the critical points and to follow their iterates for a large number of times. Obviously some critical points will be trapped inside attracting basins or parabolic domains and none of them will belong to the interior of a rotation domain, but iterating some of them will describe how the boundary of Siegel disks and Herman rings look like. This method is frequently used in computer experiments to graph a Siegel disk or a Herman ring.

Let us look now at the case of Herman rings. Before giving an example of a Herman ring, we will need some information concerning rotation numbers for homeomorphisms of the circle. Here we shall follow closely Carleson-Gamelin [8].

An orientation preserving homeomeorphism of the circle h can be written as $h(t) = e^{2\pi i H(t)}$, where $H(t)$ is an increasing homeomorphism of the real line such that $H(t+1) = H(t) + 1$.

It can be proved that the limit $\alpha(H) := lim_{n\to\infty} \frac{H^n(0)}{n}$ does exist. This is called *the rotation number of H*.

Also, the class of $\alpha(H)$ modulo 1 does not depend on the lift H and in fact only depends on h, so we will call it *the rotation number of h*.

For instance, the rotation number of the map $h(z) = e^{2\pi i\theta} \cdot z$ is $\theta(mod 1)$. Also, the rotation number of h depends continuously on whatever parameters h depends upon and is a conjugacy invariant. Rotation numbers present an example of the influence that early work of Poincare still has on several branches of mathematics more than 100 years after him!

In our example of a Herman ring, we will use the following theorem of Arnold:

Theorem 3.2.7. *Let α be Diophantine and let $\sigma > 1$. Then there is an $\epsilon > 0$ such that if f is any homeomorphism of the circle with rotation number equal to α, which extends to be analytic and injective on the annulus $\{1/\sigma < |w| < \sigma\}$ and satisfies there $|f(w) - e^{2\pi i\alpha}w| < \epsilon$, then f is conformally conjugate to the rotation $e^{2\pi i\alpha}w$ on the annulus $\{1/\sqrt{\sigma} < |w| < \sqrt{\sigma}\}$.*

The following example, given by Herman, will use Theorem 3.2.7:

Example:

For $a > 1$, consider

$$R_{a,\theta}(z) = e^{2\pi i\theta} \cdot z^2 \tfrac{z-a}{1-az}.$$

This is an orientation preserving homeomorphism of the circle whose rotation number varies continuously in a and θ. For example if $a \to \infty$, $\alpha(R_{a,\theta}) \to \theta$ (mod 1).

So for a Diophantine number α, there is a $\theta = \theta(\alpha)$ so that the corresponding $R_{a,\theta(\alpha)}$ has rotation number α and $\theta(\alpha) \to \alpha$ as $a \to \infty$.

Hence, for a large, the rotation number of $R_{a,\theta(\alpha)}$ is α and the function is close to $e^{2\pi i\alpha}z$. Therefore the hypotheses of Arnold's Theorem 3.2.7 are met and R is conjugate in an annulus around the unit circle to an irrational rotation.

This implies that the unit circle is contained in a Fatou component U of R. U cannot be attracting, parabolic; also, U cannot be Siegel disk since both 0 and ∞ are attracting. Hence the only possibility is that U is a Herman ring.

An interesting theorem of Shishikura shows that in fact there are plenty of Herman rings and that one can transform Siegel disks into Herman rings with the same rotation number.

Theorem 3.2.8. *If R is a rational function having a Siegel disk on which it is conjugated to a rotation with multiplier λ, then there exists another rational function with a Herman ring on which it is cojugate to a multiplication by the same multiplier λ.*

However it might not be always easy to construct such Herman rings explicitly.

3.3 Siegel domains; multidimensional case

In the sequel we will study the case of recurrent domains for holomorphic mappings of $\mathbf{P}^2$ as being the closest generalization of the non-wandering behaviour of the Fatou components from one variable and especially the Siegel domains, which are Fatou components corresponding to the rotation domains from one variable.

A theorem of Fornaess-Sibony [11] gives a description of the possible recurrent domains, as basins of attraction to a fixed point, basins of attraction to a Riemann surface Σ and Siegel domains (see Definition 2.2.16.

The dynamics in a basin of attraction of a fixed point is relatively simple and the dynamics in the case of a basin of attraction to a Riemann surface is based on the fact that the function will be conjugate to a rotation on Σ.

However a complete classification of the dynamics in Siegel domains in $\mathbf{P}^2$, similar to the one complex variable situation, is far from being known.

After stating this theorem, we will construct several examples of Siegel domains, first by Ueda's method using one variable examples, and then by using a partial linearization result and Brjuno conditions.

The notion of Fatou component for a holomorphic function f on $\mathbf{P}^2$ of degree $d \geq 2$ has been introduced in Definition 2.2.13.

Definition 3.3.1. *A Fatou component Ω is recurrent if for some $p_0 \in \Omega$, the limit set of forward iterates of p_0 intersects Ω. More precisely, there is $p_0 \in \Omega$ such that $\{f^{n_i}(p_0)\}$ is relatively compact in Ω for some subsequence $\{n_i\}$.*

In $\mathbf{P}^1$, the recurrent Fatou components are attractive basins, Siegel disks and Herman rings. In contradiction with the one variable case, where there can be only a finite number of recurrent domains, in two complex dimensions the situation is different. Indeed, Estela Gavosto [19] showed that holomorphic maps on $\mathbf{P}^2$ can have an infinite number of recurrent domains (specifically attracting basins).

It is easy to see that if Ω is a recurrent domain, then there exists m such that $f^m(\Omega) = \Omega$

Fornaess - Sibony [11] have proved the following theorem:

Theorem 3.3.2. *Let f be a holomorphic map of $\mathbf{P}^2$ of degree $d \geq 2$. Let Ω be a recurrent Fatou component such that $f(\Omega) = \Omega$. Then one of the following is true:*

(i) Ω is an attractive basin for a fixed point $p \in \Omega$.

(ii) There exists a Riemann surface Σ which is a closed complex submanifold of Ω and $f|_\Sigma : \Sigma \to \Sigma$ is an automorphism, moreover $d(f^n(K), \Sigma) \to 0$ for any compact set K in Ω.

The Riemann surface Σ is biholomorphic to a disk, a punctured disk or an annulus and $f|_\Sigma$ is conjugate to a rotation. The limit h of any convergent *subsequence f^{n_i} , has the same image. Any two limits h_1, h_2 differ only by a rotation in Σ.*

(iii) The domain Ω is a Siegel domain, in the sense that there exists a subsequence f^{k_i} converging locally uniformly to the identity.

In this case, any limit of a convergent subsequence of iterates is an automorphism of Ω.

The proof of the above theorem is based on considering the three possible cases for the rank of the limit h of a certain subsequence of iterates f^{n_i}. For rank $h = 0$, we get the attracting basin, for rank $h = 1$, the Riemann surface and for rank$h = 2$, the Siegel domain. For a complete proof, see [11]. The dynamics in cases (i) and (ii) above is simple, but this is not true anymore in the case of Siegel domains.

Since Ω is Kobayashi hyperbolic and of complex dimension two, a theorem of Kruzhilin says that the maximal real dimension of a commutative group in $Aut(\Omega)$ is two.

Then, similar to the general case in Chapter II, we have the proposition:

Proposition 3.3.3. *If Ω is a Siegel domain for f holomorphic on $\mathbf{P}^2$, s.t $f(\Omega) = \Omega$ and if G denotes the closure of $(f^n)_{n\geq 0}$ in the topology of uniform convergence on compact sets, then G is a sub-Lie group of $Aut(\Omega)$ and G is isomorphic to $\boldsymbol{T^k} \times \boldsymbol{F}$ where F is a finite group and k = 1 or 2.*

3.4 Examples of Siegel domains

The idea below will be to obtain several classes of holomorphic functions having Siegel domains in $\mathbf{P}^2$ from one dimensional examples, through different methods, like taking direct products, using Ueda's method or by partial linearization.

1. Products of one variable examples having rotation domains

The simplest examples of Siegel domains in $\mathbf{P}^2$ can be obtained by taking a function $(P(z), Q(w))$ which can be extended to $\mathbf{P}^2$ and such that both P and Q have Siegel disks.

