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UNSTABLE MANIFOLDS AND HÖLDER STRUCTURES ASSOCIATED WITH NONINVERTIBLE MAPS

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Abstract. We study the case of a smooth noninvertible map f with Axiom A, in higher dimension. In this paper, we look first at the unstable dimension (i.e the Hausdorff dimension of the intersection between local unstable manifolds and a basic set Λ), and prove that it is given by the zero of the pressure function of the unstable potential, considered on the natural extension $\hat{\Lambda}$ of the basic set Λ; as a consequence, the unstable dimension is independent of the prehistory $\hat{x}$. Then we take a closer look at the theorem of construction for the local unstable manifolds of a perturbation g of f, and for the conjugacy Φ_g defined on $\hat{\Lambda}$. If the map g is holomorphic, one can prove some special estimates of the Hölder exponent of Φ_g on the liftings of the local unstable manifolds. In this way we obtain a new estimate of the speed of convergence of the unstable dimension of g, when $g \to f$. Afterwards we prove the real analyticity of the unstable dimension when the map f depends on a real analytic parameter. In the end we show that there exist Gibbs measures on the intersections between local unstable manifolds and basic sets, and that they are in fact geometric measures; using this, the unstable dimension turns out to be equal to the upper box dimension. We notice also that in the noninvertible case, the Hausdorff dimension of basic sets does not vary continuously with respect to the perturbation g of f. In the case of noninvertible Axiom A maps on $\mathbb{P}^2$, there can exist an infinite number of local unstable manifolds passing through the same point x of the basic set Λ, thus there is no unstable lamination. Therefore many of the methods used in the case of diffeomorphisms break down and new phenomena and methods of proof must appear. The results in this paper answer to some questions of Urbanski ([21]) about the extension of one dimensional theory of Hausdorff dimension of fractals to the higher dimensional case. They also improve some results and estimates from [7].

1. **Introduction.** In one complex variable, it is known ([17]) that the Hausdorff dimension of the Julia set J of a hyperbolic rational map f is given by the unique zero of the pressure function $t \to P(t\varphi)$, where $\varphi(z) := -\log |Df(z)|, z \in J$.

In this paper we extend this result to the higher dimensional case, i.e that of a conformal map with Axiom A on the complex projective space $\mathbb{P}^2$, thus answering some questions from [21]. Moreover, we give some theorems (like the one about Lipschitz dependence of the unstable manifolds with respect to perturbations, and

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the one about Hölder continuity of conjugacy maps) in the smooth case, without holomorphicity conditions.

Let us introduce some notation. Many of the definitions and results will be given for smooth maps on compact Riemannian manifolds. So, let M be such a manifold and $f : M \to M$ be a $\mathcal{C}^r$ map, with $r \geq 1$. The map f is not necessarily injective. The **nonwandering set** Ω of f is defined as the set of points which come arbitrarily close to their original position, if we iterate f a sufficient number of times, $\Omega := \{y \in M, \forall U$ neighbourhood of $y, \exists n_U \geq 1, s.t f^{n_U}(U) \cap U \neq \emptyset\}$. It is easy to see that Ω is compact. Let us also assume that f has **Axiom A**. For this definition, we send to [3], or [18]. In short, Axiom A says that the periodic points of f are dense in the nonwandering set Ω of f and that we have a splitting of the tangent bundle over the natural extension (defined below) $\hat{\Omega}$ of Ω, in two subbundles, one of which is E^s, representing the contracting directions(lines) for the derivative Df, and the other E^u, representing the expanding directions for Df.

Definition 1. Let (X, d) be a compact metric space, and $f : X \to X$ a continuous map on X. Then the **natural extension** of X with respect to f is the space $\hat{X} := \{\hat{x}, \hat{x} = (x, x_{-1}, x_{-2}, ...),$ where $f(x_{-i}) = x_{-i+1}, i \geq 1\}$. The **shift map** on $\hat{X}$ is $\hat{f} : \hat{X} \to \hat{X}$, defined by $\hat{f}(\hat{x}) = (f(x), x, x_{-1}, ...)$. The **canonical projection map** $\pi : \hat{X} \to X$ is given by $\pi(\hat{x}) = x, \hat{x} \in \hat{X}$; the projection π is sometimes denoted by π_f when we want to emphasize its dependence on f. There exists also a natural metric on $\hat{X}$, for every $K > 1$, given by $d_K(\hat{x}, \hat{y}) := d(x, y) + \frac{d(x_{-1}, y_{-1})}{K} + \frac{d(x_{-2}, y_{-2})}{K^2} + ...$, if $\hat{x} = (x, x_{-1}, x_{-2}, ...)$, and $\hat{y} := (y, y_{-1}, y_{-2}, ...)$ belong to $\hat{X}$. □

Let us notice also that in the definition of hyperbolicity for noninvertible maps, the unstable tangent space $E^u_{\hat{x}}$ depends a priori on the whole prehistory $\hat{x} \in \hat{\Omega}$.

Definition 2. Let X be a nonempty Hausdorff topological space and $f : X \to X$ a continuous map on X. We will say that f is **topologically transitive** (or simply **transitive**) if for any nonempty open sets U, V, there exists $n \in \mathbb{Z}$ with $f^n(U) \cap V \neq \emptyset$.

We will say that f is **topologically mixing** (or simply **mixing**) on X if for any nonempty open sets U, V in X, there exists $N \geq 0$ such that $f^n U \cap V \neq \emptyset$, for any $n \geq N$. □

One can prove that transitivity is equivalent to the existence of a point $x \in X$ with dense full orbit in X, where by full orbit of x we understand $\mathcal{O}(x) := \{f^n(x), n \in \mathbb{Z}\}$. If there exists $x \in X$ with $\mathcal{O}^+(x) := \{f^n(x), n \geq 0\}$ dense in X, we say that f is *topologically + transitive* ([18], [19]); however in the case where we will actually need it, i.e in the Spectral Decomposition Theorem, transitivity and topological + transitivity coincide ([18]). Also, it is immediate to see that mixing implies transitivity.

If f is an Axiom A map as above, the Spectral Decomposition Theorem ([18]) says that the nonwandering set Ω can be partitioned into a finite number of f-invariant subsets, on which f is transitive. These sets are unique up to order and are called **basic sets**. We say that a basic set Λ is of **saddle type** if there are both stable and unstable directions on Λ, i.e if $\dim E^u_{\hat{x}} \geq 1, \hat{x} \in \hat{\Lambda}$ and $\dim E^s_x \geq 1, x \in \Lambda$. In the sequel we will work only with basic sets of this type.

Now, given a smooth $(\mathcal{C}^r, r \geq 2)$ map $f : M \to M$ with Axiom A, and a basic set Λ of saddle type, there exist local stable and unstable manifolds at every point

x of Λ ([18] or [5]),

$$W^s_\beta(f,x) := \{y \in M, d(f^n x, f^n y) < \beta, n \geq 0\}$$

$$W^u_\beta(f,\hat{x}) := \{y \in M, \exists \text{ a prehistory } \hat{y} = (y, y_{-1}, ...), \text{with } d(y_{-i}, x_{-i}) < \beta, i \geq 0\},$$

where $\hat{x} = (x, x_{-1}, ...) \in \hat{\Lambda}$ and β is a sufficiently small positive number.

If the map f is clear from the context, we denote these sets by $W^s_\beta(x), W^u_\beta(\hat{x})$. In case f is a holomorphic map on the complex projective space $\mathbb{P}^2$, then the local stable and unstable manifolds are embedded complex disks ([3]).

It is well-known ([18]) that, if f has Axiom A, then its nonwandering set Ω has **local product structure**, i.e for any $x \in \Omega, \hat{y} \in \hat{\Omega}$, the intersection $W^s_\beta(f,x) \cap W^u_\beta(f,\hat{y})$ has at most one point, denoted by $[x,\hat{y}]$ and this point belongs to Ω.

In the sequel, we will denote by $HD(A)$ the Hausdorff dimension of a set A. Let us define now two important notions which will be used throughout the paper:

Definition 3. In the above setting (hence with Λ a basic set of saddle type), we call the Hausdorff dimension $\delta^s(x,\beta) := HD(W^s_\beta(f,x) \cap \Lambda)$ the **stable dimension** (of size $\beta > 0$) at the point $x \in \Lambda$. Also, the Hausdorff dimension $\delta^u(\hat{x},\beta) := HD(W^u_\beta(f,\hat{x}) \cap \Lambda)$ will be called the **unstable dimension** (of size β) at the prehistory $\hat{x} \in \hat{\Lambda}$. In general, if the size $\beta > 0$ is fixed, we will not record the dependence of the stable/unstable dimension on β, and will write simply $\delta^s(x)$ or $\delta^u(\hat{x})$. □

Notation: In the sequel, we shall denote the derivative in the stable direction, $Df|_{E^s_x}$, by $Df_s(x)$, and the derivative in the unstable direction, $Df|_{E^u_{\hat{x}}}$, by $Df_u(\hat{x})$, for $\hat{x} \in \hat{\Lambda}$.

In the sequel we will also use extensively the notions of **entropy** and **topological pressure** (or simply **pressure**). These notions can be introduced for any continuous map $f : X \to X$ on a compact metric space (X,d); we refer to [5] or [22] for definitions and properties. Denote by $\mathcal{C}(X)$ the space of continuous functions defined on X and with values in $\mathbb{R}$.

Definition 4. For an integer $n > 0$ define the metric d_n on X by $d_n(x,y) := \max\{d(f^i(x), f^i(y), i = 0,..,n-1\}, x, y \in X$. We say that a set $E \subset X$ is (n,ε)-**separated** (for some positive number ε), if for any $x, y \in E, x \neq y$, we have that $d_n(x,y) \geq \varepsilon$. We will say that a subset $F \subset X$ is (n,ε)- **spanning**, if for every $x \in X$ there exists some $y \in F$ such that $d_n(x,y) < \varepsilon$. □

Definition 5. In the setting from the previous Definition, denote by $B_f(x,\varepsilon,n)$, the ball of radius ε and center x in the metric d_n. We will call it the (n,ε)**-ball** centered at x. □

Hence it follows that a subset F is (n,ε)-spanning in X iff the union of the (n,ε)-balls centered at the points of F covers X.

Definition 6. The **topological pressure** of f is a functional $P_f : \mathcal{C}(X) \to \bar{\mathbb{R}}$ defined by :

$$P_f(\varphi) := \lim_{\varepsilon \to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log \sup\{\sum_{x \in E} exp(\sum_{i=0}^{n-1} \varphi(f^i(x))), E \subset X \text{ is } (n,\varepsilon) - \text{separated}\}$$

$$= \lim_{\varepsilon \to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log \inf\{\sum_{x \in F} exp(\sum_{i=0}^{n-1} \varphi(f^i(y))), F \subset X \text{ is } (n,\varepsilon) - \text{spanning}\}$$

When we take $\varphi \equiv 0$, we will obtain the **topological entropy** of f, denoted by $h(f)$ or by $h_{top}(f)$. □

The proof that the above limits exist when $\varepsilon \to 0$, and that they are equal is done for example in [22]. When the map f was fixed and is clear from the context, we may denote the pressure of φ also by $P(\varphi)$.

In ergodic theory, another important notion is that of **measure theoretic entropy**, h_μ (denoted also by $h_\mu(f)$ when we want to emphasize the dependence on f), where μ is an f-invariant Borel probability measure on X. (we will not give the definition of h_μ here, it can be found in all texts on ergodic theory, for example in [22]).

There exists an interesting relationship between Borel invariant measures and P_f, contained in the following:

Theorem (Variational Principle)**.** *In the above setting, $P_f(\varphi) = \sup\limits_{\mu}\{h_\mu(f) + \int \varphi \, d\mu\}$, where the supremum is taken over all f-invariant Borel probability measures μ, and $h_\mu(f) =$ measure-theoretic entropy of μ.*

Definition 7. In the setting from the Variational Principle, let φ be a continuous potential from $\mathcal{C}(X)$, and assume that μ is an f-invariant Borel probability measure on X with $P_f(\varphi) h_\mu + \int \varphi \, d\mu$. Any such measure μ will be called an **equilibrium measure** (or **equilibrium state**) for φ. □

Let us list now several well-known properties of topological pressure, which will be used in the sequel ([19], [5], [22] are good references):

Theorem (Properties of Pressure)**.** *If $f\colon X \to X$ is a continuous transformation, and $\varphi, \psi \in \mathcal{C}(X)$, then:*

1) *$\varphi \le \psi \Rightarrow P_f(\varphi) \le P_f(\psi)$*
2) *$P_f(\cdot)$ is either finitely valued or constantly ∞*
3) *P_f is convex*
4) *For a strictly negative function φ, the mapping $t \to P_f(t\varphi)$ is strictly decreasing if $P(0) < \infty$.*
5) *P_f is a topological conjugacy invariant.*
6) *Assume that f is **expansive** on X, i.e there exists a small positive number ε_0 such that, if $\hat{x} := (x, x_{-1}, ...), \hat{y} := (y, y_{-1}, ...) \in \hat{X}$, (where $\hat{X}$ is the natural extension of X w.r.t f), and $d(x_i, y_i) < \varepsilon_0, \forall i \le 0, d(f^j x, f^j y) < \varepsilon_0, j \ge 0$, then $x = y$. Then there exists a bijection $\mu \to \hat{\mu}$, between f-invariant measures μ on X and $\hat{f}$-invariant measures $\hat{\mu}$ on $\hat{X}$, such that $\pi_*(\hat{\mu}) = \mu$. Under this bijection, we have that $h_\mu = h_{\hat{\mu}}$ and that $P_f(\varphi) = P_{\hat{f}}(\varphi \circ \pi)$. In particular the equilibrium states of φ with respect to f (on X) are obtained as push-forwards of the equilibrium states of $\varphi \circ \pi$ with respect to $\hat{f}$ (on $\hat{X}$), i.e if μ is an f-invariant equilibrium state for $\varphi \in \mathcal{C}(X)$ on X, then there exists a unique $\hat{f}$-invariant Borel probability measure $\hat{\mu}$ on $\hat{X}$, such that $\hat{\mu}$ is an equilibrium state for $\varphi \circ \pi$ and $\pi_*(\hat{\mu}) = \mu$.*
7) *If X can be written as a union of compact subsets, $X = \bigcup\limits_{i \in I} X_i$ and $f(X_i) \subset X_i, i \in I$, then for any $\varphi \in \mathcal{C}(X)$, we have $P_f(\varphi) = \sup\limits_{i \in I} P_{f|_{X_i}}(\varphi|_{X_i})$.*

Now let us say a few words about the special case of Axiom A holomorphic maps on $\mathbb{P}^2$. For most of this, a good reference is [3]. First of all, if f is a holomorphic map, $f : \mathbb{P}^2 \to \mathbb{P}^2$, then there exist homogeneous polynomials $P(z_0, z_1, z_2)$, $Q(z_0, z_1, z_2)$,

$R(z_0, z_1, z_2)$ of the same degree D such that $f[z_0 : z_1 : z_2] = [P(z_0, z_1, z_2) : Q(z_0, z_1, z_2) : R(z_0, z_1, z_2)]$, where $[z_0 : z_1 : z_2]$ represent the homogeneous coordinates on $\mathbb{P}^2$. We will work only with the nontrivial case when $D \geq 2$ (nondegenerate maps). If f is hyperbolic on its basic set Λ, then we know from above that the local stable/unstable manifolds are embedded complex disks, such that $W^s_\beta(f, x)$ is tangent at x to the stable space $E^s_x, x \in \Lambda$ and $W^u_\beta(f, \hat{x})$ is tangent at x to the unstable space corresponding to $\hat{x}$, $E^u_{\hat{x}}, \hat{x} \in \hat{\Lambda}$. Let us notice that the above expression of f using polynomials, implies that f is finite-to-one.

We will also denote by S_0, S_1, S_2 the sets of points in Ω with their unstable index (i.e the dimension of the unstable space) equal to 0, 1, 2 respectively.

We take this opportunity to say that the case of endomorphisms is not just a simple extension of the diffeomorphisms case,and that there appear new phenomena, which can be explained by the non-injectivity of f. Indeed, in [7], I proved that, although $\delta^s(x) \leq t^s_*$, where t^s_* is the unique zero of the pressure function $t \to P(t\phi^s)$ ($\phi^s(y) := \log |Df_s(y)|, y \in \Lambda$), still the equality does not always hold. For example, take the map $f(z, w) := (z^2 + c, w^2 + d), c \neq 0, d \neq 0$. Then, if the complex number c is chosen such that $|1 - \sqrt{1 - 4c}| = \frac{4}{5}$, we obtain $P(2\phi^s) > 0$, therefore $t^s_* > 2$, but on the other hand, $\delta^s(x) \leq 2$.

The stable dimension has been considered in the papers [9], [10]. As said before, the stable dimension cannot be written using the Bowen equation, as the unique zero of the pressure function for the stable potential. Instead we obtained estimates using a new notion, that of inverse pressure.

Another important difference from the case of diffeomorphisms is that the local unstable manifolds do not give a lamination near Λ in the case of noninvertible maps ([11]). In fact, through any point x of Λ there may pass an uncountable collection of unstable manifolds of type $W^u_\beta(f, \hat{x})$. This has the effect that, a priori, the Hausdorff dimension of Λ is not equal to the sum between the stable dimension $\delta^s(x)$ and unstable dimension $\delta^u(\hat{x})$. This gives another difference from the case of diffeomorphisms. Notice also that, although $\hat{f}$ is a homeomorphism on $\hat{X}$, it is not smooth, so many of the properties of diffeomorphisms do not extend to $\hat{f}$ (for example those related to estimates where the derivatives are used).

We proved in [12] that the stable dimension of holomorphic Axiom A maps which are open on the basic set Λ, is in fact independent of both the point $x \in \Lambda$ and of the size β (as long as β is small enough); that proof involved the sequence of inverse pressure functions corresponding to the iterates of f on Λ. We will prove the independence of the unstable dimension, by considering the (usual) topological pressure on the natural extension of Λ.

