Metric properties of some fractal sets and applications of inverse pressure

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Abstract

We consider in this paper iterations of smooth non-invertible maps on manifolds of real dimension 4, which are hyperbolic, conformal on stable manifolds, and finite-to-one on basic sets. The dynamics of non-invertible maps can be very different than the one of diffeomorphisms, as was shown for example in [4], [7], [12], [17], [19], etc. In [13] we introduced a notion of inverse topological pressure P^- which can be used for estimates of the stable dimension $\delta^s(x)$ (i.e the Hausdorff dimension of the intersection between the local stable manifold $W^s_r(x)$ and the basic set Λ, $x \in \Lambda$). In [10] it is shown that the usual Bowen equation is not always true in the case of non-invertible maps. By using the notion of inverse pressure P^-, we showed in [13] that $\delta^s(x) \le t^s(\varepsilon)$, where $t^s(\varepsilon)$ is the unique zero of the function $t \to P^-(t\phi^s, \varepsilon)$, for $\phi^s(y) := \log |Df_s(y)|, y \in \Lambda$ and $\varepsilon > 0$ small. In this paper we prove that if Λ is not a repellor, then $t^s(\varepsilon) < 2$ for any $\varepsilon > 0$ small enough. In [11] we showed that a holomorphic s-hyperbolic map on $\mathbb{P}^2\mathbb{C}$ has a global unstable set with empty interior. Here we show in a more general setting than in [11], that the Hausdorff dimension of the global unstable set $W^u(\hat\Lambda)$ is strictly less than 4 under some technical derivative condition. In the non-invertible case we may have (infinitely) many unstable manifolds going through a point in Λ, and the number of preimages belonging to Λ may vary. In [17], Qian and Zhang studied the case of attractors for non-invertible maps and gave a condition for a basic set to be an attractor in terms of the pressure of the unstable potential. In our case the situation is different, since the local unstable manifolds may intersect both inside and outside Λ and they do not form a foliation like the stable manifolds. We prove here that the upper box dimension of $W^s_r(x) \cap \Lambda$ is less than $t^s(\varepsilon)$ for any point $x \in \Lambda$. We give then an estimate of the Hausdorff dimension of $W^u(\hat\Lambda)$ by a different technique, using the Holder continuity of the unstable manifolds with respect to their prehistories.

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1 Introduction; properties of inverse topological pressure

Let us start by giving some information about the particularities of the non-invertible case. In the diffeomorphism case, Bowen ([2]) proved the following:

Theorem (Bowen)**.** *Let Λ be a basic set for a $\mathcal{C}^2$ diffeomorphism $f : M \to M$. Then the following are equivalent:*

a) Λ is an attractor;

b) $m(W^s(\Lambda)) > 0$, with m the Lebesgue measure on M and $W^s(\Lambda)$ the global stable set of Λ;

c) $P_{f|_\Lambda}(\phi^u) = 0$, where $\phi^u(y) := -\log|Df|_{E^u_y}|, y \in \Lambda$.

Nevertheless in [3], Bowen gave an example of a $\mathcal{C}^1$ map f and a basic set Ω for f such that the Lebesgue measure of Ω is **positive**; hence the $\mathcal{C}^2$ hypothesis is essential. In the diffeomorphism case, it is important for the above Bowen Theorem ([2]) that there exists a foliation with local stable manifolds near the attractor.

In the sequel we will concern ourselves with the case of a basic set Λ for a smooth (for example $\mathcal{C}^2$), possibly non-invertible map f. The non-invertible (endomorphism) case is different than the diffeomorphic one. Indeed we do not have the foliation with local unstable manifolds since now the unstable manifolds depend on entire prehistories (not only on their base points). Also we do not know in general whether the number of f-preimages belonging to Λ of a point from Λ is constant or not; this number may vary along Λ which is complicating further the study. If Λ is connected, the fact that the number of preimages is constant on Λ is related to the openness of f on Λ ([14]). For a diffeomorphism f hyperbolic on a basic set Λ, the stable dimension (i.e the Hausdorff dimension of the intersection $W^s_r(x) \cap \Lambda$) is given by the zero of the function $t \to P(t\Phi^s), \Phi^s(y) := \log|Df_s(y)|, y \in \Lambda$, as was shown by Manning and McCluskey in [8]. But for hyperbolic basic sets of endomorphisms, the stable dimension is not always equal to the zero of the pressure of the potential ϕ^s, as was proved in Example 2 of [10]. Also, by contrast with the diffeomorphic case ([8]), we showed in [12] that there exists a class of perturbations of the map $(z, w) \to (z^2 + c, w^2)$ which are homeomorphisms on their respective basic sets, and thus the stable dimension is not varying continuously with the map.

In [17], Qian and Zhang studied several properties of hyperbolic endomorphisms (i.e hyperbolic non-invertible maps), in particular the case of attractors and their relationship with the pressure of the unstable potential and the existence of an SRB measure.

Bothe proved in [4] that there exists open sets of crossed solenoids which are non-invertible on their basic sets.