It is important to note that in all the examples of this sort, P and Q must be polynomials or else the function is not extendable to $\mathbf{P}^2$. This restricts the type of Siegel domain that one may get with this method, because polynomials cannot have Herman rings (due to the maximum principle). Consequently the Siegel domains obtained with this method will contain a fixed point.

For instance, consider $f[z : w : t] = [\lambda zt + z^2 : \lambda^p wt + w^2 : t^2]$, where p is a positive integer and $\lambda = e^{2\pi i\theta}$, with θ a Diophantine number.

In a neighbourhood of (0, 0), the map $(z, w) \to (\lambda z + z^2, \lambda^p w + w^2)$ is conjugate to $(\lambda z, \lambda^p w)$; this implies that [0:0:1] belongs to a Siegel domain of f and the action of f on Ω is one-dimensional, i.e the iterates of f generate an S^1 inside the group $Aut(\Omega)$.

We can also give an example of a Siegel domain with a two dimensional real torus action.

Let $g[z : w : t] = [\lambda_1 zt + z^2 : \lambda_2 wt + w^2 : t^2]$ with $|\lambda_1| = |\lambda_2| = 1$.

We also assume a non-resonance condition of the form

$$(\star) \qquad \lambda^k - \lambda_j \neq 0,$$

for any $1 \leq j \leq 2$ and any $k \in \mathbb{N}^2$ with $|k| \geq 2$; in the expression above, if $k = (k_1, k_2) \in \mathbb{N}^2$, then λ^k means $\lambda_1^{k_1} \cdot \lambda_2^{k_2}$ and $|k|$ means $k_1 + k_2$.

Now for $m \in \mathbb{N}, m \geq 2$, define

$$\Omega(m) = inf_{j,k} |\lambda^k - \lambda_j|$$

We will say that λ_1, λ_2 satisfy the *Brjuno condition* if :

$$\sum_{k=0}^{\infty} 2^{-k} log(\Omega^{-1}(2^{k+1})) < \infty$$

We now have the theorem (see [21]):

Theorem 3.4.1. *Let f be a germ of a holomorphic function near $(0,0)$, such that the series development of f looks like $f(z,w) = (\lambda_1 z + z^2, \lambda_2 w + w^2)$ + higher order terms.*

Assume that λ_1, λ_2 are of absolute value 1 and that they satisfy the non-resonance condition $(\star)$ and the Brjuno condition above.

Then there exists a germ of a local diffeomorphism ϕ near (0,0) conjugating f to its derivative.

In fact this theorem is true in any complex dimensions. The first versions of this theorem were proved by Siegel and others by Sternberg, or Gray, under a *diophantine type condition*, i.e assuming that:

$\exists \beta \geq 0, \gamma > 0$ so that $\Omega(m) \geq \gamma \cdot m^{-\beta}, \forall m \geq 2$.

Under the Brjuno condition it was first proved by the Russian mathematician A. D. Brjuno, see for example [21]; the whole heart of the problem is undoubtely number theoretic.

Theorem 3.4.1 above shows that the map g in our Example will have a Siegel domain Ω containing the fixed point $[0 : 0 : 1]$; g is generating a $\boldsymbol{T^2}$ torus action on Ω.

2. Functions constructed by Ueda's method

We will describe now Ueda's method of constructing holomorphic mappings of $\mathbf{P}^2$ from rational functions (see Ueda [41]).

Consider the mapping $\pi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^2$ given by the formula:

$$\pi([\xi : \eta], [\xi' : \eta']) = [\xi\xi' + \eta\eta : \eta\eta' - \xi\xi' : \xi\eta' + \eta\xi'].$$

We note that $\pi([\xi : \eta], [\xi' : \eta']) = \pi([\xi' : \eta'], [\xi : \eta])$ and that π is going to be a double covering of $\mathbf{P}^2$ with $\pi(\Delta)$ as its branch locus, where $\Delta = \{(Q, Q) | Q \in \mathbf{P}^1\}$ is the diagonal of $\mathbf{P}^1 \times \mathbf{P}^1$.

For $f : \mathbf{P}^1 \to \mathbf{P}^1$, define $\hat{f} : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$ by $\hat{f} = (f, f)$. Then there will exist a unique holomorphic mapping $F : \mathbf{P}^2 \to \mathbf{P}^2$ such that $F \circ \pi = \pi \circ \hat{f}$.

In this case it is easy to prove that the Julia set of $\hat{f}$, $J_{\hat{f}}$ is equal to $(J_f \times \mathbf{P}^1) \bigcup (\mathbf{P}^1 \times J_f)$ and that $J_F = \pi(J_{\hat{f}})$.

Therefore, in order to build a Siegel domain for F, it will be enough to find a Siegel disk or a Herman ring for f.

Here we can use the one dimensional examples of Section 3.2. For example if U is a Siegel disk for f, then $\pi(U \times U)$ will be a Siegel domain for F containing one fixed point.

If we take an example of a rational function f with a Herman ring U, then F will have a Siegel domain $\pi(U \times U)$ containing no fixed points.

3. Partial Linearization

In this section we will give examples of mappings on $\mathbf{P}^2$ with Siegel domains which are not of any of the two types discussed above.

Let us start with the map

$$g[z : w : t] = [\lambda^2 zt + w^2 + z^2 : \lambda wt + w^2 : t^2]$$

Clearly g is a well-defined holomorphic function on $\mathbf{P}^2$ and it is a skew product in the w coordinate. This implies that g leaves the w-axis in $\mathbb{C}^2$ invariant.

We will denote by g also the corresponding map on $\mathbb{C}^2$ and note that the derivative of g at (0,0), has eigenvalues $\lambda_1 := \lambda^2$ and $\lambda_2 := \lambda$.

Since the non-resonance condition $(\star)$, $\lambda^k - \lambda_j \neq 0, \forall 1 \leq j \leq 2, k \in \mathbb{N}^2, |k| \geq 2$, is not satisfied, it means that a linearization like the one in Brjuno's Theorem 3.4.1 is not valid anymore. However, one may try to find a partial local linearization corresponding to a partial non-resonance condition.

Obviously, we have the following condition:

$$\lambda_1^k - \lambda_i \neq 0, k \geq 2, 1 \leq i \leq 2$$

Define the function $\omega(m) := min_{2 \leq k \leq m} min_{1 \leq i \leq 2} |\lambda_1^k - \lambda_i|$ for m an integer, $m \geq 2$.

We call λ *admissible* if it satisfies the following Brjuno-like condition:

$$\sum_{\nu \geq 0}^{\infty} 2^{-\nu} log(\omega^{-1}(2^{\nu+1})) < \infty$$

For instance, λ is admissible if it satisfies the diophantine type condition:

$$—\lambda_1^k - \lambda_i| \geq c|k|^{-N}, \forall k \geq 2, 1 \leq i \leq 2,$$

for some $c > 0$ and some large N.

In this setting, Jürgen Pöschel [31] proved in 1986 the theorem below:

Theorem 3.4.2. *If the eigenvalue λ is admissible, then there exists locally a complex analytic invariant manifold $\mathcal{M}$ of g, which is tangent to the eigenspace M of λ_1 and on which the mapping is analytically equivalent to its linear part.*

This implies that there exists a local biholomorphism $\eta : \mathcal{M} \to \Delta$, where Δ is a disk in $\mathbb{C}$, which will conjugate g to the multiplication by λ^2.

Also, $g|_{w-axis}$ is conjugate to a multiplication by λ near 0 if λ is a Diophantine number.

This conjugation of g restricted to the local w-axis to a multiplication by λ will work for example if λ satisfies

$$|\lambda^k - 1| \geq ck^{-N}, k \geq 1,$$

for some constant c and a large N. This condition is true for λ on a set of full measure on the unit circle.

But then if this condition holds, then clearly the condition $|\lambda_1^k - \lambda_i| \geq c'|k|^{-N'}$ will also be valid, perhaps for different c', N'. This will imply that Poschel's Theorem 3.4.2 will hold in this case.