In the sequel we will encounter also the notion of **Smale space**. We will give its definition and some properties, following [19]. Then we will specify a particular case of Smale space important in our applications.

Definition 8. Let (X, d) be a nonempty compact metric space with metric d, and $f : X \to X$ be a homeomorphism of X. Assume that there are given some numbers $\varepsilon_0 > 0$ and $\lambda \in (0, 1)$, and a continuous map $[\cdot, \cdot] : \{(x, y) \in X \times X, d(x, y) < \varepsilon_0\} \to X$ with the following properties:

a) $[x, x] = x, [[x, y], z] = [x, z], [x, [y, z]] = [x, z]$, whenever the two sides of the last two relations are defined.

b) Let us define $V^-_\delta(x) := \{y, y = [x, y], \text{and } d(x, y) < \delta\}$ and $V^+_\delta(x) := \{y, y = [y, x], \text{and } d(x, y) < \delta\}$, where $\delta < \varepsilon_0$ is small enough.

Then, suppose that $f[x,y] = [fx, fy], x, y \in X$ and that

$$d(f^n z, f^n y) \leq \lambda^n d(z,y), y, z \in V_\delta^-(x), n > 0$$
$$d(f^{-n} y, f^{-n} z) \leq \lambda^n d(y,z), y, z \in V_\delta^+(x), n > 0$$

A compact metric space (X,d) with a homeomorphism $f : X \to X$ for which there exist ε_0, λ with the above properties is called a **Smale space**. The sets $V_\delta^-(x), V_\delta^+(x)$ will be called, respectively, the **local stable set** (of size δ) of $x \in X$, and the **local unstable set** (of size δ) of x; they may also be denoted by $V_\delta^-(f,x)$ and $V_\delta^+(f,x)$ when we want to emphasize their dependence on f. □

If X is a Smale space as above, and $\delta > 0$ is small enough, then it follows easily that

$$V_\delta^-(x) \cap V_\delta^+(y) = [x,y]$$

Also, notice that, by replacing eventually δ with a smaller number, we get :

$$V_\delta^-(x) = \{y, d(f^n x, f^n y) < \delta, n \geq 0\}$$
$$V_\delta^+(x) = \{z, d(f^{-n} z, f^{-n} x) < \delta, n \geq 0\}$$

One can prove easily that, if X is a Smale space for the homeomorphism f, then f is expansive on X, and that the nonwandering set of f is in fact the closure of the set of periodic points. Also, we have Smale's Spectral Decomposition Theorem ([19]), saying that the nonwandering set of X is the union of finitely many disjoint compact subsets Ω_j, which are f-invariant and such that $f|_{\Omega_j}$ is topologically transitive; the sets Ω_j are called *basic sets*. Moreover each basic set Ω_j is the union of k_j disjoint subsets $\Omega_{j\ell}, 1 \leq \ell \leq k_j$, which are cyclically permuted by f and such that $f^{k_j}|_{\Omega_{j\ell}}$ is topologically mixing. Let us also add that any f-invariant measure on X has its support in the nonwandering set and that any f-invariant ergodic measure has its support in one of the basic sets.

Notation: In the sequel we shall denote the space of Hölder continuous real maps of exponent $\alpha > 0$, defined on a compact metric space (X,d), by $\mathcal{H}^\alpha(X)$. So $\mathcal{H}^\alpha(X) := \{\varphi : X \to \mathbb{R}, \exists C > 0, \text{and } \delta > 0, \text{s.t } |\varphi(x) - \varphi(y)| \leq Cd(x,y)^\alpha, \forall x, y \in X, \text{with } d(x,y) < \delta\}$. When the constant $C > 0$ is also fixed, we will denote by $\mathcal{H}_C^\alpha(X)$ the space of real functions on X, satisfying the above inequality with exponent α and multiplicative constant C. □

Another very important feature of Smale spaces is that they have Markov partitions of arbitrarily small diameter ([19]). From this it follows that there exists a symbolic dynamical space modelling the action of f on X, i.e a subshift of finite type Σ_A with a transition matrix A (this subshift is denoted below also by $\tilde{X}$ and is called a configuration space), and a projection $p : \tilde{X} \to X$ such that the following are satisfied:

Theorem (Properties of Smale spaces)**.** *If X is a Smale space for the homeomorphism f, and if $\tilde{X}$ is a configuration space for the symbolic dynamics of X , with the shift map $\tau : \tilde{X} \to \tilde{X}$, then we have the following properties:*

(a) *The canonical projection determined by the Markov partition, $p : \tilde{X} \to X$ is continuous and surjective;*
(b) *$p \circ \tau = f \circ p$;*
(c) *If f is topologically transitive (respectively mixing) on X, then τ is also topologically transitive (respectively mixing) on $\tilde{X}$; hence if f is mixing, $(\tilde{X}, \tau)$ becomes a transitive Markov chain (one says that a shift $\sigma_A : \Sigma_A \to \Sigma_A$ is a*

transitive Markov chain if A is a transitive matrix, i.e if there exists a positive integer m such that the elements of A^m are all strictly positive, [5]*).*

(d) *Assume that f is topologically mixing on X and let a real function $\varphi \in \mathcal{C}(X)$. Then $P_f(\varphi)P_\tau(\varphi \circ p)$. If $\varphi \in \mathcal{H}^\alpha(X)$, then there exists a unique equilibrium state μ_φ for φ on X which is obtained as $\mu_\varphi = p_* \nu_{\varphi \circ p}$, where $\nu_{\varphi\circ p}$ is the unique equilibrium state for $\varphi \circ p$ on $\tilde{X}$;*

(e) *Under the same assumption as in (d) (i.e topological mixing), we have that the pressure functional P_f is real analytic on $\mathcal{H}^\alpha(X)$;*

(f) *Under the same assumption as in (d), if $\varphi \in \mathcal{H}^\alpha(X)$, then supp$\mu_\varphi = X$; also, if $\varphi, \psi \in \mathcal{H}^\alpha(X)$, then their unique equilibrium states μ_φ, μ_ψ are equal if and only if there exist a constant $\chi \in \mathbb{R}$ and a continuous function Φ on X, such that*

$$\psi - \varphi = \Phi \circ f - \Phi + \chi,$$

χ is unique, and Φ is unique up to an additive constant.

Notice that in part (c) of the previous Theorem, the fact that (Σ_A, σ_A) is a transitive Markov chain is not the same as saying that (Σ_A, σ_A) is a Markov chain with topological transitivity; the first property refers to the transitivity of the matrix A and implies in fact that σ_A is topologically mixing on Σ_A. Of great importance in the theory of equilibrium states is also the notion of **specification**. There are several equivalent definitions (see for example [2], [5], [19], etc.).

Definition 9. Consider an arbitrary compact metric space (X, d) and a homeomorphism $f : X \to X$. Then we say that f satisfies **specification** on X if, for every $\varepsilon > 0$, there exists a positive integer $p(\varepsilon)$ such that the following condition holds:

if $I_1 := [a_1, b_1], ..., I_n := [a_n, b_n]$ are disjoint finite intervals in $\mathbb{Z}$, contained in a larger interval $[a, b]$, with $b_i + p(\varepsilon) < a_{i+1}, i = 1, .., n-1$, and $x_1, ..., x_n$ are arbitrary points in X, then there exists a periodic point $x \in X$ satisfying:

$$f^{b-a+p(\varepsilon)}(x) = x, \text{and}$$
$$d(f^\ell x, f^\ell x_i) < \varepsilon, \ell \in I_i, i = 1, .., n.$$

□

Remark 1: a) It is relatively easy to prove that any transitive Markov chain has the specification property. Also, if a homeomorphism of a general compact metric space X has the specification property, then it is topologically mixing.

b) Bowen ([2]) showed that, if f is an expansive homeomorphism on X satisfying the specification property, then for any Hölder continuous potential $\varphi \in \mathcal{H}^\alpha(X)$ there exists a unique equilibrium state. □

Then using the symbolic dynamics representation $(\tilde{X}, \tau)$ associated to a Smale space (X, f) (from the above Theorem on Properties of Smale Spaces), together with the above remark, we obtain the following:

Theorem (Equilibrium states on Smale spaces)**.** *Let X be a Smale space for the homeomorphism f, such that f is topologically mixing. Then for any $\alpha > 0$ and any potential $\varphi \in \mathcal{H}^\alpha(X)$, there exists a unique equilibrium state μ_φ, of φ on X.*

Next, we shall give an example of Smale space which will have important applications in the sequel. Consider as above, a smooth (i.e $\mathcal{C}^r, r \geq 2$) map $f : M \to M$, on a compact Riemannian manifold M. Let us assume that f satisfies Axiom A. The following appears in [18], and we cite it for future reference:

Theorem (Spectral Decomposition Theorem). *In the above setting, the nonwandering set Ω of f can be decomposed as the union of finitely many f-invariant disjoint closed sets Ω_j, on which f is topologically transitive; these sets are unique up to order and are called basic sets of f. Moreover, each set Ω_j can be decomposed as a disjoint union of subsets $\Omega_{j,k}, k = 1, .., n_j$, such that $f(\Omega_{j,k}) = \Omega_{j,k+1}, k = 1, .., n_j$, $\Omega_{j,n_j+1} = \Omega_{j,1}$, and f^{n_j} is topologically mixing on each subset $\Omega_{j,k}$.*

In the sequel, as announced before, we will work with $f : M \to M$ smooth map, satisfying Axiom A, and with a basic set Λ of saddle type. One can form the natural extension $\hat\Lambda := \{\hat x = (x, x_{-1}, x_{-2}, ...), f(x_{-i-1}) = x_{-i}, x_{-i} \in \Lambda, i \geq 0, x_0 = x\}$. As said before, for each number $K > 1$, there exists a metric d_K on $\hat\Lambda$, compatible with the topology induced from the product space.

In this setting, for each $\hat x \in \hat\Lambda$ and each $\delta > 0$ small, let us define the following subsets of $\hat\Lambda$:

$$V_\delta^-(\hat x) := \{\hat y \in \hat\Lambda, d_K(\hat f^n \hat x, \hat f^n \hat y) < \delta, n \geq 0\}, \text{and}$$
$$V_\delta^+(\hat x) := \{\hat y \in \hat\Lambda, d_K(\hat f^{-n} \hat x, \hat f^{-n} \hat y) < \delta, n \geq 0\}$$

Using the fact that $\hat\Lambda$ has local product structure ([18]) we can define a map $[\cdot,\cdot]$ as in the definition of Smale spaces, by putting $[\hat x, \hat y] = V_\delta^-(\hat x) \cap V_\delta^+(\hat y)$, for $d_K(\hat x, \hat y) < \delta/2$. One can check easily that the conditions in the definition of Smale spaces are satisfied, and hence $\hat\Lambda$ is a Smale space with the above relation $[\cdot,\cdot]$ and the homeomorphism $\hat f : \hat\Lambda \to \hat\Lambda$.

We know from the Spectral Decomposition Theorem that f is transitive on Λ and this implies that $\hat f$ is also transitive on $\hat\Lambda$ ([19], pg. 145). Also, we know that $\hat f$ is expansive on $\hat\Lambda$. Therefore the properties of Smale spaces from the Theorem above apply to the natural extension $\hat\Lambda$ and the homeomorphism $\hat f$. In particular there exists a symbolic representation of $\hat\Lambda$, denoted by $\tilde\Lambda$ and a projection map $p : \tilde\Lambda \to \hat\Lambda$; let us recall also the canonical projection $\pi : \hat\Lambda \to \Lambda$. Also from [19], it follows that, if $\varphi \in \mathcal{C}(\Lambda)$ then $P_f(\varphi) = P_{\hat f}(\varphi \circ \pi)$ and that π induces a bijection $\hat\mu \to \mu$, between the $\hat f$-invariant states on $\hat\Lambda$ and the f-invariant states on Λ such that $\pi_* \hat\mu = \mu$ and $h_\mu = h_{\hat\mu}$.

We study now the problem of uniqueness of equilibrium states and that of coincidence between equilibrium states and Gibbs states. First of all, a definition:

Definition 10. Consider a continuous map $f : X \to X$, where X is a compact metric space and let $\varphi \in \mathcal{C}(X)$. Then we will call a probability measure ν on X, a **Gibbs state** (or a **Gibbs measure**) for φ if and only if for each $\varepsilon > 0$, there exist $A_\varepsilon, B_\varepsilon > 0$ such that for all $y \in X$ and integer $n > 0$, we have:

$$A_\varepsilon e^{S_n\varphi(y) - nP(\varphi)} \leq \nu(B_f(y, \varepsilon, n)) \leq B_\varepsilon e^{S_n\varphi(y) - nP(\varphi)},$$

where $S_n\varphi(y) := \varphi(y) + ... + \varphi(f^{n-1}y), y \in X$ and $B_f(y, r, n)$ is the (n, r)-ball centered at y $(r > 0)$, from Definition 5. □

Regarding the construction of Gibbs states and their relation to the equilibrium states, we have the following important Theorem ([1], [5]):

Theorem (Bowen's Theorem on Construction of Gibbs/Equilibrium States). *Let (X, d) be a compact metric space and $f : X \to X$ an expansive homeomorphism satisfying the specification property. Let also $\varphi \in \mathcal{H}^\alpha(X)$, for some arbitrary $\alpha > 0$. Then there is exactly one equilibrium state μ_φ for φ on X. The measure μ_φ is a Gibbs measure for φ and it is ergodic.*

If we denote by $Fix(f^n)$ the periodic points of period n of f on X, and by $P(f,\varphi,n) := \sum_{x \in Fix(f^n)} e^{S_n\varphi(x)}$, then the measure μ_φ is obtained as:

$$\mu_\varphi = \lim_{n\to\infty} \frac{1}{P(f,\varphi,n)} \sum_{x\in Fix(f^n)} e^{S_n\varphi(x)}\delta_x,$$

where, as usual, δ_x denotes the Dirac measure at the point x.

Let us see how this may be applied to the case of Smale spaces associated to basic sets of saddle type. We have a smooth map $f : M \to M$ on a compact Riemannian manifold M satisfying Axiom A, and let Λ be a basic set of saddle type; we have also $\hat{\Lambda}$ the natural extension of Λ relative to f with the homeomorphism $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$. $\hat{\Lambda}$ has a natural structure of a Smale space, and since f is transitive on Λ, it follows that $\hat{f}$ is transitive on $\hat{\Lambda}$. Denote by $\tilde{\Lambda}$ the symbolic representation of the Smale space $\hat{\Lambda}$; we know from the Theorem on Properties of Smale Spaces that $\tilde{\Lambda}$ is also transitive, however a transitive map on a compact space does not always have the specification property. Therefore we have to decompose Λ into its mixing components, given by the Spectral Decomposition Theorem. So, we know that there exist disjoint closed subsets of Λ, denoted by $\Lambda_1, .., \Lambda_N$ such that f permutes Λ_j among themselves and f^N is topologically mixing on each Λ_j. This means that $\hat{f}^N$ is mixing on $\pi^{-1}\Lambda_j$, where we recall that $\pi : \hat{\Lambda} \to \Lambda$ is the canonical projection, $\pi(\hat{x}) = x, \hat{x} \in \hat{\Lambda}$. This implies that τ^N is mixing on $p^{-1}\pi^{-1}\Lambda_j$ and hence it is easy to prove that it has specification on that set ([1] or [5], pg. 581). But, from the definition of specification and the properties of the projection $p : \tilde{\Lambda} \to \hat{\Lambda}$, we get that also $\hat{f}^N$ has specification on $\pi^{-1}\Lambda_j$. So, the Smale space $\hat{\Lambda}$ can be written as the union of finitely many disjoint compact subsets, $\hat{\Lambda} = X_1 \cup ... \cup X_N$, $X_j = \pi^{-1}(\Lambda_j)$ such that $\hat{f}(X_j) = X_{j+1}, X_{N+1} = X_1$, and $\hat{f}^N$ satisfies specification on each X_j. We want to prove that Hölder continuous functions on $\hat{\Lambda}$ have unique equilibrium states which are also Gibbs states; the proof is based on the discussion for the mixing case.

Theorem 1. *As above, consider a smooth Axiom A map f on a compact manifold M and Λ a basic set of saddle type for f. Then, for each Hölder continuous real function $\hat{\varphi}$ on $\hat{\Lambda}$, there exists a unique equilibrium measure $\hat{\mu}_{\hat{\varphi}}$ which is also a Gibbs state for $\hat{\varphi}$.*

Proof. First, consider the topologically mixing map $\hat{f}^N$ on X_j, for some fixed $j, 1 \le j \le N$. (where we use the notations introduced before the statement of the theorem). We know that $\hat{f}^N$ is expansive on X_j and it satisfies specification, from the previous discussion. Hence we can apply Bowen's Theorem on Construction of Equilibrium/Gibbs States given before. Thus for each Hölder continuous function $\hat{\varphi}$ on $\hat{\Lambda}$, there exists an $\hat{f}^N$-invariant measure $\hat{\mu}_j$ on X_j, which is the unique equilibrium measure for $\tilde{\varphi}|_{X_j}$, with $\tilde{\varphi} := \hat{\varphi} + \hat{\varphi}\circ\hat{f} + ... + \hat{\varphi}\circ\hat{f}^{N-1}$. Define now

$$\hat{\mu}(E) := \frac{1}{N}\sum_{k=0}^{N-1} \hat{\mu}_1(X_1 \cap \hat{f}^k E),$$

for an arbitrary Borelian set $E \subset \hat{\Lambda}$. It is easy to check that $\hat{\mu}$ is an $\hat{f}$-invariant probability measure on $\hat{\Lambda}$, with $h_{\hat{\mu}_1}(\hat{f}^N) = N h_{\hat{\mu}}(\hat{f})$, and that $\int \tilde{\varphi} d\hat{\mu}_1 = N \int \hat{\varphi} d\hat{\mu}$. Therefore, if $\hat{\mu}_1$ is the unique equilibrium state for $\tilde{\varphi}$ on X_1, with respect to $\hat{f}^N$, then the measure defined above, $\hat{\mu}$, will be the unique equilibrium state for $\hat{\varphi}$ on $\hat{\Lambda}$, with respect to $\hat{f}$.