Also, in [19], Tsuji studied a class of dynamical systems generated by solenoidal maps of type $T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}, T(x, y) = (lx, \lambda y + f(x))$, with $l \geq 2$ an integer, $0 < \lambda < 1$ and f a $\mathcal{C}^2$ function on S^1. One can notice that T is a skew product Anosov endomorphism. One can then form the SBR measure associated to T, namely $\mu_T := \lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{i=0}^{n-1} \delta_{T^i(x)}$ and it is shown in [19] that we have this convergence for Lebesgue almost every point $x \in S^1 \times \mathbb{R}$. In the case $\lambda l < 1$ the SBR measure of T is totally singular with respect to the Lebesgue measure m since T contracts area; hence due to the fact that μ_T is T-invariant, we cannot have that μ_T is absolutely continuous with respect to m. However for the case $\lambda l > 1$, Tsuji proved that the set of (λ, f) for which the associated SBR measure μ_T is absolutely continuous with respect to the Lebesgue measure on $S^1 \times \mathbb{R}$, is an open and dense subset of $(\frac{1}{l}, 1) \times \mathcal{C}^2(S^1, \mathbb{R})$. Such derivative conditions will appear also in our results, in Theorems 4, 5. In a similar direction, Liu ([7]) studied invariant measures and their Lyapunov exponents and conditional measures on stable manifolds, for non-invertible maps (not necessarily uniformly hyperbolic).

Also in [15], we studied families $(F_\lambda)_\lambda$ of noninvertible hyperbolic skew products and used a transversality condition in order to prove a Bowen type formula for almost all parameters λ. This equation is considered on the natural extension $\hat{\Lambda}_\lambda$ of the basic set Λ_λ. We also proved the existence of conditional measures on the stable fibers, generated by an equilibrium measure, and then used them to obtain probability measures on the set of prehistories of points $x \in \Lambda$. This is something specific to endomorphisms, because for diffeomorphisms each point has a unique preimage.

In order to deal with the new phenomena and particularities of the endomorphism situation, we introduced and studied a notion of inverse pressure P^- in [13] and [14]. This inverse pressure takes into consideration consecutive preimages of points in Λ, rather than forward iterates like in the case of the usual pressure. Instead of covering with Bowen balls $B_n(x, \varepsilon)$ we use tubular unstable sets $\Lambda(C, \varepsilon)$ (where C is an n-prehistory in Λ) formed with points which have an n-prehistory ε-shadowed by C. These tubular unstable sets have the property that can be concatenated in order to form arbitrarily long prehistories. This property was used in [14] to prove that if f is open on Λ, and each point in Λ has d preimages in Λ, then the stable dimension is equal to the unique zero of the pressure functional $t \to P(t\phi^s - \log d)$. We also proved that the stable dimension in general (i.e without the openness condition) is smaller or equal than the unique zero of the inverse pressure functional $t \to P^-(t\phi^s)$. For non-invertible maps, the unstable manifolds $W^u_r(\hat{x})$ depend in general on the prehistories $\hat{x} \in \hat{\Lambda}$ (precise definitions are given below), and we may have several unstable manifolds (even infinitely many) going through the same point in the basic set Λ. This makes the usual proofs from the diffeomorphism case to break down and even generates new phenomena as we explained above.

In [11] we studied the case of a non-degenerate (hence non-invertible) holomorphic mapping on the 2-dimensional complex projective space $\mathbb{P}^2\mathbb{C}$ (denoted also by $\mathbb{P}^2$). Such a map has the form

$$f([z : w : t]) = [P(z, w, t) : Q(z, w, t) : R(z, w, t)], [z : w : t] \in \mathbb{P}^2,$$

where P, Q, R are homogeneous polynomials having the same degree. We then assumed that f is s-hyperbolic, a condition introduced by Fornaess and Sibony ([5]). This means: i) f has Axiom A and $f^{-1}(S_2) = S_2$; ii) there exists a neighbourhood U of S_1 such that $f^{-1}(S_1) \cap U = S_1$, and iii) there exists an analytic set of positive dimension outside S_1; here S_1, S_2 represent the sets of points from the non-wandering set of f where the unstable index is 1, respectively 2. For this type of holomorphic endomorphisms we proved that the global unstable set of the saddle part S_1 of the nonwandering set of f, namely $W^u(\hat{S}_1)$ has empty interior, extending thus in this case some results of Bedford and Smillie from the case of Henon maps (which are difeomorphisms). For this we used certain several complex variables techniques, Kontinuitatsatz, etc. However there remains the question of when is the Lebesgue measure of $W^u(\hat{S}_1)$ zero or when is the Hausdorff dimension of the same set strictly less than 4. For endomorphisms we do not have a laminar structure for the unstable manifolds, and thus there may appear a jump in the Hausdorff dimension.

In this paper we study this problem on hyperbolic basic sets for partially conformal maps. Hence in particular the results apply to the holomorphic case too. We start with the case of the stable dimension which we know that it is smaller than the zero $t^s(\varepsilon)$ of the inverse pressure functional $t \to P^-(t\phi^s, \varepsilon)$ from [13]. Then we prove that for any ε small, if Λ is not a repellor, then $t^s(\varepsilon)$ is strictly smaller than 2. Still this does not mean that the Hausdorff dimension of $W^u(\hat{\Lambda})$ is strictly less than 4. We do not know in general whether all disks transversal to the unstable directions intersect $W^u(\hat{\Lambda})$ in sets of Lebesgue measure zero. However if some technical conditions are satisfied we will prove that this is indeed the case (Theorem 5). We also show that the upper box dimension $\overline{dim}(W^s_r(x) \cap \Lambda)$ is strictly less than 2. Finally we will give an estimate for the $HD(W^u(\hat{\Lambda}))$ using the Holder dependence of local unstable manifolds with respect to their prehistories as in [10]. Examples of hyperbolic endomorphisms, where the above results can be applied will also be given namely perturbations of product maps and skew products with overlaps in their fibers.