In conclusion we can linearize g separately on two local invariant submanifolds, the first one being $\mathcal{M}$ which is tangent to the z-axis at 0, and the second one which is the local w-axis. By combining the two conjugations we can show that g can be linearized locally near (0,0).

But this actually says that $[0 : 0 : 1]$ belongs to a Siegel domain of the map g in $\mathbf{P}^2$.

Naturally one can obtain a large class of examples of this type by taking skew-products having partial non-resonance conditions and some Brjuno restrictions on the eigenvalues of their derivatives at (0,0). All these holomorphic functions will have Siegel domains containing the point $[0 : 0 : 1]$.

Chapter 4

Hyperbolicity for Maps on $\mathbb{P}^2$

4.1 Overview of the Chapter

In this chapter we will deal with different notions of hyperbolicity for maps on the complex projective space $\mathbf{P}^2$ that appear as the "opposite case" from the one of Siegel domains studied in Chapter III.

We saw in Chapter III that an example of a Stein, Kobayashi hyperbolic manifold with an S^1 action with no lines of fixed points can be obtained from Siegel domains for holomorphic maps of $\mathbf{P}^2$.

In this chapter we study the completely opposite case, that is when there can be no Siegel domains for a given map f. Such a case is when f is hyperbolic on its non-wandering set. Indeed, clearly if Siegel domains would exist, then they would belong to the non-wandering set $\Omega(f)$. However, according to Definition 4.2.2, if there is a hyperbolic structure on $\Omega(f)$, then this would contradict the existence of a subsequence of iterates $(f^{k_n})_n$ converging to the identity.

In the one variable case, hyperbolicity has a simple interpretation as expansion on the Julia set, or equivalently, the fact that the postcritical set is contained in a union of attracting Fatou components.

In two complex dimensions there are different notions of hyperbolicity, like prehyperbolicity, Axiom A, and s - hyperbolicity. In addition,the so-called no-cycle condition is in general an important tool for proving persistence of hyperbolicity under small perturbations. This condition also allows the use of filtration methods similar to those employed in the study of hyperbolic diffeomorphisms.

We also notice that a characterization of hyperbolicity on $\mathbf{P}^n$ like the one for rational maps, i.e that the critical points are contained in attracting Fatou components is not valid anymore. Indeed, we showed in Theorem 2.2.6

that a domain of holomorphy in $\mathbf{P}^n$ cannot contain an analytic set of positive dimension; since a Fatou component is a domain of holomorphy according to Theorem 2.2.14, it follows that the Julia set in $\mathbf{P}^n, n \geq 2$ will always have to intersect the critical variety.

Let f a holomorphic endomorphism of $\mathbf{P}^2$ of degree d. This implies that f can be lifted to a map (P, Q, R) of $\mathbb{C}^3 \setminus \{0\}$, where P, Q, R are homogeneous polynomials of degree d. In the sequel we will introduce the notion of ws - hyperbolicity as a slight variation of the notion of s - hyperbolicity of Fornaess - Sibony ([12]). We will show in Theorem 4.3.1 that the notion of ws-hyperbolic map is stable under perturbations if the no-cycle condition is satisfied. Examples of ws-hyperbolic maps with no superattracting periodic points are also given.

Another result concerns the set K^- of points whose preimages do not converge locally uniformly to the support of the ergodic invariant measure of f. We will show in Proposition 4.3.2 that, in the case of s-hyperbolic maps, the interior of this set does not contain non-wandering points of unstable index 1. Finally in Theorem 4.5.3 we shall prove that the repelling part S_2 of the non-wandering set for a class of ws-hyperbolic map on $\mathbf{P}^2$, has zero Lebesgue measure. In general, for all ws-hyperbolic maps we conjecture that the set S_2 has zero Lebesgue measure.

4.2 Non-wandering sets, hyperbolicity and spectral decomposition

Let us start by giving several definitions and collecting some theorems used throughout the article.

Definition 4.2.1. *The non-wandering set of f, denoted by Ω_f (or simply by Ω when no confusion arises), is by definition the set of points $x \in \mathbf{P}^2$ having no neighbourhood U such that $f^n(U) \cap U = \emptyset, \forall n \geq 1$.*

It can be easily shown that Ω is closed in $\mathbf{P}^2$, hence compact. In one complex variable all Fatou components are non-wandering, from a theorem of Dennis Sullivan, see for example [8].

Next we introduce the notion of hyperbolicity on a set K that is invariant by f. First let $\hat{K} := \{\hat{x} = (x_k)_k \in \Pi_{k \in \mathbb{Z}} K, f(x_{k-1}) = x_k, \forall k\}$ the set of complete histories of points in K. $\hat{K}$ is called the hyperbolic cover of K.

We consider now the homeomorphism $\hat{f} : \hat{K} \to \hat{K}$ defined by $\hat{f}((x_k)_k) = (f(x_k))_k$. Define also the tangent bundle to $\hat{K}$, $T\hat{K}$ as the set of pairs $(\hat{x}, \xi)$, where $\hat{x} \in \hat{K}$ and $\xi \in T_{x_0}\mathbf{P}^2$.

Definition 4.2.2. *f is called hyperbolic on K if there exists a continuous splitting of the tangent bundle $T\hat{K} = E_u \oplus E_s$, which is invariant under $D\hat{f}$ such that $\exists c \geq 0, \lambda \geq 1$ with the property that $\mid D\hat{f}^n(\xi) \mid \geq c \cdot \lambda^n \mid \xi \mid, \forall \xi \in E_u$ and $\mid D\hat{f}^n(\xi) \mid \leq c \cdot \lambda^{-n} \mid \xi \mid, \forall \xi \in E_s$.*

The unstable index of a point is the dimension of E_u at that point.

Denote by S_i the set of non - wandering points of unstable index i, with i=0, 1, 2.

We will say that f satisfies *Axiom A* when f is hyperbolic on its non-wandering set Ω, and moreover the periodic points of f are dense in Ω.

Define also $W^s_{x,R} := \{y \in \mathbf{P}^2, d(f^n(y), f^n(x)) \leq R, \forall n \geq 0\}$ and $W^u_{\hat{x},R} := \{y_0 \in \mathbf{P}^2, \exists (y_k)_{k\leq 0}$ a prehistory of y_0 in S_1 s.t$d(y_k, x_k) \leq R, \forall k\}$, the local stable and respectively unstable sets for points in K . Here d is just the usual distance on $\mathbf{P}^2$ induced by the Fubini - Study metric.

If R is small, the local stable set is defined uniquely as a submanifold passing through x, whereas the local unstable set depends on the prehistory of x_0 chosen.

Definition 4.2.3. *The set K has local product structure if there exists an $R \geq 0$ so that $\forall x \in S_1$ and $\hat{y} \in \hat{S}_1, W^s_{x,R} \cap W^u_{\hat{y},R}$ consists of at most one point and this point is contained in K.*

The following important and well - known theorem will be used throughout the paper(for reference see Ruelle [34]):

Theorem 4.2.4. *If f is Axiom A, then Ω can be uniquely decomposed as a disjoint union of invariant sets Ω_i s.t f is transitive on each one of them. The sets Ω_i are called the basic sets for f.*

If S^1_j, S^1_k are basic sets of S_1, define also $W^s(S^k_1) := \{x \in \mathbf{P}^2, d(f^n(x), S^k_1) \rightarrow 0\}$ and $W^u(S^j_1) := \{x_0 \in \mathbf{P}^2$ s.t $\exists$ a prehistory of x_0,$(x_k)_{k<0}$, s.t $d(x_k, S^j_1) \rightarrow 0$ when $k \rightarrow -\infty\}$.

More information about hyperbolicity and nonwandering sets can be found in Shub [35] or in Katok - Hasselblatt [24].

Let us remind also the natural ordering between the basic sets.

Definition 4.2.5. *If S^j_1, S^k_1 are basic sets of S_1, we say that $S^j_1 \succ S^k_1$ if $W^u(S^j_1) \cap W^s(S^k_1) \neq \emptyset$.*

f has no cycles if there is no chain of the form $S^{j_1}_1 \succ ... \succ S^{j_l}_1$ among the basic sets of unstable index 1.