We look now at the inequalities in the definition of Gibbs states. Firstly, the map $\hat{f} : X_i \to X_{i+1}$ gives a conjugation between the maps $\hat{f}^N : X_i \to X_i$ and $\hat{f}^N : X_{i+1} \to X_{i+1}$; hence from the Theorem on Properties of Pressure given above, $P_{\hat{f}^N|_{X_i}}(\tilde{\varphi}|_{X_i}) = P_{\hat{f}^N|_{X_{i+1}}}(\tilde{\varphi}|_{X_{i+1}}), i = 1, ..N$. But from the same Theorem, $P_{\hat{f}^N|_{\hat{\Lambda}}}(\tilde{\varphi}) = \sup_{1\le i\le N} P_{\hat{f}^N|_{X_i}}(\tilde{\varphi}|_{X_i})$, since $\hat{\Lambda} = \bigcup_{1\le i\le N} X_i$ and each X_i is invariated by $\hat{f}^N$. On the other hand, $P_{\hat{f}^N|_{\hat{\Lambda}}}(\tilde{\varphi}) = NP_{\hat{f}}(\hat{\phi})$, so for each i, we have

$$P_{\hat{f}^N|_{X_i}}(\tilde{\varphi}) = NP_{\hat{f}}(\hat{\varphi}) \tag{1}$$

Let us also notice that for any positive integer n,

$$\begin{aligned} S_n\tilde{\varphi}(\hat{y}; \hat{f}^N) : &= \tilde{\varphi}(\hat{y}) + \tilde{\varphi} \circ \hat{f}^N(\hat{y}) + ... + \tilde{\varphi} \circ \hat{f}^{N(n-1)}(\hat{y}) \\ &= \hat{\varphi}(\hat{y}) + .. + \hat{\varphi} \circ \hat{f}^{N-1}(\hat{y}) + \hat{\varphi} \circ \hat{f}^N(\hat{y}) \\ &\quad + \hat{\varphi} \circ \hat{f}^{N+1}(\hat{y}) + .. + \hat{\varphi} \circ \hat{f}^{Nn-1}(\hat{y}) \\ &= S_{nN}(\hat{\varphi}(\hat{y})) \end{aligned} \tag{2}$$

Denote by $\nu := \hat{\mu}_1$ the equilibrium measure for $\tilde{\varphi}|_{X_1}$on X_1, with respect to $\hat{f}^N$. Let us take now $E := B_{\hat{f}}(\hat{y}, \varepsilon, n)$, for some $\hat{y} \in \hat{\Lambda}$ and estimate $\hat{\mu}(E)$. Without loss of generality, we can assume that $\hat{y} \in X_1$. From the definition of $\hat{\mu}$, we have $\hat{\mu}(E) = \frac{1}{N}(\nu(X_1 \cap E) + ... + \nu(X_1 \cap \hat{f}^{N-1}E))$. But $X_1 \cap B_{\hat{f}}(\hat{y}, \varepsilon, n) \subset B_{\hat{f}^N|_{X_1}}(\hat{y}, \varepsilon, [\frac{n}{N}])$ (if ω is a rational number, $[\omega]$ denotes its integer part). Also, from the fact that f is smooth on M, there exists $\varepsilon' = \varepsilon'(\varepsilon) < \varepsilon$ such that $B_{\hat{f}^N|_{X_1}}(\hat{y}, \varepsilon', [\frac{n}{N}]) \subset B_{\hat{f}}(\hat{y}, \varepsilon, n)$. Let us use now the fact that ν is a Gibbs state for $\tilde{\varphi}|_{X_1}$:

$$A_{\varepsilon'} e^{S_{[\frac{n}{N}]}\tilde{\varphi}(\hat{y};\hat{f}^N) - [\frac{n}{N}]P_{\hat{f}^N|_{X_1}}(\tilde{\varphi}|_{X_1})} \le \nu(X_1 \cap E) \le B_\varepsilon e^{S_{[\frac{n}{N}]}\tilde{\varphi}(\hat{y};\hat{f}^N) - [\frac{n}{N}]P_{\hat{f}^N|_{X_1}}(\tilde{\varphi}|_{X_1})}$$

Using now relations (1) and (2) and the above inequality, we obtain that there exist constants $C_\varepsilon, D_\varepsilon > 0$ such that for all $\hat{y} \in X_1, n > 0$,

$$C_\varepsilon e^{S_n\hat{\varphi}(\hat{y}) - nP_{\hat{f}}(\hat{\varphi})} \le \nu(X_1 \cap E) \le D_\varepsilon e^{S_n\hat{\varphi}(\hat{y}) - nP_{\hat{f}}(\hat{\varphi})}$$

Repeating the above argument for $\nu(X_1 \cap \hat{f}(E)), ..., \nu(X_1 \cap \hat{f}^{N-1}(E))$, and then using the above definition of $\hat{\mu}$, we obtain that $\hat{\mu}$ is indeed a Gibbs measure for $\hat{\varphi}$ on the entire Smale space $\hat{\Lambda}$. This measure can be denoted by $\hat{\mu}_{\hat{\varphi}}$. □

Corollary 1. *In the above setting, let Λ be a basic set of saddle type for the Axiom A map $f : M \to M$. Then given any local unstable manifold $W^u_\beta(\hat{x}), \hat{x} \in \hat{\Lambda}$, and any Hölder continuous function $\varphi \in \mathcal{C}(\Lambda)$, there exists a measure $\mu = \mu(\varphi, \hat{x}, \beta)$ on $W^u_\beta(\hat{x}) \cap \Lambda$ such that for every $\varepsilon > 0$ there are positive constants $A_\varepsilon, B_\varepsilon$ so that for every $y \in W^u_\beta(\hat{x}) \cap \Lambda$ and every positive integer n, we have*

$$A_\varepsilon e^{S_n\varphi(y) - nP_f(\varphi)} \le \mu(B_f(y, \varepsilon, n) \cap W^u_\beta(\hat{x}) \cap \Lambda) \le B_\varepsilon e^{S_n\varphi(y) - nP_f(\varphi)}$$

Proof. We use the lifting to the natural extension $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$, and the previous Theorem in order to find a Gibbs measure $\hat{\mu} := \hat{\mu}_{\hat{\varphi}}$ for the Hölder continuous potential $\hat{\varphi} : \varphi \circ \pi$ on $\hat{\Lambda}$. (π is Lipschitz, hence if $\varphi \in \mathcal{H}^\alpha(\Lambda)$, for some $\alpha > 0$, it follows also that $\varphi \circ \pi \in \mathcal{H}^\alpha(\hat{\Lambda})$). We can use now the local product structure on $\hat{\Lambda}$ so any neighbourhood $B(\hat{x}, \delta)$ of a given point $\hat{x} \in \hat{\Lambda}$ can be laminated by local stable sets $V^-_\delta(\hat{y}), \hat{y} \in V^+_\delta(\hat{x})$. Hence, if we have a measure $\hat{\mu}$ on $\hat{\Lambda}$, there exists a measure $\hat{\mu}_{\hat{x},\delta}$ defined on $V_\delta^+(\hat{x})$ ($\delta > 0$ small enough), given as $\hat{\mu}_{\hat{x},\delta} = H_*(\hat{\mu})$, where $H : B(\hat{x},\delta) \to V_\delta^+(\hat{x}), H(\hat{z}) : V_\delta^-(\hat{z}) \cap V_\delta^+(\hat{x})$. One can notice that the measure $\hat{\mu}_{\hat{x},\delta}$ does not depend on δ actually, so it can be denoted also by $\hat{\mu}_{\hat{x}}$. Since $\hat{f}$ contracts distances on the leaves of V^-, we see that $\hat{\mu}_{\hat{x}}(B_{\hat{f}|_{V_\delta^+(\hat{x})}}(\hat{y},\varepsilon,n)) = \hat{\mu}(H^{-1}(B_{\hat{f}|_{V_\delta^+(\hat{x})}}(\hat{y},\varepsilon,n))) = \hat{\mu}(B_{\hat{f}}(\hat{y},\varepsilon,n)$, when $\hat{y} \in V_\delta^+(\hat{x})$. Therefore we obtain that there exist constants $A'_\varepsilon, B'_\varepsilon > 0$ such that

$$A'_\varepsilon e^{S_n\varphi(y)-nP(\varphi)} \le \hat{\mu}(B_{\hat{f}|_{V_\delta^+(\hat{x})}}(\hat{y},\varepsilon,n)) \le B'_\varepsilon e^{S_n\varphi(y)-nP(\varphi)},$$

for all $\hat{y} \in V_\delta^+(\hat{x}), n > 0$. But recall that we have a bi-Lipschitz map $L : V_\delta^+(\hat{x}) \to W_\delta^u(\hat{x}) \cap \Lambda$, given by $L(\hat{x}) = x$ (L is just the canonical projection restricted to $V_\delta^+(\hat{x})$). Indeed, if the constant K (used in the definition of the metric d_K on $\hat{\Lambda}$) is larger than 2, then $W^u_{\delta/2}(\hat{x}) \cap \Lambda \subset L(V_\delta^+(\hat{x}))$, and $d(y,z) \le d_K(\hat{y},\hat{z}) \le 2d(y,z), y,z \in W^u_{\delta/2}(\hat{x}) \cap \Lambda$ and where $\hat{y},\hat{z}$ are the unique prehistories of y,z $\delta/2$-shadowed by $\hat{x}$. (δ is small enough). Therefore L induces a measure $\mu = \mu(\hat{x},\beta)$ on $W^u_\beta(\hat{x}) \cap \Lambda$ such that for $\varepsilon < \beta < \delta$, $\hat{\mu}(B_{\hat{f}|_{V^+_{\beta/2}(\hat{x})}}(\hat{y},\varepsilon/2,n)) \le \mu(B_f(y,\varepsilon,n) \cap W^u_\beta(\hat{x}) \cap \Lambda) \le \hat{\mu}(B_{\hat{f}|_{V_\beta^+(\hat{x})}}(\hat{y},2\varepsilon,n))$. This implies that, for $\varepsilon < \beta$, there exist constants $A_\varepsilon, B_\varepsilon > 0$ satisfying

$$A_\varepsilon e^{S_n\varphi(y)-nP_f(\varphi)} \le \mu(B_f(y,\varepsilon,n) \cap W^u_\beta(\hat{x}) \cap \Lambda) \le B_\varepsilon e^{S_n\varphi(y)-nP_f(\varphi)},$$

for each $n > 0$ and $y \in W^u_\beta(\hat{x}) \cap \Lambda$. $\square$

We will consider in the next sections the intersection $W^u_\beta(\hat{x}) \cap \Lambda$ and prove that its Hausdorff dimension is given by the unique zero of the pressure function $t \to P_{\hat{f}}(t\phi^u)$, where $\phi^u(\hat{y}) := -\log|Df_u(\hat{y})|, \hat{y} \in \hat{\Lambda}$; in particular the unstable dimension is independent of $\hat{x}$ and $\beta > 0$ small.

We will show that, if g is an Axiom A perturbation of f, then the conjugating map Φ_g is α-Hölder continuous as a map from $V_\beta^+(\hat{x})$ to $W^u_\beta(\hat{x}) \cap \Lambda$. For holomorphic maps (for instance when $f : \mathbb{P}^2 \to \mathbb{P}^2$ is holomorphic and with Axiom A) we are able to give a new and precise estimate of the Hölder exponent α. In the proof of this theorem we will actually construct the unstable manifolds in the non-invertible case and using this construction, will prove the existence of a Lipschitz family of biholomorphic maps between the unstable manifolds of f and those of g. Moreover, this theorem will imply that, if $(f_a)_a$ is a family of holomorphic maps on $\mathbb{P}^2$ depending real analytically on a, then the unstable dimension varies also real analytically in a. Some estimates in the proofs will depend also on the constant K used to define the metric d_K on $\hat{\Lambda}$.

We will also notice in the sequel that the stable dimension does not depend continuously on the parameter of f_a; in particular the Hausdorff dimension of the basic sets does not vary always continuously in the case of endomorphisms. Therefore in the noninvertible case new phenomena can appear.

Lastly, using Corollary 1 we will look at the Gibbs states on the intersections of local unstable manifolds and Λ. The Gibbs state of the unstable potential will prove to be a geometric measure and we will use this fact (and a Laminated Distortion Lemma on unstable manifolds) to show that the unstable dimension is equal to the upper (and lower) box dimension.

The results above present an extension of the theory from the case of diffeomorphisms, although in this noninvertible case the proofs are different and one has to take into consideration prehistories, rather than points, and to introduce new methods in order to deal with the lack of differentiability on the natural extension.

2. **Unstable dimension is given by a Bowen type equation.** Let us take $f : M \to M$ a smooth Axiom A map on a compact Riemannian manifold M, with the (real) dimension of the unstable spaces over a basic set Λ equal to 2, and f conformal on its local unstable manifolds. Assume also that the entropy of $f|_\Lambda$ is not zero. In particular f can be a holomorphic function on the complex projective space $\mathbb{P}^2$, satisfying Axiom A.

Define the real function (called *unstable potential*) $\phi^u(\hat{y}) := -\log |Df_u(\hat{y})|$. Due to the expansion of derivative on unstable spaces, we see that ϕ^u is strictly negative on $\hat{\Lambda}$. Let also the pressure function $t \to P_{\hat{f}}(t\phi^u)$, which is continuous and strictly decreasing (from the Theorem on Properties of Pressure given in Section 1). Since $P(0) = h(f|_\Lambda) > 0$, and $P(t\phi^u) < 0$ when t is large enough, it follows that it has a unique zero t^u.

Now, the distances between iterates of points from unstable manifolds, grow exponentially. Using also the conformality of f along unstable manifolds we will obtain a Laminated Distortion Lemma, similarly to the case of local stable manifolds ([7]):

Lemma 1. *In the above setting, with f conformal on its local unstable manifolds, there exists $\beta > 0$ and $C > 0$ such that: if $\hat{x} \in \hat{\Lambda}$ and $y \in W^u_\beta(\hat{x})$, and $n > 0$ is such that $f^k(y) \in W^u_\beta(\hat{f}^k\hat{x}), 1 \le k \le n$, then*

$$e^{-C} \le \frac{|Df^n_u(\hat{y})|}{|Df^n_u(\hat{x})|} \le e^C,$$

where $\hat{y}$ is the unique prehistory of y, β-shadowed by $\hat{x}$.

We will give next a very useful Hölder continuity theorem for the unstable spaces and consequently for the unstable potential ϕ^u. For this theorem we do not need the hypothesis of conformality on unstable manifolds.

Theorem 2. *(a) Consider a smooth map $f : M \to M$ which satisfies Axiom A and let Λ be one of its basic sets of saddle type. Assume also that on the natural extension of Λ, $\hat{\Lambda}$, we take the metric d_K, for some $K > 1$. Then the unstable tangent spaces over $\hat{\Lambda}$ depend Hölder continuously on their prehistories, i.e the tangent bundle over $\hat{\Lambda}$ can be embedded in a trivial bundle $\hat{\Lambda} \times \mathbb{R}^{m_1+m_2}$, such that, if $\theta > 0$ is a number satisfying*

$$\sup_{\hat{x} \in \hat{\Lambda}} |Df_s(x)| \cdot |Df_u(\hat{x})|^{-1} \cdot K^\theta < 1,$$

then the map $E : \hat{\Lambda} \to G_q(\hat{\Lambda}), E(\hat{x}) = E^u_{\hat{x}}$ is θ-Hölder continuous (where we take on the Grassmannian $G_q(\hat{\Lambda})$ of q-dimensional subspaces of the tangent bundle over $\hat{\Lambda}$, the metric induced by the embedding of $T_{\hat{\Lambda}}$ in $\hat{\Lambda} \times \mathbb{R}^{m_1+m_2}$).

(b) If $K > 1$ and $\theta = \theta(K)$ satisfies the inequality from (a) then $\phi^u \in \mathcal{H}^\theta(\hat{\Lambda})$.

Proof. (a) The proof is very similar with that of Theorem 2 of [7] and we do not repeat it here. The condition on derivatives is the same here as in the case of unstable manifolds studied in [7].

(b) This follows from (a) and the differentiability of log and of f on M. □

We give now the main theorem of this section, about the equality between the unstable dimension and the zero of the pressure function associated to the unstable potential :

Theorem 3. *Let as above a smooth map $f : M \to M$ with Axiom A and conformal on its local unstable manifolds. Consider also Λ a basic set of saddle type and a small positive number β such that all unstable manifolds $W^u_\beta(\hat{z}), \hat{z} \in \hat{\Lambda}$ are defined. Then the unstable dimension $\delta^u(\hat{x};\beta)$ is equal to the unique zero t^u of the pressure function $t \to P_{\hat{f}}(t\phi^u)$. In particular the unstable dimension does not depend on $\hat{x}$ and β.*

Proof. Let us take $\beta > 0$ small enough such that all the local unstable manifolds of size β are defined and given as embedded smooth (poly)-disks in M. So with this fixed β we will denote $\delta^u(\hat{x};\beta)$ by $\delta^u(\hat{x})$. Let us show first that $\delta^u(\hat{x}) \leq t^u$. Denote $W^u_\delta(\hat{x}) \cap \Lambda$ by W, and take $t > t^u$ arbitrary. Then $P_{\hat{f}}(t\phi^u) < \gamma < 0$, for some negative γ. But, according to Theorem 9.8 of [22], for any $\psi \in \mathcal{C}(\hat{\Lambda}), P_{\hat{f}}(\psi) = P_{\hat{f}^{-1}}(\psi)$ ($\hat{f}$ is a homeomorphism of $\hat{\Lambda}$). For a continuous function $\psi \in \mathcal{C}(\hat{\Lambda})$, and $\varepsilon > 0$ small enough, let us denote by

$$P_n(\psi,\varepsilon) := \inf\{\sum_{\hat{y}\in F} e^{S_n\psi(\hat{y})}, F \text{ is an } (n,\varepsilon)-\text{spanning set in } \hat{\Lambda}, \text{ relative to } \hat{f}^{-1}\}$$

Then we have $P_{\hat{f}^{-1}}(t\phi^u) = \lim_{\varepsilon\to 0} \frac{1}{n}\log P_n(t\phi^u,\varepsilon) < \gamma < 0$.