Main Results: The main results of this paper are contained in Theorems 1, 2, 3, 4, and 5, and in Proposition 1. They treat conditions when the stable dimension is strictly less than 2, the stability of these conditions, the upper box dimension for the stable intersection, respectively in the first three Theorems. Theorems 4, 5 study the estimates for the Hausdorff dimension of the global unstable set of the basic set for a cf-hyperbolic map which does not have local repellors. And in Proposition 1 we give dynamical-topological and analytical conditions guaranteeing that Λ is not a local repellor. □

In the rest of this Section we give precise definitions and notations that will be used throughout the paper.

The notion of *hyperbolicity* can be extended to the non-invertible case by allowing the unstable spaces to depend on entire prehistories (for example [18]). Indeed, if M is a compact Riemannian manifold and $f : M \to M$ is a smooth map (by "smooth" in this paper we mean $\mathcal{C}^r, r \geq 2$), and Λ is an invariant set for f, then we say that f is **hyperbolic** over Λ if there exists a continuous invariant splitting of the tangent bundle $T_{\hat{\Lambda}}M$ into contracting , respectively expanding directions for Df (for more details, we refer to [10], [12], [18]); in the above, the set $\hat{\Lambda}$ denotes the **natural extension** of Λ relative to f, i.e the set of all sequences $\hat{x} := (x, x_{-1}, x_{-2}, ...)$, where $x_{-i} \in \Lambda$, and $f(x_{-i-1}) = x_{-i}, i \geq 0$.

Definition 1. The elements of $\hat{\Lambda}$ of the form $(x, x_{-1}, x_{-2}, ...)$ where $f(x_{-i-1}) = x_{-i}, i \geq 1$ and $x_0 = x$, are called *prehistories* (or *full prehistories*) of x. The n-truncation $(x, x_{-1}, ..., x_{-n})$ of a full prehistory will be called an *n-prehistory* of x. We understand by n**-prehistory** of a point $x \in \Lambda$ a finite sequence $C = (x, x_{-1}, \dots, x_{-n})$ of consecutive preimages of x, i.e $f(x_{-n}) = x_{-n+1}, \dots, f(x_{-1}) = x$. Also a point y will be called n**-preimage** of x (with respect to f) if $f^n(y) = x$ for $n \geq 1$.

We have also the shift homeomorphism $\hat f:\hat\Lambda\to\hat\Lambda$, $\hat f(\hat x)=(fx,x,x_{-1},...)$, $\hat x\in\hat\Lambda$.

Definition 2. a) Let M be a compact Riemannian manifold of real dimension 4, and $f:M\to M$ be a smooth (for example $\mathcal{C}^r, r\ge 2$) finite-to-one map, possibly non-invertible. We also assume that Λ is a compact **basic set** for f, i.e $f|_\Lambda$ is topologically transitive and there exists a neighbourhood U of Λ such that $\Lambda=\underset{n\in\mathbb{Z}}{\cap} f^n(U)$.

b) We assume that f is hyperbolic as a non-invertible map on Λ having both contracting and expanding directions, and that Λ does not intersect the critical set $\mathcal{C}_f$. We suppose that the stable index (i.e the real dimension of stable tangent spaces) over Λ is equal to 2 and that f is conformal on its stable manifolds over Λ. We will say in this case that f is **cf-hyperbolic** on Λ. □

The fact that M has real dimension 4 is not essential, we use it just to fix ideas. The same results hold in more general cases.

It is important to remark that in the case of non-invertible maps, the unstable spaces $E^u_{\hat x}$ depend in general on the entire prehistories, and not just on their base points as in the case of diffeomorphisms. One can also define **local stable and unstable manifolds**, $W^s_r(x):=\{y\in M, d(f^ix,f^iy)<r, i\ge 0\}$ and $W^u_r(\hat x):=\{y\in M, y$ has a prehistory $\hat y=(y,y_{-1},...)$, with $d(y_{-i},x_{-i})<r, i\ge 0\}$, where $\hat x=(x,x_{-1},...)\in\hat\Lambda$ and $r>0$ is some small positive number. The local stable and unstable manifolds are embedded smooth disks (since we assumed that the real dimension of $E^s_x, E^u_{\hat x}$ are both equal to 2, for all $\hat x\in\hat\Lambda$). In the case when f is a holomorphic map on $\mathbb{P}^2$ and hyperbolic on a basic set Λ, the local stable and unstable manifolds are embedded analytic disks. A priori the local unstable manifolds do not realize a lamination over Λ, (in contrast to the diffeomorphism case). Also, in the non-invertible case, we do not always have that a neighbourhood $\Lambda\cap B(x,r)$ is homeomorphic to the product $(W^s_r(x)\cap\Lambda)\times(W^u_r(\hat x)\cap\Lambda)$. So the methods from the diffeomorphism case usually break down in the non-invertible case.

For a map f as above, define also the **global stable set** of a point $x\in\Lambda$ as the union $\underset{n\ge0}{\cup} f^{-n}W^s_r(x)$, and denote it by $W^s(x)$; the global stable set of x is in fact the set $\{y, d(f^ny,f^nx)\to 0$, as $n\to\infty\}$. We also define the **global unstable set** of a prehistory $\hat x\in\hat\Lambda$ as $W^u(\hat x):=\underset{n\ge0}{\cup} f^nW^u_r(\hat x)$. The **global unstable set** of Λ, $W^u(\hat\Lambda)$, is defined as the union of all global unstable sets $W^u(\hat x)$, over all prehistories $\hat x\in\hat\Lambda$. Define also $W^u_r(\hat\Lambda):=\underset{\hat x\in\hat\Lambda}{\cup} W^u_r(\hat x)$.