Now let ω be the standard Kahler form on $\mathbf{P}^2$ normalized so that $\int \omega \wedge \omega = 1$. Then, according to Fornaess - Sibony [14], the sequence of positive closed (1,1) forms $\frac{(f^n)^\star \omega}{d^n}$ converges in the sense of currents to a positive closed current T.

The current T depends continuously on f holomorphic of degree d and it has no mass on locally pluripolar sets. It satisfies the equation $f^\star T = dT$ and the support of T is equal to the Julia set J of f.

For general information about currents and their properties, a good general reference is Klimek's book, [23].

In the case above one can define the wedge product of T with itself, $\mu := T \wedge T$ which will give an ergodic probability measure on $\mathbf{P}^2$, satisfying $f^\star \mu = d^2 \mu$. Denote the support of this measure by S_μ.

We are going to define the notion of ws - hyperbolic map. In order to do this, assume f is hyperbolic on Ω and let us denote by S_i, i = 0, 1, 2, the set of points from Ω where the dimension of the unstable bundle is equal to i. With this notation, Fornaess - Sibony proved that S_0 is a finite union of attracting periodic points and that S_2 contains the support of μ ([13]).

Definition 4.2.6. *The map f is weakly s - hyperbolic (or ws - hyperbolic) if :*

i) f is Axiom A

ii) $f^{-1}(S_2) = S_2$

iii) $\exists$ an algebraic variety A of dimension 1 s.t $A \cap S_1 = \emptyset$.

If in addition, one also has the condition

iv) $\exists U$ a neighbourhood of S_1 s.t $f^{-1}(S_1) \cap U = S_1$,

then we call f s - hyperbolic.

Remark : Maps that are weakly s - hyperbolic maps have also been studied, although under a different name in [4]; in particular for these maps we have landing of disks properties, also connectedness of S_μ in some cases, etc. We will come back to these properties later in the chapter.

It is not yet clear whether the notions of ws - hyperbolic, s - hyperbolic and Axiom A are all equivalent or not.

In the sequel we will deal also with the set of points whose preimages do not converge uniformly towards the support S_μ, of the measure μ. This set will be the analogue of the set of points with bounded backward iterates from the case of Henon diffeomorphisms ([14]). This motivation is explained below.

Indeed consider a Henon map of degree d in $\mathbb{C}^2$, $g(z, w) = (z^d + c + aw, z)$, where $a \neq 0$; this map is an automorphism of $\mathbb{C}^2$ and it extends to a holomorphic mapping of $\mathbf{P}^2$ denoted also by g. Then it is immediate to see that the point $p_+ = [1 : 0 : 0]$ is attracting for g and a point of indeterminacy for g^- and that similarly $p_- = [0 : 1 : 0]$ is attracting for g^- and a point of indeterminacy for g.

Define also temporarily the sets $K^+ := \{z \in \mathbb{C}^2 s.t g^n(z) \text{ bounded}, n \geq 0\}$ and $K^- := \{z \in \mathbb{C}^2 s.t g^{-n}(z) \text{ bounded}, n \geq 0\}$. Let $J^+ := \partial K^+$ and $J^- := \partial K^-$ their boundaries.

Define the Green functions $G^+(z, w) := lim_n \frac{1}{d^n} log^+|g^n(z, w)|$; G^- is defined similarly.

Now it can be proved ([14]) that there are positive closed (1,1) currents μ^+, μ^- on $\mathbf{P}^2$ whose restrictions to $\mathbb{C}^2$ are equal to dd^cG^+, respectively dd^cG^-. It can be shown that both μ^+ and μ^- are the trivial extensions of dd^cG^+ and dd^cG^- to $\mathbf{P}^2$.

It was proved by Fornaess-Sibony in [14] that the currents μ^+, μ^- are of mass one, that supp$\mu^+ = J^+ \cup \{p^-\}$ and supp$\mu^- = J^- \cup \{p^+\}$ and most importantly that $\mu^+ \wedge \mu^+ = \delta_{p_-}, \mu^- \wedge \mu^- = \delta_{p_+}$. ($\delta_{p_+}, \delta_{p_-}$ denote the Dirac masses at p_+, p_- respectively).

Therefore, one is led to the observation that, in fact $K^- = \{z \in \mathbb{C}^2, g^{-n}(z)$ do not converge uniformly to p_-=supp $\mu^+ \wedge \mu^+$, when $n \to \infty\}$.

Since, in the case of a holomorphic endomorphism f of $\mathbf{P}^2$, the current T is the analogue of the current μ^+ from the Henon case and the measure $\mu = T \wedge T$ is the analogue of $\mu^+ \wedge \mu^+$, the following definition makes sense:

Definition 4.2.7. *Let* $U_f^- := \{z_0, d(f^{-n}(z), S_\mu) \to 0$ *uniformly in a neighbourhood of* $z_0\}$*. Then* $K_f^- := \mathbf{P}^2 \setminus U_f^-$.

We will denote K_f^- simply by K^- when no confusion will arise. We will come back to this set later on, in Theorem 4.3.2.

The following proposition gives information about the local product structure of S_1 and the expanding set S_2. One can prove it without the help of condition *iv*) of Definition 6; we will only state the result, the idea of the proof can be found in Fornaess - Sibony ([15]).

Proposition 4.2.8. *Suppose f is ws - hyperbolic.Then each of the* S_j *is nonempty and the set* S_1 *has local product structure.*

Furthermore, if $W^s_\epsilon(S_1)$ *denotes the set of points y whose entire forward orbit* $(f^n(y))_{n\geq 0}$ *remains at a distance* $\leq \epsilon$ *from* S_1*, then there exists* $R \geq 0$ *so that* $W^s_\epsilon(S_1) = \cup_{x\in S_1} W^s_{x,R}$.

Finally the set S_2 *of unstable dimension 2 is equal to* S_μ.

In the sequel we will also need some tools concerning persistence of hyperbolic sets and fundamental neighbourhoods; the following appear in Ruelle [34] and Jonsson [22].

Proposition 4.2.9. *If* f *is hyperbolic on* $\Lambda = \Lambda_f$ *and* g *is* C^1 *- close to* f *, then there exists a continuous map* $h : \hat{\Lambda} \to \mathbf{P}^2$ *close to the projection* $\pi(\hat{x}) = x_0$ *such that* $g \circ h = h \circ \hat{f}$ *and that* g *is hyperbolic on* $\Lambda_g := h(\Lambda_f)$ *. The map* h *lifts to a homeomorphism* $\hat{h} : \hat{\Lambda}_f \to \hat{\Lambda}_g$ *with* $\hat{g} \circ \hat{h} = \hat{h} \circ \hat{f}$*. Moreover , if* $\hat{\Lambda}$ *has local product structure, then for any sufficiently small neighbourhood* U *of* Λ *, we have that, if* g *is* C^1 *- close to* f*, then the set* Λ_g *is given by* $\Lambda_g = \{x_0; (x_i)_{i\in\mathbb{Z})}$ *is a complete* g *- orbit contained in* $U\}$*. Such a neighbourhood* U *is called a fundamental neighbourhood for* f *and* Λ.

In general there is a shortage of examples of hyperbolic maps on $\mathbf{P}^2$, and in particular of maps satisfying ws - hyperbolicity.

4.3 Ws-hyperbolic maps on $\mathbb{P}^2$; stability results

In the sequel we will prove that actually the set of ws - hyperbolic maps satisfying the no cycle condition is open. Therefore, it is enough to find a ws - hyperbolic map with the no cycle condition , like the map $[z : w : t] \to [z^2 : w^2 : t^2]$ and to perturb it in order to find a large class of examples.

We are ready now to prove our first theorem in this chapter (see [27]).