Hence there exists an (n,ε)-spanning set F for $\hat{f}^{-1}$, of minimal cardinality and satisfying

$$\sum_{\hat{z}\in F} e^{S_n(t\phi^u)(\hat{z})} < e^{n\gamma} \tag{3}$$

Now, if $y \in W \subset W^u_\beta(\hat{x})$, there exists a unique prehistory of y, denoted by $\hat{y}_*$, which is β-shadowed by $\hat{x}$ and such that $\hat{y} \in \hat{\Lambda}$.

Assume that $F = \{\hat{z}^1, ..., \hat{z}^\ell\}$; then $\pi(F) = \{z^1, ..., z^\ell\}$. Consider the set $f^{n-1}(W)$ which is a subset of Λ. Hence for each $y \in W$, there must exist an i, $1 \leq i \leq \ell$, such that $\hat{f}^{n-1}\hat{y}_* \in B_{\hat{f}^{-1}}(\hat{z}^i,\varepsilon,n)$. So, this means that $d_K(\hat{f}^{n-1}\hat{y}_*, \hat{z}^i) < \varepsilon, ..., d_K(\hat{y}_*, \hat{f}^{-n+1}\hat{z}^i) < \varepsilon$. Let us take now $V_i := W \cap \pi\hat{f}^{-n+1}B_{\hat{f}^{-1}}(\hat{z}^i,\varepsilon,n), i = 1..\ell$. Since the diameter of $f^k(V_i)$ is bounded by 2ε, for all $1 \leq k \leq n-1$, and using also the Laminated Distortion, Lemma (1) we see that there exists a constant $C > 0$ with:

$$\text{diam} V_i \leq C\varepsilon |Df^{n-1}_u(\hat{y}_*)|^{-1}, 1 \leq i \leq \ell \tag{4}$$

But in (3) we have only unstable derivatives along the prehistories from F, so we have to find relations between these and the prehistories $\hat{y}_*$ appearing in (4). This will be done using Theorem 2. Indeed let us consider for each i as above, the point $\zeta^i := W \cap W^s_\varepsilon(z^i_{-n+1})$, where we denoted $\hat{z}^i := (z^i, z^i_{-1}, ...) \in \hat{\Lambda}$. Then we have

$$|\phi^u(\hat{f}^k\hat{y}_*) - \phi^u(\hat{f}^{-n+1+k}\hat{z}^i)| \leq |\phi^u(\hat{f}^k\hat{y}_*) - \phi^u(\hat{f}^k\hat{\zeta}^i_*)| + |\phi^u(\hat{f}^k\hat{\zeta}^i_*) - \phi^u(\hat{f}^{-n+1+k}\hat{z}^i)|,$$

for $1 \leq k \leq n-1$.

For the second term of the above inequality, we will use that $\zeta^i \in W^s_\varepsilon(z^i_{-n+1})$, so there exists some $\lambda \in (0,1)$ such that $d(f^k z^i_{-n+1}, f^k\zeta^i) < \lambda^k$. Therefore, using Theorem 2 and the conformality of f on unstable manifolds, we obtain:

$$|\phi^u(\hat{f}^k\hat{\zeta}^i_*) - \phi^u(\hat{f}^{-n+1+k}\hat{z}^i)| \leq C_2 d_K(\hat{f}^k\hat{\zeta}^i_*, \hat{f}^{-n+1+k}\hat{z}^i),$$

for some positive constant C_2 independent of k. But $d_K(\hat{f}^k\zeta_*^i, \hat{f}^{-n+1+k}\hat{z}^i) = d(f^k\zeta^i, z^i_{-n+1+k}) + \frac{d(f^{k-1}\zeta^i, z^i_{-n+2+k})}{K} + ... + \frac{d_K(\hat{\zeta}^i_*, \hat{f}^{-n+1}\hat{z}^i)}{K^k} \le \varepsilon\lambda^k + \varepsilon\frac{\lambda^{k-1}}{K} ... + \frac{c}{K^k} \le (\lambda')^k$, where $\lambda' \in (0,1)$, and $1 \le k \le n-1$. Hence from the previous displayed inequality we get

$$|\phi^u(\hat{f}^k\hat{\zeta}^i_*) - \phi^u(\hat{f}^{-n+1+k}\hat{z}^i)| \le C_2(\lambda')^k$$

This implies that

$$\begin{aligned}
&|\log|Df_u^{n-1}(\hat{y}_*)| - \log|Df_u^{n-1}(\hat{f}^{-n+1}\hat{z}^i)| \\
&\qquad \le |\log|Df_u^{n-1}(\hat{y}_*) - \log|Df_u^{n-1}(\hat{\zeta}^i_*)|| \\
&\qquad\qquad + |\log|Df_u^{n-1}(\hat{\zeta}^i_*)| - \log|Df_u^{n-1}(\hat{f}^{-n+1}\hat{z}^i)|| \\
&\qquad \le C_1 + \sum_{1\le k\le n-1} |\phi^u(\hat{f}^k\hat{\zeta}^i_*) - \phi^u(\hat{f}^{-n+k+1}\hat{z}^i)| \le C_1 + C_2
\end{aligned}$$

We used above the inequality $|\log|Df_u^{n-1}(\hat{y}_*) - \log|Df_u^{n-1}(\hat{\zeta}^i_*)|| \le C_1$, (for a positive constant C_1 independent of n), which follows from the Lemma 1.

So we obtained that there exists a constant $\tilde{C} > 0$, such that:

$$\frac{1}{\tilde{C}} \le \frac{|Df_u^{n-1}(\hat{y}_*)|}{|Df_u^{n-1}(\hat{f}^{-n+1}\hat{z}^i)|} \le \tilde{C} \tag{5}$$

But then , from (3) and (4) it follows that

$$\sum_i \mathrm{diam}(V_i) \le C'e^{n\gamma},$$

for a positive constant C'.

This implies that $HD(W) \le t$; but t has been chosen arbitrarily larger than t^u, so $\delta^u(\hat{x}) = HD(W) \le t^u$.

It remains to prove now the opposite inequality, i.e $t^u \le \delta^u(\hat{x})$. For this, let us consider some arbitrary $t > \delta^u(\hat{x})$ and show that $t \ge t^u$, i.e show that $P(t\phi^u) \le 0$. One can use now a Theorem of Pesin and Pitskel ([15]) which allows us to write the pressure using balls $B_{\hat{f}}(\hat{y}, \varepsilon, n_i)$ for different integers n_i. So, for a continuous potential $\psi \in \mathcal{C}(\hat{\Lambda})$, a positive ε, a positive integer N, and an arbitrary real number λ, let us denote

$$M_\varepsilon(\lambda, \psi, N) := \inf\{\sum_{\hat{y}\in F} e^{S_{n_{\hat{y}}}\psi(\hat{y}) - \lambda n_{\hat{y}}}, \text{ where } \hat{\Lambda} \subset \bigcup_{\hat{y}\in F} B_{\hat{f}}(\hat{y}, \varepsilon, n_{\hat{y}}), n_{\hat{y}} \ge N, \hat{y} \in F\}$$

Then denote by $M_\varepsilon(\lambda, \psi) := \lim_{N\to\infty} M_\varepsilon(\lambda, \psi, N)$; this limit exists since the sequence $(M_\varepsilon(\lambda, \psi, N))_N$ is increasing. Let also $P_\varepsilon(\psi) := \inf\{\lambda, M_\varepsilon(\lambda, \psi) = 0\}$; then it is shown in [15] that $P(\psi) = \lim_{\varepsilon\to 0} P_\varepsilon(\psi)$.

Therefore, in order to show that $P(t\phi^u) \le 0$, it is enough to show that $M_\varepsilon(0, t\phi^u) = 0, \varepsilon > 0$ small enough, i.e $M_\varepsilon(0, t\phi^u, N) = 0, N, \varepsilon > 0$ small. For this , it is enough to find a finite cover of $\hat{\Lambda}$ with sets $B_{\hat{f}}(\hat{y}, \varepsilon, n_{\hat{y}}), \hat{y} \in F$, such that $n_{\hat{y}} \ge N$ and $\sum_{\hat{y}\in F} e^{S_{n_{\hat{y}}}(t\phi^u)(\hat{y})} < 1$.

Consider now an integer m such that $f^{-m}W \cap \Lambda$ intersects all stable manifolds $W^s_{\varepsilon/2}(z)$, of points in Λ (this can be done as in [9]); it can be seen easily that $HD(f^{-m}W \cap \Lambda) = \delta^u(\hat{x})$. Using the inequality $t > \delta^u(\hat{x})$, we can find a finite open cover $\mathcal{U} = (U_i)i \in I$, of $f^{-m}(W) \cap \Lambda$, such that

$$\sum_{i\in I} (\mathrm{diam}U_i)^t < \tilde{\alpha} < 1, \tag{6}$$

where $\tilde{\alpha}$ is a small positive number to be determined later.

We will produce a spanning set (in the above sense) out of the cover $\mathcal{U}$. In order to do this, consider first for each $i \in I$, the set $U_i^* := \cup W^s_{\varepsilon/2}(z)$, union over all local stable manifolds of size $\varepsilon/2$ which intersect U_i. From the assumption on m, we know that $U_i^*, i \in I$ cover the entire Λ. We need now a covering of the natural extension $\hat{\Lambda}$. Let us denote the diameter of Λ by ρ and assume that $K > 1$ defines the metric d_K on $\hat{\Lambda}$; take a positive integer $s = s(\varepsilon)$ such that $\frac{\rho}{K^s} < \varepsilon$. Now, take a point y_i in each $U_i \cap \Lambda$, and cover the set $W^s_{\varepsilon/2}(y_i)$ with small open sets $V_{ij}, j \in J_i$, in such a way that $\text{diam} f_*^{-s'}(V_{ij}) < \varepsilon, j \in J_i$, for any local inverse iterate $f_*^{-s'}$, and any $0 \le s' \le s$. But s depends only on ε, hence $N_i(s) := |J_i|$ depends only on ε. Let us denote by $N(s)$ the largest such $N_i(s)$, when $i \in I$. Next, for each point z in some $V_{ij}, j \in J_i$, we will have at most d^s s-prehistories in Λ (if d denotes the largest number of f-preimages that a point from Λ can have in Λ). Consider $C = (z, z_{-1}, ..., z_{-s})$ an s-prehistory with $z_{-j} \in \Lambda, 1 \le j \le s$. We will denote by

$$\Lambda(C, \varepsilon) := \{\omega \in \Lambda, \exists \text{ an } s-\text{prehist.}\ (\omega, ..., \omega_{-s}) \text{ with } d(\omega_{-j}, z_{-j}) < \varepsilon, 1 \le j \le s\}$$

Now, fix $i \in I$ and $j \in J_i$; fix also a point $z \in V_{ij}$ and consider all the s-prehistories C of z in Λ. Then for each V_{ij} there are at most d^s such prehistories C. We will denote the set of these prehistories C corresponding to V_{ij} by Γ_{ij} (where $j \in J_i, i \in I$). One can notice also that every local unstable manifold $W^u_\varepsilon(\hat{z}'), z' \in V_{ij}$ is included in one of the sets $\Lambda(C, \varepsilon)$, for some $C \in \Gamma_{ij}$, (because of the way in which V_{ij} were taken); in particular $V_{ij} \subset \underset{C \in \Gamma_{ij}}{\cup} \Lambda(C, \varepsilon)$.

Now, for each $C \in \Gamma_{ij}$, consider some fixed complete prehistory $\hat{z}^C \in \hat{\Lambda}$ which starts with the truncated prehistory C, i.e which satisfies $z^C_{-j} = z_{-j}, 1 \le j \le s$, (where $C = (z, z_{-1}, ..., z_{-s})$). Denote the set of all these prehistories by F, i.e $F := \{\hat{z}^C, C \in \Gamma\}$, where $\Gamma := \underset{j \in J_i, i \in I}{\cup} \Gamma_{ij}$.

Define now, for each $i \in I$, the positive integer n_i with the property that

$$\text{diam} f^k(U_i) < \varepsilon, 0 \le k < n_i, \text{ but } \text{diam} f^{n_i}(U_i) > \varepsilon.$$

We want to prove that $\hat{\Lambda} \subset \underset{i \in I}{\cup} \underset{C \in \Gamma_i}{\cup} B_{\hat{f}}(\hat{z}^C, 5\varepsilon, n_i)$, where $\Gamma_i := \underset{j \in J_i}{\cup} \Gamma_{ij}$. We will also prove in the sequel that $e^{S_{n_i}(t\phi^u)(\hat{z}^C)} \le \chi_0 \text{diam} U_i, C \in \Gamma_i$, where χ_0 is a positive constant independent of C, i.

Consider then an arbitrary prehistory $\hat{\omega}$ of a point ω from U_i^*; we know that $U_i^*, i \in I$, cover the entire Λ, so for an arbitrary $\hat{\omega} \in \hat{\Lambda}$, there must exist an i as above. We know also that there exists an s-prehistory $C \in \Gamma_i$ such that $\hat{\omega}$ is ε-shadowed by C up to level s, i.e $d(\omega, z^C) < \varepsilon, ..., d(\omega_{-s}, z^C_{-s}) < \varepsilon$. Hence $d_K(\hat{\omega}, \hat{z}^C) \le d(\omega, z^C) + \frac{d(\omega_{-1}, z^C_{-1})}{K} + ... + \frac{d(\omega_{-s}, z^C_{-s})}{K^s} + \frac{\rho}{K^s} \le \varepsilon + \frac{\varepsilon}{K} + ... + \frac{\varepsilon}{K^s} + \frac{\rho}{K^s} < 3\varepsilon$.

Notice next that due to the fact that $z^C \in U_i^* \cap \Lambda$, there must exist a uniquely defined point $\xi^C \in U_i$ such that $z^C \in W^s_{\varepsilon/2}(\xi^C)$. Also, since $\omega \in U_i^* \cap \Lambda$, it follows that there exists a point $\xi \in U_i$ such that $\omega \in W^s_{\varepsilon/2}(\xi)$. So, $d(f^k\omega, f^k z^C) \le d(f^k\omega, f^k\xi) + d(f^k\xi, f^k\xi^C) + d(f^k\xi^C, f^k z^C)$. But we know that $d(f^k\omega, f^k\xi) < \varepsilon/2, k \ge 0$ and $d(f^k\xi^C, f^k z^C) < \varepsilon/2, k \ge 0$,and also from the definition of n_i, we see that $d(f^k\xi, f^k\xi^C) < \varepsilon, 0 \le k \le n_i - 1$. Hence we obtain from the above relations that $d(f^k\omega, f^k z^C) \le 2\varepsilon, 0 \le k < n_i$. Therefore $d_K(\hat{f}^k\omega, \hat{f}^k\hat{z}^C) \le 2\varepsilon + \frac{2\varepsilon}{K} + ... + \frac{\rho}{K^s} \le 5\varepsilon, 0 \le k < n_i$ (if $K > 2$). Thus we showed that $\hat{\omega} \in B_{\hat{f}}(\hat{z}^C, 5\varepsilon, n_i)$ for some $C \in \Gamma_i$. This implies that $\hat{\Lambda} = \underset{C \in \Gamma}{\cup} B_{\hat{f}}(\hat{z}^C, 5\varepsilon, n_i)$.

But we denoted by F the set of the prehistories $\hat{z}^C, C \in \Gamma$, so we obtain that F spans $\hat{\Lambda}$.

Next, let us estimate $e^{S_{n_i}(t\phi^u)(\hat{z}^C)}$. First of all, if $\hat{\xi}^C$ is the unique prehistory of ξ^C given by the fact that ξ^C belongs to a local unstable manifold which intersects U_i, we see that there exists a constant $\gamma > 0$ such that $e^{S_{n_i}(\phi^u)(\hat{\xi}^C)} \le \gamma \mathrm{diam} U_i, i \in I$. From the fact that $z^C \in W^s_{\varepsilon/2}(\xi^C)$, we will get then that $d_K(\hat{z}^C, \hat{\xi}^C) < c, d_K(\hat{f}\hat{z}^C, \hat{f}\hat{\xi}^C) \le c\lambda_s + \frac{c}{K}, ..., d_K(\hat{f}^{n_i-1}\hat{\xi}^C, \hat{f}^{n_i-1}\hat{z}^C) \le c(\lambda_s^{n_i-1} + \frac{\lambda_s^{n_i-2}}{K} + ... + \frac{1}{K^{n_i-1}}) + \frac{\rho}{K^{n_i}}$, where $\lambda_s := \sup_\Lambda |Df_s|$. But this implies that there exist constants $c > 0$, $\gamma_1 \in (0,1)$ such that $d_K(\hat{f}^k\hat{z}^C, \hat{f}^k\hat{\xi}^C) < c\gamma_1^k, 0 \le k < n_i$.