Definition 3. Given a cf-hyperbolic map f on a basic set Λ and a point $x\in\Lambda$, let us denote by $\delta^s(x):=HD(W^s_r(x)\cap\Lambda)$, for some fixed small positive r (HD stands for the Hausdorff dimension). We shall say that $\delta^s(x)$ is the **stable dimension** of Λ at x (with respect to f). Also we call **stable upper box dimension** the upper box dimension of the intersection $W^s_r(x)\cap\Lambda$ for $x\in\Lambda$ and $r>0$ small and fixed.

Notation: Denote the derivative in the stable direction at x, $Df|_{E^s_x}$, by $Df_s(x)$, and the derivative in the unstable direction, $Df|_{E^u_{\hat x}}$, by $Df_u(\hat x)$ for any $\hat x\in\hat\Lambda$. $Df_s(x)$ will be called the **stable derivative** at x, and $Df_u(\hat x)$, the **unstable derivative** at $\hat x\in\hat\Lambda$. Define also the **stable potential** ϕ^s on Λ by $\phi^s(y):=\log|Df_s(y)|, y\in\Lambda$, where $|Df_s|$ represents the norm of Df_s as an

$\mathbb{R}$-linear transformation. Due to the condition $\mathcal{C}_f \cap \Lambda = \emptyset$, we know that $-\infty < \phi^s < 0$. Similarly we have the **unstable potential** on $\hat{\Lambda}$, $\phi^u(\hat{x}) := -\log|Df_u(\hat{x})|, \hat{x} \in \hat{\Lambda}$. □

From the properties of topological pressure ([20]), it follows that $t \to P(t\phi^s)$ is strictly decreasing and this function will have then a unique zero, denoted by t_*. In [10] we showed that $\delta^s(x) \leq t_*$, but we gave also examples of hyperbolic non-invertible maps where the inequality is strict. In particular in [10], there is an example when $t_* > 2$ (while $\delta^s(x) \leq 2$, being the Hausdorff dimension of a subset of a disk). In order to give a better estimate for the stable dimension, we introduced in [13] the concept of **inverse topological pressure**. We recall here for the convenience of the reader the definition and some useful properties:

Consider (X, d) a compact metric space and $f : X \to X$ a continuous surjective map. The surjectivity of f implies the existence of n-prehistories $C = (x, x_{-1}, ..., x_{-n})$ of any point $x \in X$. Given a prehistory $C = (y, y_{-1}, ...y_{-n})$, we denote by $n(C)$ its length, i.e $n(C) = n$. In the case $n(C) = \infty$, C is a full prehistory. Denote by $\mathcal{C}_n$ (or more precisely $\mathcal{C}_n(X)$), the set of prehistories of length n with elements from X, and let also $\mathcal{C}_* := \underset{n \geq 0}{\cup} \mathcal{C}_n$ (this set may also be denoted by $\mathcal{C}_n(X)$ when need be). We do not record dependence on f unless necessary. Let also $\mathcal{C}(X, \mathbb{R})$ be the space of real continuous functions on X. Let $C = (x, x_{-1}, ..., x_{-n}) \in \mathcal{C}_n$ and $\varepsilon > 0$; then we define $X(C, \varepsilon)$ to be the set of points which are ε-**shadowed by** C, defined by: $X(C, \varepsilon) := \{y \in B(x, \varepsilon), \exists y_{-1} \in f^{-1}(y) \cap B(x_{-1}, \varepsilon), ..., \exists y_{-n} \in f^{-1}(y_{-n+1}) \cap B(x_{-n}, \varepsilon)\}$. The set $X(C, \varepsilon)$ will be called also **tubular unstable set** of size ε generated by C, since for the case when f is smooth and hyperbolic on X, $X(C, \varepsilon)$ is a tubular set around an unstable manifold $W^u_\varepsilon(\hat{x})$, for a prehistory $\hat{x}$ of x starting with the elements of C. For a function $\phi \in \mathcal{C}(X, \mathbb{R})$ and an n-prehistory $C = (x, x_{-1}, ...x_{-n})$, define the consecutive sum $S_n^- \phi(C) := \phi(x) + \phi(x_{-1}) + ... + \phi(x_{-n})$. We define now the notion of **inverse topological pressure** (introduced in [13]); this notion takes into consideration the many different prehistories of points instead of their (uniquely determined) forward orbits. Thus, given a continuous surjective map $f : X \to X$, take $\phi \in \mathcal{C}(X, \mathbb{R}), \lambda \in \mathbb{R}, \varepsilon > 0$ small and N positive integer; assume also that Y is a subset of X. Then define the quantity $M_f^-(\lambda, \phi, Y, N, \varepsilon) := \inf\{\sum_{C \in \Gamma} \exp(-\lambda n(C) + S^-_{n(C)}\phi(C), \Gamma \subset \mathcal{C}_*, \text{s.t } Y \subset \underset{C \in \Gamma}{\cup} X(C, \varepsilon)$, and $n(C) \geq N, C \in \Gamma\}$. $M_f^-(\lambda, \phi, Y, N, \varepsilon)$ will also be denoted by $M^-(\lambda, \phi, Y, N, \varepsilon)$ when the map f is clear from the context. If a collection $\Gamma \subset \mathcal{C}_*$ has the property that $Y \subset \underset{C \in \Gamma}{\cup} X(C, \varepsilon)$, we will say that Γ ε-**covers** Y. Now, keep $\lambda, \phi, Y, \varepsilon$ fixed as above and let N increase. Then $\lim\limits_{N \to \infty} M^-(\lambda, \phi, Y, N, \varepsilon)$ exists as the limit of an increasing sequence; it will be denoted by $M^-(\lambda, \phi, Y, \varepsilon)$. Next let $P^-(\phi, Y, \varepsilon) := \inf\{\lambda, M^-(\lambda, \phi, Y, \varepsilon) = 0\}$. Let us also remark that, if $M^-(\lambda, \phi, Y, \varepsilon) = 0$, then $M^-(\lambda, \phi, Y, N, \varepsilon) = 0, \forall N \geq 1$. Also the limit $\lim\limits_{\varepsilon \to 0} P^-(\phi, Y, \varepsilon)$ exists and will be denoted by $P^-(\phi, Y)$.