Theorem 4.3.1. *The set of ws - hyperbolic maps on* $\mathbf{P}^2$ *having no cycles is open.*

Proof : Assume f is a ws - hyperbolic map which does not have any cycles

Then any map which is C^1 - close to f will also satisfy Axiom A and have no cycles (see Ruelle [34] or Moriyasu [29]). Moreover the non - wandering sets of such maps f, g are close if the maps are close and $\hat{f}\mid_{\hat{\Omega}_f}$ is conjugate to $\hat{g}\mid_{\hat{\Omega}_g}$.

In particular, if f_ϵ is close to f and $S_{i,\epsilon}, i = 0, 1, 2$ denotes the unstable set of f_ϵ of unstable index i, a modification of the filtration method argument from the diffeomorphism case ([35] and [22]) will show that $S_{i,\epsilon}$ is close to S_i. Moreover, in [22] it is shown that the set $S_{i,\epsilon}$ is actually the set Λ_{f_ϵ} from Proposition 4.2.9, as long as $\Lambda = S_i(f)$.

Because the condition about the existence of an analytic set outside S_1 is then clearly true for maps in a neighbourhood of f, it is enough to prove only the persistence of the property of total invariance for $S_{2,\epsilon}$.

Let now U be a fundamental neighbourhood of S_2 in the sense of Proposition 4.2.9.

Let δ be a small positive number so that $U = S_2(\delta)$. Now, because $f^{-1}(S_2) = S_2$, and S_2 is repelling, there will exist some $0 < \alpha < 1$ such that $f^{-1}(U) \subset S_2(\alpha\delta)$.

We want to prove that the preimages of points from U by maps f_ϵ which are uniformly C^1 - close to f on $\mathbf{P}^2$, are always contained in U. Assume that ϵ is fixed and it is so small that Proposition 4.2.9 is satisfied for functions f_ϵ.

Suppose $x \in U$, but there exists $y \notin U$ s.t $f_\epsilon(y) = x$. Obviously, if C denotes the critical set of f, $C \cap S_2 = \emptyset \Rightarrow f(C) \cap S_2 = \emptyset$.

WLOG one may assume that $f(C) \cap \bar{U} = \emptyset$.

Then,since x is not a critical value, one can find $k = d^2$ f-preimages $x_1, x_2, ..., x_k$ of x and neighbourhoods $U_1, ..., U_k$ of these preimages and $V \subset U$ a neighbourhood of x so that $f : U_i \to V$ is a biholomorphism for any i. Let assume U_i is a ball of radius $\leq \frac{(1-\alpha)\delta}{3}$ centered at $x_i, \forall i$.

However, $f_\epsilon(y) = x$, so if ϵ is small enough , $f(y) \in V$, from the uniform closeness between f and f_ϵ. But any point from V has k different f-preimages contained in $U_1 \cup ... \cup U_k$. By assumption, $y \notin U$, hence $d(y, S_2) > \delta$. Moreover, $x_i \in f^{-1}(U) \subset S_2(\alpha\delta)$ for any i.

Therefore, if $z \in U_i$ for some i, then $d(z, S_2) < \alpha\delta + \frac{(1-\alpha)\delta}{3}$ and $\alpha < 1$, hence $d(z, S_2) < \delta$ so we cannot have $y \in U_1 \cup ... \cup U_k$.

Thus we arrived to a contradiction, meaning that $f_\epsilon^{-1}(U) \subset U$.

Now if x_0 is a point from $f_\epsilon^{-1}(S_{2,\epsilon})$, then all the forward f_ϵ iterates of x_0 are in $S_{2,\epsilon}$ and also all its f_ϵ-prehistories will still remain in U by our result above.

Therefore, if in Proposition 4.2.9 we take $g = f_\epsilon$ and $\Lambda = S_2(f)$, then x_0 is satisfying the condition from the end of Proposition 4.2.9, hence it should be in $S_{2,\epsilon}$. This concludes our proof showing that ws - hyperbolicity is stable under small perturbations.

$\diamond$

If one looks at the usual examples of hyperbolic maps (like the ones studied in Fornaess - Sibony [15]), they all have superattracting periodic points because these 2 -dimensional examples are basically obtained from polynomials, all of which have a superattracting fixed point. We can ask if this would be the case in general for ws - (or s -) hyperbolic maps of $\mathbf{P}^2$. As an example of a ws-hyperbolic map with no superattracting periodic points we will take in the sequel the following $f_\epsilon[z:w:t] = [z^2+\epsilon w^2 : w^2+\epsilon t^2 : t^2+\epsilon z^2]$, where $\epsilon > 0$ is small.

Then f_ϵ is close to $f[z:w:t] = [z^2 : w^2 : t^2]$ which is Axiom A and has no cycles. Indeed one can check that $S_0(f) = \{[1:0:0],[0:1:0],[0:0:1]\}, S_1(f) = \{|z| = |w|, t = 0\} \cup \{|z| = |t|, w = 0\} \cup \{|t| = |w|, z = 0\}$ and $S_2(f) = \{|z| = |w| = |t|\}$. It is easy to see that there are no cycles among the basic sets for f and that f is ws - hyperbolic.

Also, one can check that the critical set of f_ϵ is still $(z=0)\cup(w=0)\cup(t=0)$ and that there are no fixed points on this set. Consequently , f_ϵ has no superattracting fixed points and it is ws-hyperbolic if ϵ is small.

Let us notice also that, since $S_0(f_\epsilon)$ is close to $S_0(f)$ and $\hat{f}_\epsilon|S_0\hat{(}f_\epsilon)$ is conjugated to $\hat{f}|S_0(f)$ (this follows from the proof of Theorem 4.3 of [22]), we will have that actually the only periodic points of f_ϵ are three fixed points close to the original fixed points of f.

So no periodic point of f_ϵ is superattracting.

Let us look now at the problem whether the set K^- can have nonempty interior..

If S_1^j is minimal for the ordering between basic sets, then a theorem of Fornaess - Sibony [15] states that $W^u(S_1^j)\backslash S_1^j$ is contained in the basin of attraction of S_0. It makes sense then to look at the dynamics of f on K^- near S_1. We will show that K^- does not have interior points from S_1 ([27]), although a priori it could have other interior points.

Proposition 4.3.2. *In the above notation, if f is an s - hyperbolic map of $\mathbf{P}^2$ (in the sense of Fornaess - Sibony [15]), and if the critical set of f does not intersect $S_1(f)$, then K_f^- cannot have interior points in $S_1(f)$.*

Proof If f is s - hyperbolic, then one can prove that $K^- = \cup_{\hat{x}\in\hat{S_1}} W^u(\hat{x}) \cup S_0$ (Fornaess - Sibony [15]).

Now, if K^- does have interior points on S_1 , then a stable disk of a point from S_1, $W^s(x)$, will be contained in $W^u(\hat{S_1})$. However from the local product structure (Proposition 4.2.3), this is possible only if $W^s(x) \subset S_1$. So we have to show that there are no stable disks in S_1.

Actually from the discussion above and because periodic points are dense in Ω one can see that it is enough to show that there are no local stable disks of periodic points contained in S_1. Assume then that $\exists p \in S_1$ s.t $\Sigma := W^s_\epsilon(p) \subset S_1$; without loss of generality suppose p is fixed and that Σ is simply connected.

Define g_1 to be a local branch of f^{-1}, s.t $g_1(p) = p$ and g_1 takes the part of Σ where it is defined, into S_1. Let now $\gamma|[0,1]$ a path in Σ connecting p to some other point. Assume we can extend g_1 analytically on a union of balls W so that $g_1(\Sigma \cap W) \subset S_1$ and $g_1(W) \subset U$, where U is the neighbourhood of S_1 from the definition of s-hyperbolicity ; assume also that $\gamma(t_0) \in \partial W$ for some $t_0 \in (0,1)$.

We want to prove that g_1 can be extended analytically past t_0 s.t it takes points from Σ to S_1, which will imply that actually g_1 can be extended to Σ s.t $g_1(\Sigma) \subset S_1$.

Let then $x_1, ..., x_m$ the f - preimages of $\gamma(t_0)$ which are in S_1. According to Definition 4.2.6, since f is s - hyperbolic, we know that these are the only preimages of $\gamma(t_0)$ in U.