Now let us apply the Hölder continuity of ϕ^u from Theorem 2. So, we get $|\phi^u(\hat{f}^k\hat{z}^C) - \phi^u(\hat{f}^k\hat{\xi}^C)| \le C_1\gamma_2^k, 0 \le k < n_i$, where $C_1 > 0$ and $\gamma_2 \in (0,1)$. Thus $|S_{n_i}(t\phi^u)(\hat{z}^C) - S_{n_i}(t\phi^u)(\hat{\xi}^C)| \le C_1(1 + \gamma_2 + ... + \gamma_2^{n_i-1}) < L_1$, for some positive constant L_1. Therefore there exists a constant $L_2 > 0$ such that

$$\frac{1}{L_2} \le e^{S_{n_i}(t\phi^u)(\hat{z}^C) - S_{n_i}(t\phi^u)(\hat{\xi}^C)} \le L_2 \tag{7}$$

Let us come back now to the sum $\sum\limits_{C\in\Gamma} e^{S_{n_i}(t\phi^u)(\hat{z}^C)}$ and use (7) and the fact that $e^{S_{n_i}(\phi^u)(\hat{\xi}^C)} \le \gamma \mathrm{diam} U_i, i \in I$. Recall that $\Gamma = \bigcup\limits_{i\in I, j\in J_i} \Gamma_{ij}$, and $|J_i| \le N(s), |\Gamma_{ij}| \le d^s, i \in I, j \in J_i$.

So, we have $\sum\limits_{C\in\Gamma} e^{S_{n_i}(t\phi^u)(\hat{z}^C)} \le L_3 d^s N(s) \sum\limits_{i\in I} (\mathrm{diam} U_i)^t$, for some positive constant L_3. Recall also that s depends only on ε and that $N(s)$ depends only on s. Therefore, if in the begining we take $\tilde{\alpha} < \frac{1}{L_3 \cdot d^s N(s)}$ we get

$$\sum_{C\in\Gamma_i, i\in I} e^{S_{n_i}(t\phi^u)(\hat{z}^C)} < 1$$

Recalling also that we showed that $F = \{\hat{z}^C, C \in \Gamma\}$ spans $\hat{\Lambda}$ in the sense that $\hat{\Lambda} \subset \bigcup\limits_{i\in I} \bigcup\limits_{C\in\Gamma_i} B_{\hat{f}}(\hat{z}^C, 5\varepsilon, n_i)$, we can conclude that $P_\varepsilon(t\phi^u) \le 0$, hence:

$$P_{\hat{f}}(t\phi^u) \le 0$$

Thus $t \ge t^u$. But t has been chosen arbitrarily larger than $\delta^u(\hat{x})$, thus $\delta^u(\hat{x}) \ge t^u$. Corroborating with the other inequality proved earlier, we obtain finally that $t^u = \delta^u(\hat{x})$. □

In particular we notice that $\delta^u(\hat{x}; \beta)$ does not depend on β (small), nor on $\hat{x} \in \hat{\Lambda}$, so it can be denoted simply by δ^u, the **unstable dimension** of Λ.

As said above, the stable dimension cannot be written similarly using a Bowen type equation, and $\delta^s(x)$ is in general only strictly smaller than the zero t^s of the pressure function associated to the stable potential ϕ^s (see [7] for further details). However in the estimates of $\delta^s(x)$, a very important role is played by the inverse pressures P^- and P_-, as was observed in [9], [10], [12].

3. **Construction and properties of unstable manifolds, α-Hölder continuity of Φ_g along unstable manifolds, estimates for the exponent α in the holomorphic case.** We will now prove that if g is a perturbation of an Axiom A noninvertible map f, then the unstable manifolds of g depend Lipschitz continuously on g. This proof will also help us in proving a theorem of uniform α-Hölder

continuity of the conjugating map Φ_g along the unstable set $V_\beta^+(\hat{x}), \hat{x} \in \hat{\Lambda}$. In the special case of holomorphic maps on $\mathbb{P}^2$, the method of proof will give a new precise estimate for α. As a remark, the unstable manifolds depend smoothly on a (in a certain sense), when $(g_a)_a$ depends smoothly on the parameter a.

First, we need to define the notions of **continuous family of submanifolds** and **Lipschitz family of diffeomorphisms**.

Definition 11. (a) Let (X, d) be a compact metric space and M a smooth Riemannian manifold. Assume that to each $x \in X$ we can associate a submanifold $\mathcal{F}_x$ of M in such a way that this correspondence is continuous, i.e for each $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that if $x, y \in X, d(x, y) < \eta(\varepsilon)$, then $d(\mathcal{F}_x, \mathcal{F}_y) < \varepsilon$, where by $d(\mathcal{F}_x, \mathcal{F}_y)$ we understand the distance in the Hausdorff metric between the respective sets. Then we call $\mathcal{F} = \{\mathcal{F}_x\}_x$ a **continuous family of submanifolds** indexed by X.

(b) Let now E be a metric space and $Y \subset E$ an open subset of E such that to each $g \in Y$ we can associate a continuous family of submanifolds of M indexed by X, denoted $\mathcal{G}(g)$. Assume that we fix an $f \in Y$ and denote $\mathcal{G}(f)$ by $\mathcal{F}$. Assume also that for each $g \in Y$, there exists a family of smooth diffeomorphisms $\Psi^g := \{\Psi_x^g\}_{x \in X}, \Psi_x^g : \mathcal{F}_x \to \mathcal{G}(g)_x, x \in X$ (we assume that the diffeomorphisms Ψ_x^g are in the same category $\mathcal{C}^r, r \geq 2$, as M and f). Then we say that a family $(\Psi^g)_{g \in Y}$ is a **Lipschitz family of diffeomorphisms** if there exists a positive constant C such that $d(\Psi_x^g, \Psi_x^\kappa) \leq C \cdot d_{\mathcal{C}^0}(\kappa, g), g, \kappa \in Y$, and $x \in X$. (the distance $d(\Psi_x^g, \Psi_x^\kappa)$ is the usual supremum distance in $\mathcal{C}^0(\mathcal{F}_x, M)$).

We are ready to state now a theorem about perturbations of f and the corresponding conjugacy maps Φ_g, giving also a Lipschitz family of diffeomorphisms between local unstable manifolds. The Lipschitz continuity of Φ_g (in g) and the existence of a Lipschitz family of unstable diffeomorphisms are new facts in the case of endomorphisms (but items 1)-3) are known, for example [18]).

Theorem 4. *Let M be a compact Riemannian manifold, and $f : M \to M$ a smooth map on M (M and f are of the same order $\mathcal{C}^r, r \geq 2$). Let also Λ be a compact subset of M such that $f(\Lambda) = \Lambda$, $f|_\Lambda$ is transitive and f is hyperbolic (as an endomorphism) and has local product structure over Λ. Assume also that g is a perturbation of f, i.e g belongs to a small neighbourhood $\mathcal{U}$ of f in $\mathcal{C}^1(M, M)$. Then:*

1) *There exists a continuous map $\Phi : \mathcal{U} \to \mathcal{C}^0(\hat{\Lambda}, M)$ such that $\Phi(g) \circ \hat{f} = g \circ \Phi(g), g \in \mathcal{U}$ and $\Phi(f) = \pi_f$, where $\pi_f : \hat{\Lambda} \to \Lambda$ is the canonical projection*
2) *If $g \in \mathcal{U}$ and Φ is the map from 1), let us denote by $\Phi_g := \Phi(g)$. Let $\Lambda_g := \Phi_g(\hat{\Lambda})$; then $g(\Lambda_g) = \Lambda_g$ and g is hyperbolic over Λ_g.*
3) *Φ_g can be lifted to a homeomorphism $\hat{\Phi}_g : \hat{\Lambda} \to \hat{\Lambda}_g$, which conjugates the actions of $\hat{f}$ and $\hat{g}$ on $\hat{\Lambda}$, $\hat{\Lambda}_g$ respectively ($\hat{\Lambda}_g$ represents the natural extension of Λ_g with respect to g).*
4) *There is a constant $C > 0$ such that $d_{\mathcal{C}^0(\hat{\Lambda}, M)}(\Phi_g, \Phi_\kappa) \leq C d_{\mathcal{C}^0(M,M)}(g, \kappa)$, $g, \kappa \in \mathcal{U}$.*
5) *There exists $\beta > 0$ such that for every $g \in \mathcal{U}$, there are local unstable (stable) manifolds of size β at all points $\hat{x} \in \hat{\Lambda}_g$ ($x \in \Lambda_g$ respectively), and there exists also a Lipschitz family of diffeomorphisms $\{\Theta_{\hat{x}}^u(g) : W_\beta^u(f, \hat{x}) \to W_\beta^u(g, \hat{\Phi}_g(\hat{x}))\}_{\hat{x} \in \hat{\Lambda}}$ (resp. $\{\Theta_{\hat{x}}^s(g) : W_\beta^s(f, x) \to W_\beta^s(g, \pi_g \hat{\Phi}_g(\hat{x}))\}_{\hat{x} \in \hat{\Lambda}}$) such that $\Theta_{\hat{x}}^u(g)(x) = \Phi_g(\hat{x})$ and $\Theta_{\hat{x}}^s(g)(x) = \Phi_g(\hat{x}), \hat{x} \in \hat{\Lambda}$.*

Proof. The proof of item 1)-3) follows the general ideas from [20] and [4] adapted to the natural extension $\hat{\Lambda}$ and to the hyperbolicity of f as an endomorphism (i.e where the unstable manifolds depend on the entire prehistory of the base point). It is good to recall it here since it will be used in the proof of 4) and 5).

Proof of 1), 2), 3): Let us consider $\mathcal{C}(\hat{\Lambda}, M)$ be the metric space of continuous maps from $\hat{\Lambda}$ to M, with the sup metric. For each $g \in \mathcal{U}$ define the map $\mathcal{L}_g : \mathcal{C}(\hat{\Lambda}, M) \to \mathcal{C}(\hat{\Lambda}, M), \mathcal{L}_g(h) := g \circ h \circ \hat{f}^{-1}$. Notice that $\mathcal{L}_f(h) = f \circ h \circ \hat{f}^{-1}$ and that $\mathcal{L}_f$ has a hyperbolic fixed point at π_f. We can apply now the procedures from Theorem 7.8 of [20]. The idea is the following: first linearize $\mathcal{C}(\hat{\Lambda}, M)$ by replacing it with the space of continuous sections $\Gamma(\hat{\Lambda}, T_{\hat{\Lambda}}M)$, where $T_{\hat{\Lambda}}M := \{(\hat{x}, v), \hat{x} \in \hat{\Lambda}, v \in T_x M\}$ is the tangent bundle over $\hat{\Lambda}$; this replacement can be done using the exponential maps. One then replaces $\mathcal{L}_g$ by its expression in exponential coordinates $\bar{\mathcal{L}_g} : \Gamma_r(\hat{\Lambda}, T_{\hat{\Lambda}}M) \to \Gamma(\hat{\Lambda}, T_{\hat{\Lambda}}M)$, where for small $r > 0$, $\Gamma_r(\hat{\Lambda}, T_{\hat{\Lambda}}M)$ denotes the bundle of balls of radius r in $\Gamma(\hat{\Lambda}, T_{\hat{\Lambda}}M)$. Hence $\bar{\mathcal{L}_g}(\sigma)(\hat{x}) = exp_x^{-1}(g(exp_{x_{-1}}\sigma(\hat{f}^{-1}\hat{x})))$, where $\hat{x} = (x, x_{-1}, ...)$. Next we show easily that $\bar{\mathcal{L}_g}$ is Lipschitz close to a hyperbolic linear operator F on the section space $\Gamma(\hat{\Lambda}, T_{\hat{\Lambda}}M)$, i.e that for a small $\varepsilon > 0$, $\mathrm{Lip}(\bar{\mathcal{L}_g} - F) < \varepsilon$. It can be shown that $\bar{\mathcal{L}_g}$ is a Lipschitz perturbation of its derivative F at the zero section ($\mathrm{Lip}(F)$ denotes in general the smallest Lipschitz constant of a Lipschitz map F). In this case, Proposition 7.7 of [20] shows that $\bar{\mathcal{L}_g}$ has a fixed point Z_g near the zero section. But this is equivalent to $\mathcal{L}_g$ having a fixed point $\Phi_g \in \mathcal{C}(\hat{\Lambda}, M)$, close to the canonical projection $\pi_f : \hat{\Lambda} \to \Lambda$. This implies that $\Phi_g \circ \hat{f} = g \circ \Phi_g$.

Next, we define the lifting $\hat{\Phi}_g$ of Φ_g. Let $\hat{x} \in \hat{\Lambda}$ and $y := \Phi_g(\hat{x})$. Let also $y_{-1} := \Phi_g(\hat{f}^{-1}\hat{x})$; we see that $g(y_{-1}) = g\Phi_g(\hat{f}^{-1}\hat{x}) = \Phi_g \circ \hat{f}(\hat{f}^{-1}\hat{x})\Phi_g(\hat{x}) = y$; similarly we can define $y_{-i} := \Phi_g(\hat{f}^{-i}), i \geq 1$. One can prove as above that $\hat{y} := (y, y_{-1}, y_{-2}, ...)$ is indeed a prehistory of y in $\hat{\Lambda}_g$. Then it follows easily that $\hat{\Phi}_g$ is a homeomorphism of $\hat{\Lambda}_g$. The fact that the hyperbolic structure of f on $\hat{\Lambda}$ transfers to a hyperbolic structure of g on $\hat{\Lambda}_g$ is proved similarly as in Prop. 7.6 of [20]. This concludes the proof of items 1), 2), 3).

Proof of 4): Let us estimate now $d(\mathcal{L}_{g_1}, \mathcal{L}_{g_2})$ for $g_1, g_2 \in \mathcal{U}$. We know that $\mathcal{L}_{g_1}, \mathcal{L}_{g_2} : \mathcal{C}(\hat{\Lambda}, M) \to \mathcal{C}(\hat{\Lambda}, M)$, $\mathcal{L}_{g_1}(h) = g_1 h \hat{f}^{-1}, \mathcal{L}_{g_2}(h) = g_2 h \hat{f}^{-1}$; hence $(\mathcal{L}_{g_1} - \mathcal{L}_{g_2})(h) = (g_1 - g_2) \circ h\hat{f}^{-1}$. Thus we obtain $\sup\limits_{h \in \mathcal{C}(\hat{\Lambda}, M)} |\mathcal{L}_{g_1} - \mathcal{L}_{g_2}| \leq d(g_1, g_2)$ since M is a compact manifold.

Now, let us recall how the map $\bar{\mathcal{L}_g}$ was formed; we took χ the chart defined on a neighborhood $\mathcal{V}$ of π_f in $\mathcal{C}(\hat{\Lambda}, M)$ given by $\chi : \mathcal{V} \to \Gamma_{r'}(\hat{\Lambda}, T_{\hat{\Lambda}}M), \chi(h)(\hat{x}) = exp_x^{-1}h(\hat{x}), \hat{x} \in \hat{\Lambda}$, where exp_x denotes the exponential map at x. This works since, if h is close to π_f, then $h(\hat{x})$ is close to x, so we can apply the exponential map. Then $\bar{\mathcal{L}_g} = \chi\mathcal{L}_g\chi^{-1}$. Thus $d(\bar{\mathcal{L}}_{g_1}, \bar{\mathcal{L}}_{g_2}) \leq C_1 d(\mathcal{L}_{g_1}, \mathcal{L}_{g_2}) \leq C_1 d_{\mathcal{C}^0(M,M)}(g_1, g_2)$, for some positive constant C_1. But then from Proposition 7.7 of [20], we have $|Z_{g_1} - Z_{g_2}| \leq C_2 d(g_1, g_2), g_1, g_2 \in \mathcal{U}$. So, if $Z_g \in \Gamma_r(\hat{\Lambda}, T_{\hat{\Lambda}}M)$, we will define $\Phi_g(\hat{x}) := exp_x Z_g(\hat{x}), \hat{x} \in \hat{\Lambda}$.

Since exp is a local diffeomorphism we also get from above that there exists a constant $C > 0$ such that $d(\Phi_{g_1}, \Phi_{g_2}) \leq Cd(g_1, g_2)$.

So we showed that the conjugating map Φ_g depends Lipschitz continuously on $g \in \mathcal{U}$.

Proof of 5): We shall now prove the existence of a Lipschitz family of diffeomorphisms $\{\Theta^u_{\hat{x}}\}_{\hat{x}\in\hat{\Lambda}}$ satisfying the conditions from the statement. In the proof of 1), 2), 3), we defined the map $\mathcal{L}_g : \mathcal{C}(\hat{\Lambda}, M) \to \mathcal{C}(\hat{\Lambda}, M), \mathcal{L}_g(h) = gh\hat{f}^{-1}, h \in \mathcal{C}(\hat{\Lambda}, M)$, and we proved that it has a hyperbolic fixed point denoted by Φ_g, which is close to the projection $\pi_f : \hat{\Lambda} \to M$; we also denoted $\Lambda_g := \Phi_g(\hat{\Lambda})$. Take now $\mathcal{W}^u_g \subset \mathcal{C}_\varepsilon(\hat{\Lambda}, M)$ be a local unstable manifold of the fixed hyperbolic point Φ_g, where $\mathcal{C}_\varepsilon(\hat{\Lambda}, M)$ is a neighbourhood of the canonical projection $\pi_f : \hat{\Lambda} \to M$. In the same way as in Theorem 3.2 of [4], one can prove that $\mathcal{W}^u_g(\hat{y}) := \{h(\hat{x}), h \in \mathcal{W}^u_g\}$ is a local unstable manifold corresponding to the prehistory $\hat{y} = \hat{\Phi}_g(\hat{x}) \in \hat{\Lambda}_g$. But from the Unstable Manifold Theorem for a Hyperbolic Point of [4], it follows also that $\mathcal{W}^u_g$ is the graph of an unstable function $\mathcal{G}_g : \mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}}) \to \mathcal{C}_\varepsilon(E^s_{\hat{\Lambda}})$, where $\varepsilon > 0$ is small and $\mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}})$ represents the bundle of balls of radius ε centered at the zero section, inside the bundle of continuous sections of $E^u_{\hat{\Lambda}}$. Notice also that $\mathcal{C}_\varepsilon(\hat{\Lambda}, M)$ can be identified with $\mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}}) \times \mathcal{C}_\varepsilon(E^s_{\hat{\Lambda}})$ by exponential coordinates. We will recall how the unstable function $\mathcal{G}_g$ was obtained, from the general Unstable Manifold Theorem for Banach spaces ([4]), as the unique fixed point of a graph transform. Indeed, we defined earlier the map χ (giving the exponential chart) and $\bar{\mathcal{L}}_g : \Gamma_\varepsilon(\hat{\Lambda}, T_{\hat{\Lambda}}M) \to \Gamma(\hat{\Lambda}, T_{\hat{\Lambda}}M), \bar{\mathcal{L}}_g = \chi\mathcal{L}_g\chi^{-1}$. The map $\bar{\mathcal{L}}_g$ has an associated graph transform $\Gamma_g : \mathcal{M} \to \mathcal{M}$, with $\mathcal{M} := \{H : \mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}}) \to \mathcal{C}_\varepsilon(E^s_{\hat{\Lambda}}), H(\mathbf{0}) = \mathbf{0}$, and $\mathrm{Lip}(H) \leq 1\}$, where $\mathbf{0}$ = zero section.