Definition 4. The quantity $P^-(\phi, Y)$ introduced above is called **the inverse topological pressure** of ϕ on Y (relative to the map f), and $P^-(\phi, Y, \varepsilon)$ is called the ε-**inverse pressure** of ϕ on Y. When we want to emphasize the dependence of P^- on f, we will denote it by $P_f^-(\phi, Y)$, respectively $P_f^-(\phi, Y, \varepsilon)$.

When $Y = X$, we will denote the inverse pressure of ϕ on X by $P^-(\phi)$, and the ε-inverse pressure by $P^-(\phi, \varepsilon)$. A useful property which was proved in [13] says that the inverse pressure can also be computed by employing at each step only prehistories of the same length.

Proposition ([13])**.** *Assume that X is a compact metric space and $f : X \to X$ is a continuous and surjective map on X. Let $P_n^-(\phi, \varepsilon) := \inf\{\sum\limits_{C\in\Gamma} \exp(S_n^-\phi(C)), X = \bigcup\limits_{C\in\Gamma} X(C, \varepsilon), \Gamma \subset \mathcal{C}_n\}$, where $\phi \in \mathcal{C}(X, \mathbb{R})$. Then $P^-(\phi, \varepsilon) = \overline{\lim\limits_{n\to\infty}} \frac{1}{n} \log P_n^-(\phi, \varepsilon)$ and $P^-(\phi) = \lim\limits_{\varepsilon\to 0} \overline{\lim\limits_{n\to\infty}} \frac{1}{n} \log P_n^-(\phi, \varepsilon)$.*

The proof of this proposition is quite technical and is based on the fact that we can concatenate two sets $X(C, \varepsilon), C \in \mathcal{C}_n$, respectively $X(C', \varepsilon), C' \in \mathcal{C}_m$, in order to form a set $X(C", 2\varepsilon)$, with $C" \in \mathcal{C}_{n+m}$. We proved also the following property of inverse pressure:

Proposition ([13])**.** *If $\phi < 0$ on X, then the map $t \to P^-(t\phi, Y, \varepsilon)$ is strictly decreasing; if $P^-(\cdot, Y)$ is finitely valued, then also $t \to P^-(t\phi, Y)$ is strictly decreasing.*

P^- defines also a notion of *inverse entropy* $h^- := P^-(0)$. Let us remark that $0 \le h^- \le \min\{h_i, h_{top}\}$, where h_i is a notion of preimage entropy ([16]) and h_{top} is the usual topological entropy. In particular, using a theorem from [16], we see that $h^- = 0$ for a continuous function f defined on a finite graph X (for example when X is a circle or a Jordan curve). Since $\phi^s(y) := \log |Df_s(y)|, y \in \Lambda$, we see that $\phi^s < 0$; also since f is smooth we have that its topological entropy on Λ is finite, hence by the above remark, $h^-(f|_\Lambda)$ is finite; then by the previous Proposition, the function $t \to P^-(t\phi^s, \varepsilon)$ has a unique zero $t^s(\varepsilon)$, for $\varepsilon > 0$ small. Similarly as in [13], we prove:

Theorem (Estimate of the stable dimension)**.** *Let f be cf-hyperbolic on a basic set of saddle type Λ. Then $\delta^s(x) \le t^s(\varepsilon), x \in \Lambda$, for any $\varepsilon > 0$ small.*

In the sequel we shall use the estimate $\delta^s(x) \le t^s(\varepsilon)$ in order to prove that $\delta^s(x) \le t^s(\varepsilon) < 2$. Then we will use this inequality to study the Hausdorff dimension and the Lebesgue measure of the global unstable set of Λ.

2 The stable dimension $\delta^s(x)$ and the stable upper box dimension are strictly smaller than 2

We consider as before a compact Riemannian manifold M of real dimension 4 and a smooth (for example $\mathcal{C}^2$) map $f : M \to M$ which is cf-hyperbolic on a basic set Λ, according to Definition 2. We work in general with basic sets of **saddle type**, i.e for which there are both stable and unstable directions, as follows from Definition 2. We shall prove that if Λ is not a local repellor for f, then the unique zero of the ε-inverse pressure of the stable potential on such a set, $t^s(\varepsilon)$, is strictly smaller than 2. Let us give first two lemmas which will be used throughout the paper.