Since $C \cap S_1 = \emptyset, x_1, ..., x_m$ are not critical, so we can find neighbourhoods U_i of these points respectively and N a neighbourhood of $\gamma(t_0)$, so that $f : U_i \to N$ is a biholomorphism $\forall i$. Now g_1 is defined on $N \cap W$, and $g_1(N \cap W)$ is connected; hence it is contained in only one of the U_i 's.

Then there is a branch of f^{-1} in N coinciding with the old g_1 in $N \cap W$. Also, since $\forall i = 1..m, U_i \subset U, \Longrightarrow g_1(\Sigma \cap (W \cup N)) \subset f^{-1}(S_1) \cap U$.

Therefore using again condition iv) of Definition 4.2.6, we get $g_1(\Sigma) \subset S_1$.

Since Σ is simply connected , the function g_1 will be defined on a whole neighbourhood $\tilde{U_1}$ of Σ and it is single - valued, analytic, and $g_1(\Sigma) \subset S_1$.

A similar argument will enable us to define for all n, a branch g_n of f^{-n} on a neighbourhood $\tilde{U}_n$ of Σ such that $g_n(\Sigma) \subset S_1$. The neighbourhoods $\tilde{U}_n$ are shrinking towards Σ when $n \to \infty$.

Now by replacing eventually the analytic set A or C by a close analytic set of high enough degree, one can assume, using a result of Siu - Yeung [38], that $\mathbf{P}^2 \setminus A$ is *Brody hyperbolic*, i.e there is no nonconstant holomorphic function from $\mathbb{C}$ to $\mathbf{P}^2 \setminus A$.

The following argument also gives a complete and direct proof for a fact that is employed in [15].

Let us take two small neighbourhoods V_1, V_2 of A s.t $A \subset V_1 \subset V_2$ and assume that $\mathbf{P}^2 \setminus V_1$ and $\mathbf{P}^2 \setminus V_2$ are connected. We want to show that $\mathbf{P}^2 \setminus V_2$ is Kobayashi hyperbolic.

To prove this we will modify an idea of Brody ([7]). Assume we have chosen a Hermitian metric $|v|$ on the complex manifold $\mathbf{P}^2 \setminus V_1$ and consider the Hermitian metric on $\mathbf{P}^2 \setminus V_2$ induced by the metric$|v|$ considered on $\mathbf{P}^2 \setminus V_1$.

According to Lemma 1.1 of Brody ([7]), if M denotes the complex manifold $\mathbf{P}^2 \setminus V_2$ with the above metric, then M is Kobayashi hyperbolic iff $sup|f'(0)| < \infty$, $f \in$ Hol(Δ, M). (Hol(Δ, M) is the set of all holomorphic mappings from Δ to M)

So, if M is not hyperbolic, there exists a sequence of maps $f_i \in$Hol(Δ, M) with $|f_i'(0)| \to \infty$. Obviously, by scaling, we can take $f_i \in$ Hol(Δ_{r_i}, M) s.t $|f_i'(0)| = 1$ and $r_i \to \infty$.

Also, according to Lemma 2.1 of [7], given a holomorphic map f from the disk of radius r to M, $f \in$ Hol(Δ_r, M), there will exist $\tilde{f} \in$ Hol(Δ_r, M) with

$$sup_{z\in\Delta_r}|\tilde{f}'(z)|\frac{r^2-|z|^2}{r^2} = |\tilde{f}'(0)| = |f'(0)|.$$

Hence, in our case, there will exist maps $\tilde{f}_i \in$Hol(Δ_{r_i}, M) satisfying

$$sup_{z\in\Delta_{r_i}}|\tilde{f_i}'(z)|\frac{r_i^2-|z|^2}{r_i^2} = |\tilde{f_i}'(0)| = |f_i'(0)| = 1.$$

But in this case, because the Hermitian metric on the tangent bundle TM is just the one induced from the tangent bundle of $\mathbf{P}^2 \setminus V_1$, it follows that there is a uniform upper bound for $|\tilde{f_i}'(z)|$ in Δ, hence $(\tilde{f}_i)_i$ are equicontinuous as maps from Δ to $\mathbf{P}^2 \setminus V_1$. On the other hand, since M is relatively compact in $\mathbf{P}^2 \setminus V_1$, it follows also that for any $z \in \Delta$, $(\tilde{f}_i(z))_i$ forms a relatively compact set in $\mathbf{P}^2 \setminus V_1$. Therefore some subsequence of $\tilde{f}_i$ is convergent uniformly on compacts in Δ.

A further refinement will give another subsequence converging on Δ_2 and then by a diagonal process we get the limit as a holomorphic map ϕ defined on all of $\mathbb{C}$ with values in $\mathbf{P}^2 \setminus A$. Also, $|\phi'(0)| = lim_{i\to\infty}|\tilde{f_i}'(0)| = 1$, so ϕ is not constant.

But then ϕ is a nonconstant entire map with values in $\mathbf{P}^2 \setminus A$ and this is a contradiction to the fact that, according to Siu - Yeung's result [38], $\mathbf{P}^2 \setminus A$ is Brody hyperbolic. Hence M $= \mathbf{P}^2 \setminus V_2$ is Kobayashi hyperbolic.

Then, using the Kobayashi distance on M, we have a family of holomorphic maps $g_n : \Sigma \to \mathbf{P}^2 \setminus V_2$ which is equicontinuous and s.t $g_n(\Sigma) \subset S_1 \subset\subset \mathbf{P}^2 \setminus V_2$. So, according to Arzela - Ascoli Theorem [1], $(g_n)_n$ is a normal family; but this would be a contradiction with the fact that, from the hyperbolicity condition of f, the derivatives of its backward iterates, Dg_n, are increasing indefinitely in the stable direction, hence on $T\Sigma$.

In conclusion, we cannot have local stable disks in S_1, and $\overset{\circ}{K^-} \cap S_1 = \emptyset$.

$\diamond$

One will notice that a proof of the normality of the sequence $(g_n)_n$ can be given above by using a normality result of Ueda. We preffered to give a perhaps longer proof because it also shows that the complement of a neighbourhood V_2 of A is Kobayashi hyperbolic.

In the sequel we shall give several examples of ws-hyperbolic maps for which the non-wandering sets can actually be computed explicitly. More examples of ws-hyperbolicity can be obtained from these by way of small perturbations, according to Theorem 4.3.1. These examples also show that a more general theorem about the interior of the set K^-, on the lines of Proposition 4.3.2 might actually be true.

4.4 Examples of ws-hyperbolic mappings

In this section, we shall give several classes of examples of ws(s)-hyperbolic mappings on $\mathbf{P}^2$ obtained from products of one variable expanding maps, by Ueda's method or by taking perturbations of real solenoids.

1. Product functions

Let the holomorphic function f of $\mathbf{P}^2$ be given by

$$f[z:w:t] = [P(z,t) : Q(w,t) : t^d],$$

where P and Q are polynomials of degree d in z, respectively w. Assume also that P and Q are hyperbolic on their Julia sets, J_P and J_Q. This happens if and only if all critical points are in the basin of attraction of attracting cycles ([8]).

In this case, it is clear that the line at infinity ($t = 0$) becomes totally invariant for f and that there exists a basin of attraction to ($t = 0$). Since P and Q are expanding on their respective Julia sets, it follows that $S_2 \cap (t = 0) = \emptyset$ and that in $t = 1$, $S_2 = J_P \times J_Q$.

The basic sets for S_1 are, in $t = 1$, $\{periodic\ sinks\ of P\} \times J_Q$ and $J_P \times \{periodic\ sinks\ of Q\}$. The basic set for $f|_{t=0}$ is the Julia set for the one variable map $[P(z,0) : Q(w,0) : 0]$. Finally, $S_0 = \{periodic\ sinks\ of P\} \times \{periodic\ sinks\ of Q\}$.

Then f is ws-hyperbolic and it does not have any cycles. Hence we can apply Theorem 4.3.1 to get other examples which may not be product functions, but which are ws-hyperbolic and have no cycles as well.

One also notices that for product functions all the unstable manifolds for all prehistories of a point in S_1 coincide and are contained in complex lines.