In the sequel we will find an expression for Γ_g starting from the decomposition of $\bar{\mathcal{L}}_g$ as $\tilde{T}_1 \times \tilde{T}_2$, where $\tilde{T}_1 : \mathcal{C}_\varepsilon(T_{\hat{\Lambda}}M) \to \mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}})$ and $\tilde{T}_2 : \mathcal{C}_\varepsilon(T_{\hat{\Lambda}}M) \to \mathcal{C}_\varepsilon(E^s_{\hat{\Lambda}})$. Now, if $H \in \mathcal{M}$, define $S_1(H) := \tilde{T}_1 \circ (id, H)$, where id denotes here the identity of the bundle $\mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}})$ and $S_2(H) := \tilde{T}_2 \circ (id, H)$. Because Dg is expanding in the unstable direction, we have that $S_1(H)$ is an injective map $\mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}}) \to \mathcal{C}(E^u_{\hat{\Lambda}})$ and $S_2(H) : \mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}}) \to \mathcal{C}(E^s_{\hat{\Lambda}})$.Then, it can be shown that $S_1(H)(\mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}})) \supset \mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}})$, and that $S_1(H)$ is a Lipschitz homeomorphism onto its image, with Lipschitz inverse. Therefore $\left(S_1(H)|_{\mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}})}\right)^{-1}$ takes values into $\mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}})$; hence it makes sense to apply $S_2(H)$ to $\left(S_1(H)|_{\mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}})}\right)^{-1}$, thus obtaining that:

$$\Gamma_g(H) = S_2(H) \circ \left(S_1(H)|_{\mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}})}\right)^{-1}$$

Since the Lipschitz constant of $S_2(H)$ is easily seen to be strictly less than 1, we conclude that $\Gamma_g(H) \in \mathcal{M}$, so Γ_g is a well defined map $\mathcal{M} \to \mathcal{M}$. Using the above expression, it can be proved also that the graph transform Γ_g is a contraction. Thus Γ_g has a unique fixed point $\mathcal{G}_g : \mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}}) \to \mathcal{C}_\varepsilon(E^s_{\hat{\Lambda}})$. The graph of $\mathcal{G}_g$ gives, by exponential coordinates, the local unstable set $\mathcal{W}_g$ of Φ_g.

Let us prove next that this unstable function $\mathcal{G}_g$ depends Lipschitz continuously on g, i.e that there exists a positive constant K_0 such that

$$\sup_{\sigma\in\mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}}), \hat{x}\in\hat{\Lambda}} |\mathcal{G}_g(\sigma)(\hat{x}) - \mathcal{G}_\kappa(\sigma)(\hat{x})| \leq K_0|g - \kappa|, \forall g, \kappa \in \mathcal{U}$$

We will prove this property in the general setting, i.e for an arbitrary map $h : E(r) \to E$, with E a Banach space and $E(r)$ the closed ball of radius r centered at 0 in E. We assume that there exists a linear operator $T = T_1 \times T_2 : E \to E$, $T_i : E_i \to E_i, i = 1, 2$ and T is a hyperbolic operator, expanding on E_1 and contracting on E_2; assume also that $\mathrm{Lip}(h - T) < \varepsilon < 1, |h(0)| < \delta < 1$, for some small constants ε, δ. Let us define $h_i := p_i \circ h, i = 1, 2$, with p_i the projection from E to E_i and $\mathcal{M} := \{H : E_1(r) \to E_2(r), H(0) = 0, \mathrm{Lip}(H) \le 1\}$. Then let $S_1^h(H) := h_1 \circ (id, H), S_2^h(H) = h_2 \circ (id, H)$; using the definitions, it can be shown that $S_1^h(H)(E_1(r)) \supset E_1(r)$ and that the Lipschitz constant of the map $\left(S_1^h(H)|_{E_1(r)}\right)^{-1}$ is strictly less than 1. So, one can define the map $\Gamma_h(H) := S_2^h(H) \circ \left(S_1^h(H)|_{E_1(r)}\right)^{-1}$. It can be checked that it is a contraction, hence from the Contraction Principle it has a unique fixed point $\mathcal{G}_h \in \mathcal{M}$. We prove now that $\mathcal{G}_h$ depends Lipschitz continuously on h. Indeed since $\mathcal{G}_h$ gives the fixed point for Γ_h, we know that $|\mathcal{G}_\kappa - \mathcal{G}_h| = |\Gamma_\kappa(\mathcal{G}_\kappa) - \Gamma_h(\mathcal{G}_h)| \le |\Gamma_\kappa(\mathcal{G}_\kappa) - \Gamma_\kappa(\mathcal{G}_h)| + |\Gamma_\kappa(\mathcal{G}_h) - \Gamma_h(\mathcal{G}_h)| \le \alpha'|\mathcal{G}_\kappa - \mathcal{G}_h| + |\Gamma_\kappa(\mathcal{G}_h) - \Gamma_h(\mathcal{G}_h)|$, with $\alpha' \in (0, 1)$ the contraction constant of Γ_κ. But from definition, $\Gamma_\kappa(H) = S_2^\kappa(H) \circ \left(S_1^\kappa(H)|_{E_1(r)}\right)^{-1}$ and similarly for $\Gamma_h(H)$. In the sequel we will write for simplicity $S_1^\kappa(H)^{-1}$ (and $S_1^h(H)^{-1}$) instead of $\left(S_1^\kappa(H)|_{E_1(r)}\right)^{-1}$ (respectively $\left(S_1^h(H)|_{E_1(r)}\right)^{-1}$). So $|\Gamma_\kappa(H) - \Gamma_h(H)| \le |S_2^\kappa(H) \circ S_1^\kappa(H)^{-1} - S_2^\kappa(H) \circ S_1^h(H)^{-1}| + |S_2^\kappa(H) \circ S_1^h(H)^{-1} - S_2^h(H) \circ S_1^h(H)^{-1}| \le \mathrm{Lip}(S_2^\kappa(H))|S_1^\kappa(H)^{-1} - S_1^h(H)^{-1}| + |S_2^\kappa(H) - S_2^h(H)|$, and $\mathrm{Lip}(S_2^\kappa(H)) < 1$. But recall that $S_2^\kappa(H) = \kappa_2 \circ (id, H)$, where here id denotes the identity of $E_1(r)$. Hence $|S_2^\kappa(H) - S_2^h(H)| = |\kappa_2 \circ (id, H) - h_2 \circ (id, H)| \le |\kappa - h|$. Also, $|S_1^\kappa(H)^{-1} - S_1^h(H)^{-1}| = |S_1^\kappa(H)^{-1} \circ S_1^h(H) \circ S_1^h(H)^{-1} - S_1^\kappa(H)^{-1} \circ S_1^\kappa(H) \circ S_1^h(H)^{-1}| \le \mathrm{Lip}(S_1^\kappa(H)^{-1})|S_1^h(H) - S_1^\kappa(H)| \le C_0|\kappa - h|$, for some constant $C_0 > 0$ and any h, κ. In conclusion we proved the general statement that the fixed point $\mathcal{G}_h$ of Γ_h depends Lipschitz continuously on h.

Hence, also in our case of the unstable function $\mathcal{G}_g$, there exists a positive constant K_0 such that $|\mathcal{G}_g - \mathcal{G}_\kappa| \le K_0 \cdot |\kappa - g|, \forall g, \kappa \in \mathcal{U}$.

In our case, the unstable function $\mathcal{G}_g$ gives the graph (modulo exponential coordinates) of the unstable set $\mathcal{W}_g^u, \forall g \in \mathcal{U}$; recall also that $\mathcal{W}_f^u, \mathcal{W}_g^u \subset \mathcal{C}_\varepsilon(\hat{\Lambda}, M)$ and we can assume that these are local unstable sets of size $\beta > 0$ ($\mathcal{W}_f^u$ is the local unstable set of size β for the projection π_f and $\mathcal{W}_g^u$ is the local unstable set of size β of the conjugacy map Φ_g). We will define then a map $\Theta^u(g) : \mathcal{W}_f^u \to \mathcal{W}_g^u$ in the following fashion:

$$\Theta^u(g)(v, \mathcal{G}_f(v)) := (v, \mathcal{G}_g(v)), v \in \mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}})$$

Now, for a prehistory $\hat{x} \in \hat{\Lambda}$, let $\hat{y} := \hat{\Phi}_g(\hat{x}) \in \hat{\Lambda}_g$. Recall also that $W^u_\beta(f, \hat{x}) = \{w(\hat{x}), w \in \mathcal{W}_f^u\}$ and $W^u_\beta(g, \hat{y}) = \{w'(\hat{y}), w' \in \mathcal{W}_g^u\}$. Then define a map between local unstable manifolds of f and g by:

$$\Theta^u_{\hat{x}}(g) : W^u_\beta(f, \hat{x}) \to W^u_\beta(g, \hat{y}), \Theta^u_{\hat{x}}(g)(w(\hat{x})) := \Theta^u(g)(w)(\hat{y}), w \in \mathcal{W}_f^u$$

It is proved in [4] that $\mathcal{G}_g$ is of order $\mathcal{C}^1$ if g is $\mathcal{C}^1$, and g is $\mathcal{C}^1$-close to f. Hence $\Theta^u_{\hat{x}}(g)$ is also a $\mathcal{C}^1$ diffeomorphism. Notice also that from the definitions it follows that $\Theta^u_{\hat{x}}(f) =$ the identity on $W^u_\beta(f, \hat{x})$. We showed before that there exists $K_0 > 0$ such that $|\mathcal{G}_g - \mathcal{G}_\kappa| \le K_0|g - \kappa|, g, \kappa \in \mathcal{U}$, hence $|(v, \mathcal{G}_g(v)) - (v, \mathcal{G}_\kappa(v))| \le K_0|\kappa - g|, v \in \mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}})$. Thus by passing through exponential coordinates we obtain $|\Theta^u_{\hat{x}}(\kappa)(\xi) - \Theta^u_{\hat{x}}(g)(\xi)| \le K_0 d(\kappa, g), \xi \in W^u_\beta(f, \hat{x}), \hat{x} \in \hat{\Lambda}$. This shows that $\Theta^u_{\hat{x}}(g)$ depends Lipschitz continuously on g, hence the family $\{\Theta^u_{\hat{x}}(g)\}_{\hat{x} \in \hat{\Lambda}}$ is a Lipschitz family of diffeomorphisms. Similarly it can be shown that $(\Theta^u_{\hat{x}}(g))^{-1}$ depends Lipschitz continuously on $g \in \mathcal{U}$.

Let us also show that $\Theta^u_{\hat{x}}(g)(x)\Phi_g(\hat{x}) = y, \hat{x} \in \hat{\Lambda}$. First of all, $\mathcal{W}_f^u$ is the local unstable set of the canonical projection $\pi_f : \hat{\Lambda} \to \Lambda$. So $\Theta^u_{\hat{x}}(f)(x) = \Theta^u(f)(\pi_f)(\hat{x})$. Now, for $g \in \mathcal{U}$, we have $\mathcal{G}_g$ the unstable function, giving the graph of the local unstable set $\mathcal{W}_g^u$ of the fixed hyperbolic point Φ_g. So $\Phi_g = (\mathbf{0}, \mathcal{G}_g(\mathbf{0}))$, where $\Phi_g \in \mathcal{C}_\varepsilon(\hat{\Lambda}, M)$, $\mathbf{0}$ is the zero section over $\hat{\Lambda}$ of the bundle $\mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}})$, and where we identified again $\mathcal{C}_\varepsilon(\hat{\Lambda}, M)$ with $\mathcal{C}_\varepsilon(E^u_{\hat{\Lambda}}) \times \mathcal{C}_\varepsilon(E^s_{\hat{\Lambda}})$ with the help of exponential coordinates. But $\Phi_f = \pi_f = (\mathbf{0}, \mathcal{G}_f(\mathbf{0}))$, so $\Phi_g = (\mathbf{0}, \mathcal{G}_g(\mathbf{0})) = \Theta^u(g)(\mathbf{0}, \mathcal{G}_f(\mathbf{0}))$, i.e $\Phi_g = \Theta^u(g)(\pi_f)$. This implies that $\Theta^u_{\hat{x}}(g)(x) = \Theta^u(g)(\pi_f)(\hat{x})\Phi_g(\hat{x}) = y, \hat{x} \in \hat{\Lambda}$.

Similarly, the conclusion of 5) can also be proved in the case of families of local stable manifolds. □

Remark 2: Assume that A is a non-empty open set in $\mathbb{R}^m$. From the previous proof it follows that if $(g_a)_{a\in A}$ is a family of perturbations of f (in $\mathcal{C}^r(M, M), r \geq 1$), which depend smoothly ($\mathcal{C}^r$) on the parameter a, then the unstable manifolds of g_a depend smoothly ($\mathcal{C}^r$) on a. This means that for any $\hat{x} \in \hat{\Lambda}$, the map $(a, z) \to \Theta^u_{\hat{x}}(g_a)(z)$ is a $\mathcal{C}^r$ map from $A \times W^u_\beta(f, \hat{x})$ to M, and its derivatives depend continuously on $\hat{x} \in \hat{\Lambda}$. This is proved in the same way as before, using also the Implicit Function Theorem to show that the fixed point of Γ_{g_a} depends smoothly on $a \in A$. Then we use the maps $\Theta^u(g_a)$ to define the diffeomorphisms $\Theta^u_{\hat{x}}(g_a) : W^u_\beta(f, \hat{x}) \to W^u_\beta(g_a, \hat{\Phi}_{g_a}(\hat{x}))$. □

Corollary 2. *Suppose that M is a compact complex manifold and $f : M \to M$ is a holomorphic Axiom A map, and Λ is one of its basic sets of saddle type. Let also another holomorphic map g on M, which belongs to a close neighbourhood $\mathcal{U}$ of f inside $\mathcal{C}^0(M, M)$.*

(a) Then g has a basic set Λ_g on which it is hyperbolic and there exists a surjective map $\Phi_g : \hat{\Lambda} \to \Lambda_g$, commuting with $\hat{f}$ and g. The maps f and g have systems of local unstable/stable manifolds $W^u_\beta(f, \hat{x}), W^s_\beta(f, x), \hat{x} \in \hat{\Lambda}$, respectively $W^u_\beta(g, \hat{y}), W^s_\beta(g, y), \hat{y} \in \hat{\Lambda}_g$, for some $\beta > 0$; these manifolds are embedded complex disks, with $T_x W^u_\beta(f, \hat{x}) = E^u_{\hat{x}}, T_x W^s_\beta(f, x) = E^s_x, \hat{x} \in \hat{\Lambda}$ and similarly for g.

(b) There exists a Lipschitz family of biholomorphic maps $\Theta^u_{\hat{x}}(g) : W^u_\beta(f, \hat{x}) \to W^u_\beta(g, \hat{\Phi}_g(\hat{x})), \hat{x} \in \hat{\Lambda}$, and $\Theta^s_{\hat{x}}(g) : W^s_\beta(f, x) \to W^s_\beta(g, \Phi_g(\hat{x})), \hat{x} \in \hat{\Lambda}$ such that $\Theta^u_{\hat{x}}(g)(x) = \Phi_g(\hat{x}), \Theta^s_{\hat{x}}(g)(x) = \Phi_g(\hat{x}), \hat{x} \in \hat{\Lambda}$. Moreover there exists a constant $K_0 > 0$ satisfying:

$|\Theta^u_{\hat{x}}(g)(\xi) - \Theta^u_{\hat{x}}(\kappa)(\xi)| \leq K_0|\kappa - g|, |\Theta^s_{\hat{x}}(g)(\xi') - \Theta^s_{\hat{x}}(\kappa)(\xi')| \leq K_0|\kappa - g|, \forall g, \kappa \in \mathcal{U}, \xi \in W^u_\beta(f, \hat{x}), \xi' \in W^s_\beta(f, x)$.

Similar inequalities are also true for $(\Theta^u_{\hat{x}}(g))^{-1}, (\Theta^s_{\hat{x}}(g))^{-1}$.

Proof. The proof follows along the same lines as in the previous Theorem. Now we have to work with complex Banach spaces and with bundle maps which are holomorphic on fibers. One constructs the diffeomorphisms $\Theta^u_{\hat{x}}(g)$ as in the previous Theorem and shows that they are now bi-holomorphic based on the fact that $\mathcal{G}_g$ is holomorphic. The Lipschitz continuity of $\Theta^u_{\hat{x}}(g)$ follows directly from the Theorem. □

We are ready now to prove that the conjugacy map Φ_g is Hölder continuous on certain subsets of unstable manifolds, when g is close to f, thus extending a result of Palis and Viana ([14]). In the case of holomorphic maps on $\mathbb{P}^2$, we can actually estimate the Hölder exponent in terms of the distance $d(f, g)$, between f and g, in the uniform convergence metric. For this last estimate the holomorphicity hypothesis is essential. In the sequel, a map $\Psi : X \to Y$, between two metric spaces (X, d) and (Y, d'), is called (C, α)-*Hölder continuous* if there exists some $\delta_0 > 0$

and constants $C > 0, \alpha > 0$ such that $d'(\Psi(x), \Psi(y)) \leq C d(x,y)^\alpha, x, y \in X$ with $d(x,y) < \delta_0$.