Lemma 1 (Laminated Distortion Lemma)**.** *Let $f : M \to M$ be a cf-hyperbolic map on a basic set of saddle type Λ. Consider also an n-prehistory $C = (x, x_{-1}, ..., x_{-n})$ in $\mathcal{C}_n(\Lambda)$ and consider a point*

$y \in \Lambda(C, \varepsilon)$, for $\varepsilon > 0$ small; then, there is a constant $C_0 > 1$ independent of n, x, y, such that, if $(y, y_{-1}, ..., y_{-n})$ is the n-prehistory of y ε-shadowed by C, then we have:

$$\frac{1}{C_0} \le \frac{|Df_s^n(y_{-n})|}{|Df_s^n(x_{-n})|} \le C_0$$

This lemma is proved in [14]. The next lemma is similar to the Volume Lemma of Bowen ([2]); the proof uses the same arguments as in [2] and [17]. We denote in general by $\mathcal{H}^s$ the s-dimensional Hausdorff measure, for $s > 0$ arbitrary. In the sequel we use the Riemannian metric on M, or the induced metric on submanifolds of M to define $\mathcal{H}^s$ in the case of subsets of M. Recall also that M has real dimension 4 and that $\mathcal{H}^4$ is equivalent as measures with the Lebesgue (volume) measure on M. If $Y \subset M$ is a submanifold of real dimension m, then if we restrict $\mathcal{H}^m$ to Y, we obtain a measure which is equivalent to the m-dimensional Lebesgue (area) measure on the submanifold Y.

Lemma 2 (Volume Lemma)**.** *In the above setting, consider $C = (x, x_{-1}, ..., x_{-n}) \in \mathcal{C}_n(\Lambda)$; then there is a constant $C_1 > 0$, independent of n or C, such that for all $\varepsilon > 0$ small,*

$$\frac{1}{C_1}\varepsilon^4 |Df_s^n(x_{-n})|^2 \le \mathcal{H}^4(M(C, \varepsilon)) \le C_1 \varepsilon^4 |Df_s^n(x_{-n})|^2$$

If Δ is any local embedded smooth disk (i.e $\Delta \subset B(x, \varepsilon)$ for some $x \in \Lambda$), transversal to the unstable directions, then $\frac{1}{C_1}\varepsilon^2 |Df_s^n(x_{-n})|^2 \le \mathcal{H}^2(\Delta \cap M(C, \varepsilon)) \le C_1 \varepsilon^2 |Df_s^n(x_{-n})|^2$, where $\mathcal{H}^2$ denotes the area measure on the disk Δ.

We will also need the following topological condition:

Definition 5. Let f be a continuous map on a compact metric space X, $f : X \to X$; we will say that f is **preimage-transitive** if any point $y \in X$ has the set of all its preimages $\mathcal{F}(y) := \{z \in X, \exists n \ge 0, f^n(z) = y\}$ dense in X.

For instance the map $f(z, w) = (z^2 + c, w^2), (z, w) \in \mathbb{C}^2$ and $|c|$ small, is preimage-transitive on $J_c \times \{0\}$, (where J_c is the Julia set of $z \to z^2 + c$). We will now give the definition of repellor slightly differently than the usual one for diffeomorphisms and will explain later the advantages of this definition in the case of endomorphisms.

Definition 6. Let a smooth ($\mathcal{C}^2$) map on a Riemannian manifold M, $f : M \to M$, and assume that f is hyperbolic on a basic set Λ. Then we say that Λ is a **local repellor** for f if there exist local stable manifolds of f contained in Λ.

For diffeomorphisms, a basic set Λ is said to be a repellor if there exists a neighbourhood U of Λ such that $f(U) \supset \bar{U}$. So Λ is not a repellor if and only if such a neighbourhood U does not exist. For endomorphisms this condition alone does not guarantee a priori that all of the local stable manifolds are not contained in Λ. This happens because of the subtle structure of foldings and overlappings for endomorphisms, which may take a point outside Λ into a point from Λ. If we want to have equivalence between our Definition 6 and the fact that there exists a neighbourhood U of Λ with $\bar{U} \subset f(U)$, then we have to assume in addition that f is preimage-transitive or that $f|_\Lambda$ is open on Λ. On the other hand one can notice that these two conditions are not stable under perturbations. Indeed let us consider the examples of maps from [12], $f_\varepsilon(z,w) = (z^2 + c + a\varepsilon z + b\varepsilon w + d\varepsilon zw + e\varepsilon w^2, w^2)$ for $b \neq 0$, $|c|$ small and ε small also, $0 < \varepsilon < \varepsilon(a,b,c,d,e)$. We showed that f_ε has a basic set Λ_ε close to $\{p_0(c)\} \times S^1$ (with $p_0(c)$ the fixed attracting point of $f_0(z) := z^2 + c$), and that f_ε is hyperbolic and a homeomorphism on Λ_ε. Due to the conjugacy between the liftings $\hat{f}|_{\hat{\Lambda}}$ and $\hat{f}_\varepsilon|_{\hat{\Lambda}_\varepsilon}$, there must exist fixed points for f_ε inside Λ_ε. But in this case such fixed points $w \in \Lambda_\varepsilon$ have only one prehistory in Λ_ε, namely $(w, w, w, \ldots)$, since f_ε is a homeomorphism on Λ_ε. So the set of preimages $\mathcal{F}(w)$ contains only the point w and thus it cannot be dense in Λ_ε. So f_ε is not preimage-transitive on Λ_ε, although f_0 is preimage-transitive on Λ.

The following Proposition gives several cases when Λ cannot be a local repellor for f.