2. Examples obtained from Ueda's method

Here we will use again Ueda's method of constructing an endomorphism f of $\mathbf{P}^2$ from a one variable function f_0, [41]

Indeed, if π is the projection map from $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^2$ used in Chapter III, then if $f_0 : \mathbf{P}^1 \to \mathbf{P}^1$ is a hyperbolic map, and if f is the uniquely determined map of $\mathbf{P}^2$ such that $\pi(f_0, f_0) = f \circ \pi$, it follows that f is ws-hyperbolic.

The basic sets for S_1 are of the form $\pi(J_0, periodic\ sinks\ of f_0)$, where J_0 denotes the Julia set of f_0.

The repelling part, $S_2 = \pi(J_0, J_0)$; this follows from the fact that f_0 is expanding on its Julia set J_0, hence $\pi(J_0, J_0)$ will have unstable index 2 and this is indeed the entire repelling part of the non-wandering set of f. In this case, again the set K^- is an algebraic variety.

3. Examples obtained from real 2-dimensional ones

The following example appears in Fornaess-Sibony [15]. Consider the complexified perturbation of the real solenoid:

$$f_\epsilon[z:w:t] = [z^2 : \tfrac{1}{10}wt + \tfrac{1}{2}zt + \epsilon w^2 : t^2].$$

There are only three basic sets for S_1, namely a circle in (t = 0), a quasicircle in (z = 0) and a solenoid in (t = 1). The real version of the solenoid is discussed in several dynamical systems texts, for example in Robinson [33] or Katok-Hasselblatt [24].

What is important for us here is that the map f_ϵ is injective in a neighbourhood of the solenoid, so again there is just one prehistory belonging entirely to S_1 for any given point. Hence K^- has empty interior in this case too.

All these examples might then conjecture that for any given (w)s-hyperbolic map, the set K^- does not have "many" unstable manifolds for a point in S_1, and that K^- has empty interior. Let us notice also that for the separate case of Henon maps g, if $|detDg| < 1$, then int $K^- = \emptyset$ and if g is hyperbolic and $|detDg| = 1$, then int $K^- =$ int $K^+ = \emptyset$ (for reference see [5]).

4.5 The repelling part of the non-wandering set of a ws-hyperbolic map

In this final section we will state an interesting result that was proved in Bedford - Jonsson [4] in connection to topological properties for the repelling part of the non-wandering set of ws-hyperbolic mappings. Afterwards, we will also prove that, for a particular class of functions, a repellor with empty interior has zero Lebesgue measure.

Theorem 4.5.1 ([4]). *Let f be a ws-hyperbolic endomorphism of $\mathbf{P}^2$ such that there exists a hyperplane Π totally invariant to f (i.e $f^{-1}(\Pi) = \Pi$) and on which f is hyperbolic. Assume also that no critical point is attracted to the Julia set J_Π of the restriction $f|_\Pi$.*

In this case, if J_Π is connected, then also S_2 will be connected. If J_Π is locally connected, then so is S_2.

The theorem is proved by obtaining a natural surjective map from $J_\Pi \times S^1$ onto S_2 corresponding to the "landing" of stable disks of points in J_Π onto S_2.

More precisely, they show that there exists a homeomorphism $\Psi : J_\Pi \times (\hat{\mathbb{C}} - \bar{\Delta}) \to W^s(J_{Pi})$, whose restriction $\Psi(a, \cdot)$ maps $\hat{\mathbb{C}} - \bar{\Delta}$ conformally onto the stable disk W_a, for any point a in the Julia set J_Π of the restriction to Π of our function f (Δ is supposed to be the unit disk).

Furthermore Ψ extends to a continuous map of $J_\Pi \times (\hat{\mathbb{C}} - \Delta)$ into $W^s(J_\Pi) \cup J$.

This will give then S_2 as a topological quotient of $J_\Pi \times S^1$, hence transfering connectedness properties from J_Π to S_2.

One may ask if there is an actual homeomorphism between the non-wandering sets of holomorphic maps of $\mathbf{P}^2$ which are close by, on the lines of Proposition 4.2.9.

In general this is not true, still in the case of the repelling parts S_2, we will have a holomorphic motion which will provide us with an actual homeomorphism without recurring to the hyperbolic covers. This would give topological information about the sets S_2 for maps that are close to a simpler ws-hyperbolic one, in the light of Theorem 4.3.1.

In the sequel we will also need the notion of *repellor*, i.e., a compact invariant set K of unstable index 2 for a holomorphic map on $\mathbf{P}^2$. We will study the question whether a repellor has zero Lebesgue measure.

In the one variable case, it can be shown ([8]) that, if R is a rational function expanding on its Julia set J and such that $\infty \notin J$, then J has zero area. In fact it is conjectured that for any rational map, either J has zero area or $J = \bar{\mathbb{C}}$. The proof in the two dimensional case is going to be similar to the one dimensional argument.

First, let us consider a special class of functions. We will ask in what circumstances the norm of a product AB of derivatives of each of these functions at different points will be equal to the norm of the reversed product BA.

Example 4.5.2. *$f(z,w) = (Q(w), P(z))$, with P and Q polynomials of degree $d \geq 2$, so that f can be extended holomorphically to the entire $\mathbf{P}^2$.*

Then the derivative of f is equal to $f'(z,w) = \begin{pmatrix} 0 & Q'(w) \\ P'(z) & 0 \end{pmatrix}$

But, for two matrices of the form $A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & a' \\ b' & 0 \end{pmatrix}$, it is noticed that $AB = \begin{pmatrix} ab' & 0 \\ 0 & ba' \end{pmatrix}$ and $BA = \begin{pmatrix} a'b & 0 \\ 0 & b'a \end{pmatrix}$, hence the norm of AB is the same as the norm of BA.

We also make the observation that there exist cross-product functions having totally invariant repellors, see the definition of a repellor before Theorem **??**. Indeed, it is enough to see that the function $[z:w:t] \to [w^2 : z^2 : t^2]$ has a repellor equal to the set $\{[z:w:1], |z| = 1, |w| = 1\}$.

Let us also notice that in general there is no relationship between the non-wandering sets of f and of the function $\tilde{f}[z:w:t] = [P(z) : Q(w) : t^d]$. Also, the eigenvalues of $Df(z,w)$ and $D\tilde{f}(z,w)$ are different in general, so a repellor for f is not necessary a repellor for $\tilde{f}$ and vice-versa.

We are now ready to prove that the Lebesgue measure of a repellor with empty interior is equal to zero in the case of functions of this type.

Theorem 4.5.3. *Consider a function f like in Example 4.5.2 which has a totally invariant repellor K with empty interior. In this case the Lebesgue measure of K is zero.*

Proof

Because K is a repellor, there will exist constants $c > 0$ and $\lambda > 1$, so that $|f_\star^n v| \geq c\lambda^n |v|, \forall n \geq 1$ and all tangent vectors v over S_2.

Since K is compact, it follows that the second derivative and the inverse of any first derivative of f are bounded on K.

Let now a neighbourhood U of K such that $f^{-1}(U) \subset U$; such a neighbourhood exists because of the repelling structure and because the set K is totally invariant. Therefore, there will exist a positive constant M so that

$||f'(f^{-k}(z)) - f'(f^{-k}(w)|| \leq M \cdot |f^{-k}(z) - f^{-k}(w)| \leq M \cdot \frac{1}{\lambda^k} \cdot d(z,w), \forall k \geq 1$, z,w in a small ball V contained in U and f^{-k} a fixed analytic branch of the inverse k iterate of f.

So, since f' is bounded also on U, then perhaps for a different constant M, we have that

(1) $||I - f'(f^{-k}(w)) \cdot (f'(f^{-k}(z)))^{-1}|| \leq M \cdot \frac{1}{\lambda^k} d(z,w), \forall k \geq 1$, and $z, w \in V$.

Let us estimate the norm $||((f^{-n})'(w))^{-1} \cdot (f^{-n})'(z)||$, when $n \to \infty$.