Theorem 5. *Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be an Axiom A holomorphic map on the complex projective space $\mathbb{P}^2$, and Λ a basic set of saddle type for f. Let $g : \mathbb{P}^2 \to \mathbb{P}^2$ be another holomorphic map such that we can define families of local unstable manifolds for f and g of size β. Let also $\eta = \eta(g) := d_{\overline{B(\Lambda, 2\beta)}}(f, g) < \varepsilon$ for some small $\varepsilon > 0$, where $d_{\overline{B(\Lambda,2\beta)}}(f,g)$ denotes the distance between f, g in the sup metric on the neighbourhood $\overline{B(\Lambda, 2\beta)}$ of Λ. Then there exist $C > 0$, and $\alpha = \alpha(g) \in (0,1)$ such that for any $\hat{x} \in \hat{\Lambda}$, the restriction of Φ_g to $V^+_\beta(\hat{f}, \hat{x})$ induces a map on $W^u_\beta(f, \hat{x}) \cap \Lambda$, denoted by $\Phi_g|_{W^u_\beta(f,\hat{x})\cap\Lambda}$, which is (C, α)-Hölder continuous ; the inverse $(\Phi_g|_{W^u_\beta(f,\hat{x})\cap\Lambda})^{-1}$ is also (C,α)- Hölder continuous.*

Moreover we can take above $\alpha = \alpha(g) = \frac{\log(\lambda_u - \vartheta\eta)}{\log \lambda_u}$, where ϑ is a fixed positive constant, independent of g or η, and $\lambda_u := \inf\limits_{\hat{x}\in\hat{\Lambda}} \{|Df_u(\hat{z})|, z \in W^u_\beta(f, \hat{x})\}$, and $\hat{z}$ is the unique prehistory of z, β-shadowed by $\hat{x}$.

Proof. If y is an arbitrary point in $W^u_\beta(f, \hat{x}) \cap \Lambda$, for some $\hat{x} \in \hat{\Lambda}$, then there exists a unique prehistory $\hat{y}$ such that $d(y_{-i}, x_{-i}) < \beta, i \geq 0$, where $\hat{y} = (y, y_{-1}, ...)$ and $\hat{x} = (x, x_{-1}, ...)$. Hence $d(y_{-i}, \Lambda) < \beta, i \geq 0$, and also $f^k(y) \in \Lambda, k \geq 0$. So, from the local maximality of Λ ([18]) it follows that, if β is small enough, then $y_{-i} \in \Lambda, i \geq 0$. So for this particular prehistory we have $\hat{y} \in \hat{\Lambda}$. Thus we can define $\Phi_g|_{W^u_\beta(f,\hat{x})\cap\Lambda}(y)$ as being $\Phi_g(\hat{y})$, which is a point in Λ_g. It also follows from Theorem 4 that $\Phi_g(\hat{y}) \in W^u_{\beta'}(g, \hat{\Phi}_g(\hat{x}))$, for some small β'. Let now $\lambda_u := \inf\limits_{\hat{x}\in\hat{\Lambda}} \{|Df_u(\hat{z})|, z \in W^u_\beta(f, \hat{x}), \text{and } \hat{z} \text{ is the unique prehistory of } z,\ \beta-\text{shadowed by } \hat{x}\}$.

Assume that y, z are points in a local unstable manifold $W^u_\beta(f, \hat{x}), \hat{x} \in \hat{\Lambda}$; then there exists a positive constant χ independent of $\hat{x} \in \hat{\Lambda}$ such that

$$|Df_u(\hat{y}) - Df_u(\hat{z})| = |D(f|_{W^u_\beta(f,\hat{x})})(y) - D(f|_{W^u_\beta(f,\hat{x}}(z)| \leq \chi d_u(y,z), \tag{8}$$

where $d_u(\cdot,\cdot)$ represents the metric induced by the metric from $\mathbb{P}^2$ on $W^u_\beta(f,\hat{x})$. This is true by applying the Mean Value Inequality on the complex disk $W^u_\beta(f, \hat{x})$ and recalling that $W^u_\beta(f,\hat{x})$ varies continuously with $\hat{x} \in \hat{\Lambda}$.

Assume that the distance η between f and g is small. From Corollary 2 and Theorem 4 item 4), it follows that there exists a positive constant $\chi_1 > K_0$ such that:

$$d_u(y, \Theta^u_{\hat{x}}(g)(y)) \leq \chi_1\eta, \text{and } d_u(y, (\Theta^u_{\hat{x}}(g))^{-1}\Phi_g(\hat{y})) \leq \chi_1\eta, y \in W^u_\beta(f,\hat{x}), \hat{x} \in \hat{\Lambda} \tag{9}$$

The last inequality in (9) follows from Corollary 2 and the inequality $d(y, \Phi_g(\hat{y})) = d(\Phi_g(\hat{y}), \pi_f(\hat{y})) \leq C \cdot d(f,g) = C\eta$ (Theorem 4, 4)).

Let us recall now that the maps f, g and $\Theta^u_{\hat{x}}(g), (\Theta^u_{\hat{x}}(g))^{-1}$ are all holomorphic, for every $\hat{x} \in \hat{\Lambda}$. Thus we can apply Cauchy's inequalities on the complex disks $W^u_\beta(f,\hat{x})$ in order to bound the difference between derivatives by the difference between the original maps. By using also inequalities (8) and (9) we will then obtain:

$$|D((\Theta^u_{\hat{f}\hat{x}}(g))^{-1} \circ g \circ \Theta^u_{\hat{x}}(g))(y) - Df_u(y)| \leq \chi_2\eta, \forall y \in W^u_\beta(f,\hat{x}), \hat{x} \in \hat{\Lambda}, \tag{10}$$

where $\chi_2 > 0$ is a constant independent of $y, g, \hat{x}$.

Let two points $y, z \in W^u_\beta(f, \hat{x})$, with $0 < d_u(y,z) < \delta$ for some fixed small $\delta > 0$. Consider N to be the largest integer such that $d(f^k y, f^k z) < \delta, k \le N$. Then $d_u(f^{N+1}y, f^{N+1}z) > \delta$. From the conformality of f on $W^u_\beta(f, \hat{x})$ and the Laminated Distortion Lemma on unstable manifolds, we get that

$$d_u(f^n y, f^n z) = d_u(y,z) \cdot \prod_{j=0}^{n-1} |Df_u(f^j \xi)| \le M_1 d_u(y,z) \cdot \prod_{j=0}^{n-1} |Df_u(f^j y)|, \qquad (11)$$

where ξ is a point inside the ball $B_u(y, 2d_u(y,z))$ in the metric d_u, and M_1 is a positive universal constant. Using (8), (9), (10), we can assume that δ, ε are of the form $\upsilon_1 \cdot \eta$, respectively $\upsilon_2 \cdot \eta$, for some constants $\upsilon_1, \upsilon_2 > 0$ (independent of η), and that they satisfy:

i) if $d_u(y,z) \le 4\delta \Rightarrow |Df_u(y) - Df_u(z)| \le \varepsilon, y, z \in W^u_\beta(f, \hat{x})$;

ii) $d_u(\Theta^u_{\hat{x}}(g)^{-1}\Phi_g(\hat{y}), y) \le \delta/2, y \in W^u_\beta(f, \hat{x}) \cap \Lambda$;

iii) $|D(\Theta^u_{\hat{f}\hat{x}}(g)^{-1} \circ g \circ \Theta^u_{\hat{x}}(g))(y) - Df_u(y)| \le \varepsilon, y \in W^u_\beta(f, \hat{x})$,

for all $\hat{x} \in \hat{\Lambda}$ and g holomorphic on $\mathbb{P}^2$, g in the neighbourhood $\mathcal{U}$ of f.

Let us denote now $\Theta^u_{\hat{f}^n\hat{x}}(g)$ by Θ_n for a given map g inside $\mathcal{U}$, $n > 0$ integer, and $\hat{x} \in \hat{\Lambda}$. Then, from ii), and the way in which y, z, N were chosen above, we get that

$$d_u(\Theta_n^{-1}\Phi_g(\hat{f}^n\hat{y}), \Theta_n^{-1}\Phi_g(\hat{f}^n\hat{z})) \le 2\delta, 0 \le n \le N \qquad (12)$$

But on the other hand, $(\Theta_n^{-1} \circ g \circ \Theta_{n-1}) \circ ... \circ (\Theta_2^{-1} \circ g \circ \Theta_1)(\Theta_1^{-1} \circ g \circ \Phi_g) = \Theta_n^{-1} \circ g^n \circ \Phi_g = \Theta_n^{-1} \circ \Phi_g \circ \hat{f}^n$. So, if $y, z \in W^u_\beta(f, \hat{x}) \cap \Lambda$, and $0 \le n \le N$, we have:

$$\begin{aligned} &d_u(\Theta_n^{-1}\Phi_g(\hat{f}^n\hat{y}), \Theta_n^{-1}\Phi_g(\hat{f}^n\hat{z})) \\ &\qquad = d_u(\Theta^u_{\hat{x}}(g)^{-1}\Phi_g(\hat{y}), \Theta^u_{\hat{x}}(g)^{-1}\Phi_g(\hat{z})) \cdot \prod_{j=0}^{n-1} |D(\Theta_{j+1}^{-1} \circ g \circ \Theta_j)(\xi_j)|, \end{aligned}$$

where $\xi_j \in B_u(\Theta_j^{-1}\Phi_g(\hat{y}), d_u(\Theta_j^{-1}\Phi_g(\hat{y}), \Theta_j^{-1}\Phi_g(\hat{z}))), 0 \le j \le n-1$.

Hence using the last formula, together with (11) and (12), it follows that

$$\begin{aligned} &d_u(\Theta^u_{\hat{x}}(g)^{-1}\Phi_g(\hat{y}), \Theta^u_{\hat{x}}(g)^{-1}\Phi_g(\hat{z})) \cdot \prod_{j=0}^{N-1} |D(\Theta_{j+1}^{-1} \circ g \circ \Theta_j)(\xi_j)| \\ &\qquad \le 2\delta \le 2\delta^\alpha \le 2 \cdot d_u(y,z)^\alpha \cdot \prod_{j=0}^{N} |Df_u(f^j y)|^\alpha, \end{aligned} \qquad (13)$$

where $\alpha = \alpha(g) \in (0,1)$ is chosen such that $\lambda_u - 2\varepsilon\lambda_u^\alpha$ (we recall that $\varepsilon = \upsilon_2 \cdot \eta$).

But now $d_u(\xi_j, f^j y) < 2\delta$, so $|D(\Theta_{j+1}^{-1} \circ g \circ \Theta_j)(\xi_j)| \ge |Df_u(\xi_j)| - \varepsilon \ge |Df_u(f^j y)| - 2\varepsilon \ge |Df_u(f^j y)|^\alpha$. Indeed, if we define the real map $h : [\lambda_u, \infty) \to \mathbb{R}, h(x) = x^\alpha - x$, then $h'(x) = \alpha x^{\alpha - 1} - 1$, and since $\alpha \in (0,1)$ and $x \ge \lambda_u > 1$, we get $x^{\alpha-1} < 1$, hence h is strictly decreasing; so, if α has been chosen such that $\lambda_u - 2\varepsilon = \lambda_u^\alpha$, then $|Df_u(z)|^\alpha - |Df_u(z)| \le \lambda_u^\alpha - \lambda_u = -2\varepsilon, z \in W^u_\beta(\hat{\zeta}), \hat{\zeta} \in \hat{\Lambda}$.

Therefore, from (13), we obtain:

$$d_u(\Theta^u_{\hat{x}}(g)^{-1}\Phi_g(\hat{y}), \Theta^u_{\hat{x}}(g)^{-1}\Phi_g(\hat{z})) \le 2 \cdot |Df_u(f^N y)|^\alpha \cdot d_u(y,z)^\alpha$$

Notice however that $y, z \in W^u_\beta(f, \hat{x})$, and that for their prehistories $\hat{y}, \hat{z}$ shadowed by $\hat{x}$, we get $\Phi_g(\hat{y}), \Phi_g(\hat{z}) \in W^u_\beta(g, \hat{\Phi}_g(\hat{x}))$. Now one can apply the bi-Lipschitz continuity of the biholomorphic map $\Theta^u_{\hat{x}}(g) : W^u_\beta(f, \hat{x}) \to W^u_\beta(g, \hat{\Phi}_g(\hat{x}))$. Hence it follows that there exists a positive constant C (independent of g and $\hat{x} \in \hat{\Lambda}$) such that

$$d_u(\Phi_g(\hat{y}), \Phi_g(\hat{z})) \le C \cdot d_u(y,z)^\alpha,$$

for all $y, z \in W^u_\beta(f, \hat{x}), \hat{x} \in \hat{\Lambda}$, where the prehistories $\hat{y}, \hat{z}$ are the unique prehistories of y, z, β-shadowed by $\hat{x}$.

Lastly, notice that α was chosen such that $\lambda_u - 2\varepsilon = \lambda_u^\alpha$, where $\varepsilon = \upsilon_2 \cdot \eta$; hence $\log(\lambda_u - 2\upsilon_2 \cdot \eta) = \alpha \cdot \log \lambda_u$. Thus we obtain the formula for α,

$$\alpha = \frac{\log(\lambda_u - \vartheta \cdot \eta)}{\log \lambda_u},$$

where $\vartheta := 2\upsilon_2$ is a positive constant independent of $g \in \mathcal{U}$, and η denotes the distance in the sup metric between f and g on a neighbourhood of Λ. □

In the case of holomorphic Axiom A endomorphisms, we found therefore an estimate for the Hölder exponent α; it was proved using the fact that $\{\Theta^u_{\hat{x}}(g), \hat{x} \in \hat{\Lambda}, g \in \mathcal{U}\}$ is a Lipschitz family of biholomorphic maps. The estimate for α is not true without the holomorphicity condition. Using the previous Theorem, it is easy to prove now the following

Corollary 3. *In the setting of Theorem 5, denote by $\delta^u(g) := HD(W^u_\beta(g, \hat{x}_g) \cap \Lambda_g), \hat{x}_g \in \hat{\Lambda}_g$. Then if g is close enough to f, we have*

$$\alpha(g) \cdot \delta^u(f) \leq \delta^u(g) \leq \frac{1}{\alpha(g)} \cdot \delta^u(f),$$

where $\alpha(g) = \frac{\log(\lambda_u - \vartheta \cdot d(f,g))}{\log \lambda_u}$ and $\vartheta > 0$ is a constant independent of g. In particular $\delta^u(g) \to \delta^u(f)$ when $g \to f$.

4. **Real analyticity of the unstable dimension and existence of Gibbs states for noninvertible maps. Differences from the case of diffeomorphisms.** In the one dimensional case, Ruelle [17] showed that the Hausdorff dimension of the Julia sets of hyperbolic rational maps, depends real analytically on parameters. As explained in [9], Mihailescu and Urbanski have found examples that prove that the Hausdorff dimension of the intersection between local stable manifolds and basic sets (called *stable dimension*) does not vary continuously in the case of rational maps on $\mathbb{P}^2$, even if the parameters vary real analytically. In the sequel we will prove that, by contrast, the unstable dimension varies real analytically when the parameters vary real analytically. As proved in Section 2, the unstable dimension $\delta^u(\beta; \hat{x})$ does not depend either on $\beta > 0$ (small), nor on $\hat{x} \in \hat{\Lambda}$.

We will need to use the following fact:

Lemma 2. *Let M be a compact complex manifold and $f : M \to M$ a holomorphic map satisfying Axiom A; let Λ be one of its basic sets of saddle type. Consider now $g : M \to M$ another holomorphic map which is $\mathcal{C}^0$-close to f on Λ, i.e $g \in \mathcal{U}$, where $\mathcal{U}$ is a neighbourhood of f in the sup metric on a neighbourhood $B(\Lambda, 2\beta)$. Denote by $\Phi_g : \hat{\Lambda} \to \Lambda_g$ the conjugacy map given by Theorem 4. Then there exists $\gamma'(g) \in (0,1)$ such that the lifting $\hat{\Phi}_g : \hat{\Lambda} \to \hat{\Lambda}_g$ is $\gamma'(g)$-Hölder continuous for certain metrics d_K on $\hat{\Lambda}$, $\hat{\Lambda}_g$. The exponent $\gamma'(g)$ can be obtained as $min\{\alpha(g), \gamma\}$ where $\alpha(g)$ is given in Theorem 5, and $\gamma \in (0,1)$ such that $K^\gamma \cdot (\sup_\Lambda |Df_s| + \frac{1}{K}) < 1$. In particular $\Phi_g : \hat{\Lambda} \to \Lambda_g$ is $\gamma'(g)$-Hölder continuous, for any $g \in \mathcal{U}$.*

Let us make the observation that, if both f and g are holomorphic, it is enough to have g in $\mathcal{U}$ in order to prove that also the derivative in the stable directions of g, Dg_s, is close to Df_s. (use the Cauchy formulas for the derivative of a holomorphic function and the fact that the stable directions of g are close to the stable directions of f).