Proposition 1. *a) Let $f : M \to M$ be a cf-hyperbolic map on Λ, a basic set of f which does not have any neighbourhood U with $f(U) \supset \bar{U}, \bar{U} \neq M$. Assume also that f is preimage-transitive on Λ, and denote by $r > 0$ the uniform size of local stable manifolds along Λ. Then for any $\tau \in (0, r)$ there exists $\gamma = \gamma(\tau) > 0$ such that for any $z \in \Lambda$ there exists $z' \in W^s_\tau(z)$ with $d(z', \Lambda) > \gamma$. In particular it follows that Λ is not a local repellor.*

b) In case $f : \mathbb{P}^2 \to \mathbb{P}^2$ is a holomorphic map on the 2-dimensional complex projective space and s-hyperbolic on a basic set of saddle type Λ, it follows that we always have the conclusion of a) along Λ.

c) The same conclusion as in a) is true if we replace the condition f preimage-transitive on Λ, with the condition $f|_\Lambda : \Lambda \to \Lambda$ is open.

Proof. As before r denotes the uniform size of local stable manifolds on Λ.

a) Suppose that the conclusion is wrong and let us prove that this will lead to a contradiction. So assume that there exists a small positive number $\varepsilon_1 \in (0, r)$ such that for any $\gamma > 0$ small, there exists $z = z(\gamma) \in \Lambda$ with $d(W^s_{\varepsilon_1}(z), \Lambda) < \gamma$.

But now, by taking a sequence of γ's of the form $(\frac{1}{n})_n$, we get a sequence of z_n's in Λ.

Due to the compactness of Λ, choose among these points a convergent sequence, which for convenience will be denoted also by $(z_n)_n$, and assume that $z_n \to w$. Now, by continuity of the stable lamination, we have that $W^s_{\varepsilon_1}(z_n) \to W^s_{\varepsilon_1}(w)$, thus $W^s_{\varepsilon_1}(w) \subset \Lambda$. So we found a local stable manifold entirely contained in Λ.

Assume now that η is a small positive number smaller than ε_1.

Let us take now an arbitrary point $y \in \Lambda$ and assume that its local stable manifold $W^s_\eta(y)$ is not contained in Λ; hence there exists a point $\zeta \in W^s_\eta(y) \setminus \Lambda$. Take the largest disk centered at ζ which does not intersect Λ and denote it by $\Delta(\zeta)$; then on the boundary of this disk there will exist at least a point $\xi \in \Lambda$. Then we can apply the preimage-transitivity for ξ. Indeed ξ has m-preimages ξ_{-m} as close as we want to w when m increases. But $W^s_{\varepsilon_1}(w) \subset \Lambda$, so through any point of $W^s_{\varepsilon_1}(w)$ there passes at least an unstable manifold which will intersect transversaly $W^s_{\varepsilon_1}(\xi_{-m})$ in a point $\chi(m)$ belonging to Λ (since Λ has local product structure as being a basic set, [6]). In fact we see that in $W^s_{\varepsilon_1}(\xi_{-m})$ there may exist only disks of radius less or equal than $Cd(\xi_{-m}, w)$, which disks do not contain some point $\chi(m)$ obtained in this fashion. Now if m is large enough, then

$d(f^m(\chi(m)), f^m(\xi_{-m}))$ can be made as small as we want and since Λ is f-invariant, it follows that $f^m(\chi(m))$ is in Λ. But this implies that for any $\rho > 0$ small, there exist points from Λ in any disk of radius ρ contained in a larger disk of radius $r(\rho)$ centered at ξ. This contradicts the fact that ξ is on the boundary of the disk $\Delta(\zeta)$ with $\Delta(\zeta) \cap \Lambda = \emptyset$. But y was taken arbitrarily in Λ, so for any point $y \in \Lambda$, there exists a stable manifold $W^s_\eta(y)$ which is contained in Λ (with $\eta > 0$ fixed, and $\eta < \varepsilon_1$); we assume also without loss of generality that $\eta < \varepsilon_0$, where ε_0 is the injectivity constant of f near Λ.

Using the above property, we will prove that there exists a neighbourhood V of Λ such that $\bar{V} \subset\subset f(V)$. Let two positive numbers $\rho \in (0, \eta)$ and $\rho' = \rho'(\rho) \in (\rho, \eta)$ so that, if $\hat{x}$ is a prehistory in $\hat{\Lambda}$, and $y \in B(x, \rho) \setminus \Lambda$ and y_{-1} is the preimage of y close to x_{-1}, then $y_{-1} \in B(x_{-1}, \rho')$; moreover we assume that $B(z, r') \cap W^s_{\varepsilon_1}(z) \subset W^s_\eta(z), z \in \Lambda$. Due to the fact that $W^s_\eta(x) \subset \Lambda$, we know that the point y from $B(x, \rho) \setminus \Lambda$, has a preimage y_{-1} such that $d(y_{-1}, W^s_\eta(x_{-1})) \le \lambda \cdot d(y, W^s_\eta(x))$ for some fixed $\lambda \in (0, 1)$ independent of x, y, η. This holds because of the hiperbolicity of f on Λ and since all the contracting directions are contained in Λ, so the (uniform) unstable directions are transversal to $W^s_\eta(x)$ and hence distances between preimages of f decrease. If $y \in B(x, \rho) \setminus \Lambda$ then we saw that $y_{-1} \in B(x_{-1}, \rho')$ for a preimage x_{-1} in Λ of x. But recall that the entire $W^s_\eta(x_{-1})$ is contained in Λ, so there must exist a point $\zeta \in W^s_\eta(x_{-1})$ with $d(\zeta, y_{-1}) < \rho$; hence there exists a preimage y_{-2} of y_{-1}, with $y_{-2} \in B(\zeta_{-1}, \rho')$ for some preimage ζ_{-1} in Λ of ζ. Recall also that $d(y_{-2}, W^s_{\rho'}(\zeta_{-1})) \le \lambda \cdot d(y_{-1}, W^s_\eta(x_{-1}))$. In conclusion, by repeating this procedure, we shall find a sequence of consecutive preimages $(y, y_{-1}, y_{-2}, ...)$, with $d(y_{-n}, \Lambda) < \eta, n \ge 0$. Thus, from the local maximality of Λ, it follows that $y \in W^u_\eta(\hat{\Lambda})$. So we proved that there exists a neighbourhood V of Λ such that $\bar{V} \subset f(V)$, and $V \subset W^u_\eta(\hat{\Lambda})$. This implies then a contradiction with the hypothesis.