Now, $(f^{-n})'(w) = (f')^{-1}(f^{-(n-1)}(w)) \cdot ... \cdot (f')^{-1}(w)$, so

$((f^{-n})'(w))^{-1} \cdot (f^{-n})'(z) = f'(w) \cdot ... \cdot f'(f^{-(n-1)}(w) \cdot (f')^{-1}(f^{-(n-1)}(z)) \cdot ... \cdot (f')^{-1}(z)$.

Then we can use the inequality (1) from above repeatedly, starting from the center of the right term in the previous equation towards the extremities, because of the commutativity of the derivatives up to taking norms that was proved in Example 4.5.2. In conclusion we get that, when z, w belong to a small ball V in U, $||((f^{-n})'(w))^{-1} \cdot (f^{-n})'(z)|| \leq 1 + Cd(z,w)$,, where $C > 0$ is independent of n and $z, w \in U$.

Therefore, in the limit, the branches f^{-n} are uniformly close to being affine on small balls in U.

We also get from above that $((f^{-n})'(w))^{-1} \cdot (f^{-n})'(z) = I + Md(z,w)$, with M independent of z, w, n.

This means that the second derivatives are also bounded uniformly in z, w, n on small balls in U.

Let now an arbitrary point z_0 in K and fix a small $\epsilon > 0$. All the sets of the form $f^{-n}(B(f^n(z_0, \epsilon))$ will be open and will contain the point z_0.

Since K is compact with empty interior, it follows that for any small ball of the form $B(z, \epsilon), z \in S_2$, at least a fraction $\delta > 0$ from its area will be contained in $U \setminus S_2$.

Also notice that, because K, U are totally invariant, we have $f^{-1}(U \setminus K) = U \setminus K$.

Now, because f^n is expanding, we have that diam $f^{-n}(B(f^n(z_0, \epsilon))) \to 0$, when $n \to \infty$.

We will use now the fact proved above, that the branches f^{-n} are uniformly close to being affine on small balls. This implies that they will take $B(f^n(z_0), \epsilon)$ into a ball-like open set centered at z_0 which contracts towards z_0 when $n \to \infty$.

Hence it follows that at least a fraction of $\frac{\delta}{2}$ of the areas of $f^{-n}(B(f^n(z_0, \epsilon)))$ will be in $U \setminus S_2$.

Taking a break from the main proof, let us now look at how is f^{-n} deforming a small ball of the form $B(z, \epsilon')$.

The eigenvalues of a matrix of the form $A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ are equal to $\overset{+}{-} \sqrt{ab}$. If $f = (P, Q)$ is assumed to be repelling on K, the eigenvalues of $f'(z,w)$ will not be zero. Their values will be $\overset{+}{-}\sqrt{P'(z)\cdot Q'(w)}$ and will be different, though equal in absolute value.

It follows that the matrix is diagonalizable and two linearly independent eigenvectors at (z,w) are $\begin{pmatrix} 1 \\ \sqrt{\frac{P'(z)}{Q'(w)}} \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -\sqrt{\frac{P'(z)}{Q'(w)}} \end{pmatrix}$.

Denote the eigenvalues of $f'(z,w)$ by $\lambda_1(z,w)$ and $\lambda_2(z,w)$; it is clear that they depend continuously in (z,w) and that their corresponding unit eigenvectors $e_1(z,w), e_2(z,w)$ depend also continuously in (z,w). If $< \cdot, \cdot >$ represents the scalar product on $\mathbb{C}^2$, then for $(z,w) \in K$, there will exist $0 \leq \eta_0 < 1$ so that $0 \leq Re < e_1(z,w), e_2(z,w) > \leq \eta_0 < 1$.

We have this since e_1, e_2 are linearly independent unit vectors in $\mathbb{C}^2$ depending continuously on a point in a compact set.

Fix now a point (z,w); we will use the notations $\lambda_1, \lambda_2, e_1, e_2$ without recording the point (z,w) again. Consider also a tangent vector which can be written as $v = \alpha e_1 + \beta e_2$.

Then $f_\star^n(v) = f_\star^n(\alpha e_1 + \beta e_2) = \alpha f_\star^n(e_1) + \beta f_\star^n(e_2)$. But e_1 and e_2 are eigenvectors, hence from the above, $f_\star^n(v) = \alpha\lambda_1^n e_1 + \beta\lambda_2^n e_2$.

We know that $\lambda_2 = -\lambda_1$, so the only thing that remains to be proven is that there exists a constant $C > 0$, such that $||\alpha e_1 - \beta e_2|| \leq C||\alpha e_1 + \beta e_2||, \forall\alpha$ and β.

But $||\alpha e_1 + \beta e_2|| = \alpha^2 + \beta^2 + 2\alpha\beta Re < e_1, e_2 >$ and then, since $\exists\eta_0 < 1$ so that $Re < e_1, e_2 >< \eta_0$, it follows that there is a constant $M > 0$, independent on (z,w), such that $||\alpha e_1 + \beta e_2|| \leq M||\alpha e_1 - \beta e_2||$.

This means that for any tangent vector v, $f_\star^n(v) = |\lambda_1^n|(\alpha e_1 \overset{+}{-} \beta e_2)$ and so there is a constant $C > 0$ independent on (z,w) and n, for which the following condition is satisfied:

$$\frac{1}{C} \cdot |\lambda_1^n| \cdot ||v|| \leq ||f_\star^n(v)|| \leq C \cdot |\lambda_1^n \cdot ||v||, \forall n \geq 1.$$

So the inverse iterate f^{-n} is deforming approximately the same in every direction. Therefore we can conclude that for a small ball, $B(z,\epsilon')$, the preimage $f^{-n}(B(z,\epsilon'))$ will look approximately like a ball whose diameter converges to 0 when $n \to \infty$.

This will assure that the preimages $f^{-n}(B(f^n(z_0),\epsilon))$ are still open sets "round" in shape, containing z_0 and that they are not ellipses with excentricities diverging to infinity.

The reason for this precaution is that Lebesgue's Differentiation Theorem does not work with any kind of sequence of open sets contracting towards z_0; we actually need balls or cubes or other kind of objects for which there is some control over their "excentricities"; we thank J. E Fornaess for this observation.

Now, if the open sets $f^{-n}(B(f^n(z_0),\epsilon))$ were proved to be approximately balls whose diameters converge to zero, we can apply Lebesgue Differentiation Theorem. Indeed, since f^{-n} were proved to be uniformly close to being affine on small balls, at least a fixed fraction $\frac{\delta}{2}$ of the areas of the sets $f^{-n}(B(f^n(z_0), \epsilon))$ is in $U \setminus K$.

This means that $z_0 \in K$ is not a point of full area density. By the Lebesgue Differentiation Theorem, the set K will have then zero area.

$\diamond$

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Periodic points and hyperbolicity in higher dimensional complex dynamics Eugen Mihailescu Chair: John E. Fornaess

Higher dimensional complex dynamics has experienced a tremendous growth in the past decade and much of this has been inspired by its connections with other fields. In Chapter II we study the periodic points that can appear for a torus action on a Kobayashi hyperbolic Stein manifold. Here we prove that under certain restrictions, there exist only a finite number of periodic points of all periods. Then we study the fixed points for an S^1 action on a Stein manifold; multidimensional residue theory is used to prove the existence of at most one fixed point in many cases. We also give counterexamples of manifolds with actions having an arbitrary number of fixed points. In Chapter III, we generalize an important Corollary from Chapter II to manifolds having any finite number of generators for their second cohomology group. There are also given several applications to Siegel domains for holomorphic endomorphisms of $\mathbf{P}^2$; a new class of such Siegel domains is constructed.

In Chapter IV we study "the opposite case", hyperbolic maps on $\mathbf{P}^2$, which are proved to have no Siegel domains. We show that the set of ws-hyperbolic maps without cycles is open, which further provides a large class of examples of ws-hyperbolic maps constructed from simpler ones. We also conjecture that the interior of the set K^- of points with "bounded backward iterates", is empty, at least for certain classes of strongly hyperbolic maps on $\mathbf{P}^2$. In the final section we prove that a repellor with empty interior for a special class of functions, will have Lebesgue measure zero.