Proof. Take $\beta > 0$ as in Corollary 2, i.e such that one can form families of stable/unstable manifolds of size β for all $g \in \mathcal{U}$. According to Proposition 19.1.1 from [5], in order to prove that $\hat{\Phi}_g$ is Hölder continuous, it is enough to show that $\hat{\Phi}_g$ is Hölder continuous on the stable/unstable sets $V_\delta^-(\hat{f}, \hat{x}), V_\delta^+(\hat{f}, \hat{x})$, of its Smale space structure ($0 < \delta < \beta$). But we know from Theorem 5 that Φ_g is Hölder when restricted to $W_\beta^u(f, \hat{x}) \cap \Lambda, \hat{x} \in \hat{\Lambda}$; using the bi-Lipschitz map between $W_\beta^u(f, \hat{x}) \cap \Lambda$ and $V_\delta^+(\hat{f}, \hat{x})$ (for some appropiate δ), given by $y \to \hat{y}$, (where $\hat{y} \in \hat{\Lambda}$ is the unique prehistory of y β-shadowed by $\hat{x}$), we see that $\hat{\Phi}_g$ is Hölder on $V_\delta^+(\hat{f}, \hat{x})$.

So, it remains only to show that $\hat{\Phi}_g$ is Hölder when restricted to the stable set $V_\delta^-(\hat{f}, \hat{x})$. To this end, consider prehistories $\hat{y}, \hat{z} \in V_\delta^-(\hat{g}, \hat{x}')$, for some $\hat{x}' \in \hat{\Lambda}_g$. Then $d_K(\hat{g}\hat{y}, \hat{g}\hat{z}) = d(gy, gz) + \frac{d(y,z)}{K} + \frac{d(y_{-1},z_{-1})}{K^2} + ... \geq \frac{d(y,z)}{K} + \frac{d(y_{-1},z_{-1})}{K^2} + ... = \frac{1}{K} d_K(\hat{y}, \hat{z})$, for $g \in \mathcal{U}$; hence

$$d_K(\hat{g}\hat{y}, \hat{g}\hat{z}) \geq \frac{1}{K} d_K(\hat{y}, \hat{z}) \tag{14}$$

Let us now take a positive number $\lambda \in (0,1)$ such that $\lambda > |Dg_s(y)|$, for all $y \in W_\beta^s(g, x'), x' \in \Lambda_g, g \in \mathcal{U}$. If $\hat{x}' \in \hat{\Lambda}_g$, it follows that $\pi_g(V_\delta^-(\hat{f}, \hat{x}')) \subset W_\delta^s(g, x')$. Therefore using the above definition of λ, the Mean Value Inequality on stable disks, and the fact that $\delta < \beta$, we see that for any $\hat{y}, \hat{z} \in V_\delta^-(\hat{g}, \hat{x}')$:

$$d_K(\hat{g}\hat{y}, \hat{g}\hat{z}) \leq \lambda d(y, z) + \frac{d(y, z)}{K} + \frac{d(y_{-1}, z_{-1})}{K^2} + ... \leq (\lambda + \frac{1}{K}) d_K(\hat{y}, \hat{z}) \tag{15}$$

So, it is enough to assume that $K > 1$ is chosen so that $\Xi := \lambda + \frac{1}{K} < 1$. K is independent of $g \in \mathcal{U}$.

Next, let us recall that, for every $\varepsilon_0 > 0$, there exists $\delta_0 > 0$ such that, if $\hat{x}, \hat{y} \in \hat{\Lambda}$ and $d_K(\hat{x}, \hat{y}) < \delta_0$, then $d_K(\hat{\Phi}_g(\hat{x}), \hat{\Phi}_g(\hat{y})) < \varepsilon_0$. Now, consider $\hat{y} \in V_\delta^-(\hat{f}, \hat{x}), \hat{y} \neq \hat{x}$. Let an integer $n > 0$ such that $d_K(\hat{f}^{-n}\hat{y}, \hat{f}^{-n}\hat{x}) \leq K^n d_K(\hat{x}, \hat{y}) < \delta_0 \leq K^{n+1} d_K(\hat{x}, \hat{y})$; the first inequality follows from (14). Therefore, from the way ε_0, δ_0 were chosen, we would have $d_K(\hat{\Phi}_g \hat{f}^{-n}\hat{y}, \hat{\Phi}_g \hat{f}^{-n}\hat{x}) < \varepsilon_0$. Now we will use the conjugacy property of $\hat{\Phi}_g$: $\hat{g} \circ \hat{\Phi}_g = \hat{\Phi}_g \circ \hat{f}$, hence $\hat{\Phi}_g = \hat{g}^n \hat{\Phi}_g \hat{f}^{-n}$, for all integers $n > 0$. Thus $\hat{\Phi}_g \hat{f}^{-n}\hat{y} \in V_{\delta'}^-(\hat{g}, \hat{\Phi}_g \hat{f}^{-n}\hat{x})$, for some $\delta' > 0$ small. This, and (15) imply then the following:

$$\begin{aligned}
&d_K(\hat{\Phi}_g\hat{x}, \hat{\Phi}_g\hat{y}) \\
&= d_K(\hat{g}^n\hat{\Phi}_g\hat{f}^{-n}\hat{y}, \hat{g}^n\hat{\Phi}_g\hat{f}^{-n}\hat{x}) \quad \leq \quad \Xi^n \cdot \varepsilon_0 \\
&= \Xi^n \cdot \varepsilon_0 \cdot \frac{\delta_0^\gamma}{\delta_0^\gamma} \quad \leq \quad \Xi^n \cdot \frac{\varepsilon_0}{\delta_0^\gamma} \cdot K^{\gamma(n+1)} d_K(\hat{x}, \hat{y})^\gamma \\
&= (\Xi \cdot K^\gamma)^n \cdot \frac{\varepsilon_0 K^\gamma}{\delta_0^\gamma} \cdot d_K(\hat{x}, \hat{y})^\gamma
\end{aligned}$$

Supposing that $\gamma \in (0,1)$ is taken such that $K^\gamma \cdot \Xi < 1$, we will get from the last inequality that:

$$d_K(\hat{\Phi}_g\hat{x}, \hat{\Phi}_g\hat{y}) \leq C' \cdot d_K(\hat{x}, \hat{y})^\gamma,$$

for all $\hat{y} \in V_\delta^-(\hat{f}, \hat{x})$ and $\hat{x} \in \hat{\Lambda}$, where C' is a positive constant. In conclusion we showed that $\hat{\Phi}_g$ is (C', γ)-Hölder continuous on all stable sets $V_\delta^-(\hat{f}, \hat{x}), \hat{x} \in \hat{\Lambda}$.

To recap, we chose the constants $\lambda \in (0,1), K > 1, \gamma \in (0,1)$ and $C' > 0$ in the following fashion:

1) $\lambda > |Dg_s(y)|, y \in W^s_\beta(g, x'), x' \in \Lambda_g$;
2) $K > 1$ such that $\lambda + \frac{1}{K} < 1$;
3) $\gamma \in (0,1)$ such that $K^\gamma(\lambda + \frac{1}{K}) < 1$;
4) $C' = \frac{\varepsilon_0 \cdot K^\gamma}{\delta_0^\gamma}$.

So C' and γ depend on the constant K. We proved in Theorem 5 that $\hat{\Phi}_g$ is Hölder continuous of exponent α on $V^+_\delta(\hat{f}, \hat{x})$. Therefore, if $\gamma'(g)$ denotes $\min\{\alpha(g), \gamma\}$, then the map $\hat{\Phi}_g$ is Hölder continuous of exponent $\gamma'(g)$ on $\hat{\Lambda}$, for $g \in \mathcal{U}$. □

Theorem 6. *Let f be a holomorphic Axiom A map of degree $D \geq 2$ on $\mathbb{P}^2$, and Λ a basic set of saddle type for f. Let also $(f_a)_{a\in V}$ be a family of perturbations of f, where a is a p-multi-variable parameter, and f_a depends real analytically on a. Then if $V \subset \mathbb{R}^p$ is a small neighbourhood of 0, and $f_0 = f$, it follows that f_a has a basic set Λ_a close to Λ, f_a is hyperbolic on Λ_a, and the map $a \to \delta^u_a$ is real-analytic, where $\delta^u_a := HD(W^u_\beta(f_a, \hat{x}_a) \cap \Lambda_a), \hat{x}_a \in \hat{\Lambda}_a$.*

Proof. The number β above can be taken as in Corollary 2. Next, for $a \in V$, denote by $\phi^u_a(\hat{z}) := -\log|D(f_a)_u(\hat{z})|, \hat{z} \in \hat{\Lambda}_a$.

We use now the theorem of Hölder continuity for the unstable spaces with respect to the prehistories , see [7]; this theorem implies also the Hölder continuity of ϕ^u_a on $\hat{\Lambda}$. From that theorem, it also follows that the Hölder exponent of ϕ^u_a can be taken independent of a. Recall also the conclusion of Lemma 2 which says that for all $a \in V$, the conjugacy $\hat{\Phi}_{f_a} : \hat{\Lambda} \to \hat{\Lambda}_a$ is γ'-Hölder continuous for some $\gamma' \in (0,1)$. Thus there exists $\theta \in (0,1)$ such that for all $a \in V$, $\phi^u_a \circ \hat{\Phi}_{f_a} \in \mathcal{H}^\theta(\hat{\Lambda}, \mathbb{R})$.

We apply now a Theorem of [19], to get that the map $\psi \to P_{\hat{f}}(\psi)$ is real analytical when considered as a map $\mathcal{H}^\theta(\hat{\Lambda}, \mathbb{R}) \to \mathbb{R}$. Using a similar proof as in [6], one can also show that the map $\mathcal{U} \to H^\theta(\hat{\Lambda}, \mathbb{R}), f_a \to \phi^u_a \circ \hat{\Phi}_{f_a}$ is real analytical. Thus the composition $a \to f_a \to \phi^u_a \circ \hat{\Phi}_{f_a} \to P_{\hat{f}}(t\phi^u_a \circ \hat{\Phi}_{f_a})$ is real analytical.

So, by using The Implicit Function Theorem we see that the unique zero t^u_a of the equation $P_{\hat{f}}(t\phi^u_a \circ \hat{\Phi}_{f_a}) = 0$, depends real analytically on $a \in V$. But recall from Theorem 3 that $\delta^u_a := HD(W^u_\beta(f_a, \hat{x}_a) \cap \Lambda_a)$ is equal to t^u_a. Therefore we obtained the conclusion of the statement, i.e δ^u_a depends real analytically on a. □

We show next the existence of a geometric measure on the intersection $W^u_\beta(f, \hat{x}) \cap \Lambda$; this will imply the equality between δ^u_a and the corresponding upper (and lower) box dimension. For the definition of a *geometric measure* we refer to [16]. We say that a measure m on a metric space (X, d) is **a geometric measure of exponent t**, if there exists a number t and a constant $c > 1$ with $c^{-1}r^t \leq m(B(x,r)) \leq cr^t, x \in X, r > 0$.

Theorem 7. *In the setting of Theorem 3, and for β small, there exists a geometric measure on $W^u_\beta(f, \hat{x}) \cap \Lambda$, of exponent t^u, and the unstable dimension δ^u is equal to the upper (and lower) box dimension of the intersection $W^u_\beta(f, \hat{x}) \cap \Lambda$.*

Proof. It is enough to show the existence of a geometric measure of exponent t^u on each intersection $W^u_\beta(f, \hat{x}) \cap \Lambda$. Indeed, this would imply the equality $HD(W^u_\beta(f, \hat{x}) \cap \Lambda) = \overline{\dim}(W^u_\beta(f, \hat{x}) \cap \Lambda)\underline{\dim}(W^u_\beta(f, \hat{x}) \cap \Lambda)$, see for example [16]. Then we use the equality $t^u = HD(W^u_\beta(f,\hat{x}) \cap \Lambda), \hat{x} \in \hat{\Lambda}$. Moreover the geometric measure and the t^u-Hausdorff measure are equivalent, with bounded Radon-Nikodym derivatives.

Hence, let us prove now the existence of a geometric measure of exponent t^u on the intersection $W^u_\beta(f,\hat{x}) \cap \Lambda$. We will use Corollary 1. Since the unstable spaces depend Hölder continuously on prehistories ([7]), it follows that the potential $t^u \cdot \phi^u$ is Hölder continuous on $\hat{\Lambda}$, hence there exists a unique equilibrium measure for $t^u \cdot \phi^u$, denoted by $\hat{\mu}_u$ (in fact the measure depends also on $\hat{x}$, but to simplify notation we do not record this anymore, since $\hat{x}$ is fixed in $\hat{\Lambda}$). From Corollary 1 it also follows that $\hat{\mu}_u$ is a Gibbs state, i.e for every $\varepsilon > 0$ small, there exist positive constants $A_\varepsilon, B_\varepsilon$ with $A_\varepsilon \cdot e^{S_n(t^u\phi^u(\hat{y}))} \le \hat{\mu}_u(B_{\hat{f}}(\hat{y},\varepsilon,n)) \le B_\varepsilon \cdot e^{S_n(t^u\phi_u(\hat{y}))}$ for every $\hat{y} \in \hat{\Lambda}$. But we know that $\hat{\Lambda}$ has local product structure ([18]), so for small $\beta > 0$, we can define a map $\Psi : B(\hat{x},\beta) \to V^+_\beta(\hat{f},\hat{x}), \Psi(\hat{z}) = V^-_\beta(\hat{f},\hat{z}) \cap V^+_\beta(\hat{f},\hat{x})$. Then take $\tilde{\mu}_u := \Psi_*\hat{\mu}_u$, a measure on $V^+_\beta(\hat{f},\hat{x})$. Recall now the canonical projection $\pi_f : \hat{\Lambda} \to \Lambda$ and the fact that $d(y,z) \le d_K(\hat{y},\hat{z}) \le 2d(y,z), \hat{y},\hat{z} \in V^+_\beta(\hat{f},\hat{x})$ and K large. So, we define $\mu_u := (\pi_f)_*\tilde{\mu}_u$ which is a measure on $W^u_\beta(f,\hat{x}) \cap \Lambda$. This measure μ_u is the one given in the proof of Corollary 1. If y is a point in $W := W^u_\beta(f,\hat{x}) \cap \Lambda$, and $\hat{y}$ is the unique prehistory of y β-shadowed by $\hat{x}$, we see that given any r small, $r < \varepsilon$, there exists a unique positive integer n such that $B_f(y,\varepsilon,n) \cap W \subset B^u(y,r) \cap W \subset B_f(y,\varepsilon,n-1) \cap W$ (where $B^u(y,r)$ denotes the ball of center y and radius r in the metric d_u induced on $W^u_\beta(f,\hat{x})$). Hence, using also Lemma 1, there must exist positive constants L_3, L_4 with $L_3 \cdot \varepsilon \le |Df^n_u(\hat{y})| \cdot r \le L_4 \cdot \varepsilon$. Therefore, since μ_u comes from a Gibbs measure and since $P(t^u\phi^u) = 0$, the proof of Corollary 1 shows that there are positive constants A', B' so that for any $r > 0$ small, and $y \in W$,

$$A' \cdot r^{t^u} \le \mu_u(B^u(y,r) \cap W) \le B' \cdot r^{t^u}$$

This implies that μ_u is indeed a geometric measure of exponent t^u, on the intersection $W^u_\beta(f,\hat{x}) \cap \Lambda$. □

In the end, let us emphasize an important difference between the case of non-invertible maps and that of diffeomorphisms. For this we will use Theorem 4.1 of [9], which says that, given the holomorphic map (extendable to $\mathbb{P}^2$), $f_\varepsilon(z,w) = (z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2, w^2)$, there exist small positive constants $c(a,b,d,e)$ and $\varepsilon(a,b,c,d,e)$ such that, if $b \ne 0, 0 < |c| < c(a,b,d,e), 0 < \varepsilon < \varepsilon(a,b,c,d,e)$, we have that f_ε is injective on its basic set Λ_ε; Λ_ε being the basic set of f_ε close to $\{p_0(c)\} \times S^1$ (where $p_0(c)$ denotes the attracting fixed point of $z^2 + c$).

Since f_ε is a homeomorphism on Λ_ε for $\varepsilon > 0$, we see that $HD(\Lambda_\varepsilon) = \delta^u(f_\varepsilon) + \delta^s(f_\varepsilon)$, by a similar argument as for diffeomorphisms. Let us denote $f(z,w) = (z^2 + c, w^2)$ which can be extended as an Axiom A holomorphic map on $\mathbb{P}^2$. If $\Lambda = \{p_0(c)\} \times S^1$, denote by $\delta^u(f) := HD(W^u_\beta(f,\hat{x}) \cap \Lambda)$; then $\delta^u(f) = 1$.

From Corollary 3 it follows that $\delta^u(f_\varepsilon) \to \delta^u(f) = 1$, when $\varepsilon \to 0$.

As far as the stable dimension is concerned, from Corollary 4.2 of [9], $\delta^s(f_\varepsilon)$ does not converge towards 0, instead $\delta^s(f_\varepsilon) > \frac{\log 2}{\log|\frac{1+\sqrt{1-4c}}{2c}|}$, with c fixed, $0 < |c| < c(a,b,d,e)$. Therefore $HD(\Lambda_\varepsilon)$ does not converge towards $HD(\Lambda) = 1$. This can be summarized in the following:

Corollary 4. *Let $f_\varepsilon(z,w) = (z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2, w^2)$; then there exist small positive constants $c(a,b,d,e), \varepsilon(a,b,c,d,e)$ such that if $b \ne 0, 0 < |c| <$*

$c(a,b,d,e), 0 < \varepsilon < \varepsilon(a,b,c,d,e)$, *we have*

$$HD(\Lambda_\varepsilon) > 1 + \frac{\log 2}{\log |\frac{1+\sqrt{1-4c}}{2c}|},$$

where Λ_ε *is the basic set of* f_ε *close to* $\{p_0(c)\} \times S^1$, *and* $p_0(c)$ *is the attracting fixed point of* $z \to z^2 + c$. *In particular the Hausdorff dimensions of basic sets of perturbations do not always vary continuously, in the case of noninvertible maps.*

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