Therefore, for any small $\tau > 0$ there is $\gamma = \gamma(\tau) > 0$ such that for any $z \in \Lambda$, there is $z' \in W^s_\tau(z)$ with $d(z', \Lambda) > \gamma$; the conclusion of (a) is proved.

b) We are now in the case when $f : \mathbb{P}^2 \to \mathbb{P}^2$ is holomorphic. Redoing the argument in the proof of (a), we see that through any point y of Λ there passes a complex analytic disk $W^s_\eta(y)$, contained in Λ. Hence from a theorem of Takeuchi, $\mathbb{P}^2 \setminus \Lambda$ is a domain of holomorphy ([5] for references). But by hypothesis we have that $\Lambda \cap \mathcal{C}_f = \emptyset$, so $\mathcal{C}_f \subset \mathbb{P}^2 \setminus \Lambda$, which is a contradiction with another theorem of Takeuchi which says that a domain of holomorphy in $\mathbb{P}^2$ (different from $\mathbb{P}^2$) cannot contain a complex variety of positive dimension (like $\mathcal{C}_f$). So the conclusion of (a) is true in this case too.

c) For the case when $f|_\Lambda$ is open, we see that this condition implies that there exists $\varepsilon_1 > 0$ such that $f^{-1}(\Lambda) \cap B(\Lambda, \varepsilon_1) = \Lambda$ (where $B(\Lambda, \varepsilon_1)$ stands for the union of the balls centered at points of Λ, of radii ε_1). Indeed we have the following:

Topological Fact:

If $f|_\Lambda$ is open, then there exists $\varepsilon_1 > 0$ small enough such that $f^{-1}(\Lambda) \cap B(\Lambda, \varepsilon_1) = \Lambda$.

Proof of Topological Fact :

Denote by ε_0 the injectivity constant of f near Λ, i.e a number $\varepsilon_0 > 0$ so that f is injective on balls of radius ε_0 centered on Λ. Let us assume that the Topological Fact is not true; then for any $\varepsilon > 0$ small, there exists a point $z_\varepsilon \in B(y_\varepsilon, \varepsilon) \setminus \Lambda$ such that $f(z_\varepsilon) = x_\varepsilon \in \Lambda$ and $y_\varepsilon \in \Lambda$.

But then when $\varepsilon \to 0$, it follows that one can extract a subsequence of points z_ε converging towards a point z, and a subsequence of y_ε converging towards a point $y \in \Lambda$; without loss of generality these subsequences can be denoted in the same way, i.e $z_\varepsilon \to z, y_\varepsilon \to y$ and $x_\varepsilon \to x$, with $z_\varepsilon \in B(y_\varepsilon, \varepsilon) \setminus \Lambda, f(z_\varepsilon) = x_\varepsilon \in \Lambda$. From this we obtain that $f(y) = x$, and since $f|_\Lambda : \Lambda \to \Lambda$ is open, there exists $\eta(\varepsilon)$ small with $\eta(\varepsilon) \underset{\varepsilon \to 0}{\to} 0$ such that we can find $\xi_\varepsilon \in B(y, \eta(\varepsilon)) \cap \Lambda$ satisfying $f(\xi_\varepsilon) = x_\varepsilon$. But on the other hand we have $f(z_\varepsilon) = x_\varepsilon$ and $z_\varepsilon \in B(y, \varepsilon_0/2)$ for ε small. Therefore we found two points $z_\varepsilon, \xi_\varepsilon$ in $B(y, \varepsilon_0)$ with $f(z_\varepsilon) = f(\xi_\varepsilon) = x_\varepsilon \in \Lambda$, which is a contradiction with the injectivity of f on $B(y, \varepsilon_0)$. So there must exist a positive number ε_1 with $f^{-1}(\Lambda) \cap B(\Lambda, \varepsilon_1) = \Lambda$. This finishes the proof of the Topological Fact. □

Coming back to the proof of c), the idea is that we can use this Topological Fact and the transitivity of f on the basic set Λ to pull back (i.e to take preimages) of local stable manifolds near any point of Λ. By this pull back, the size of the local stable manifolds is enlarged.

More precisely, if the conclusion of a) would be wrong then there would exist a local stable manifold $W^s_\tau(z) \subset \Lambda$. Take now another point y of Λ and fix some η, with $0 < \eta < \varepsilon_1$. Consider also a neighbourhood $B(z, \varepsilon) \cap \Lambda$, of z in Λ, for some small $0 < \varepsilon < \eta$. Then, from transitivity, there is a point $\xi \in B(z, \varepsilon) \cap \Lambda$, a large integer m, and a preimage ξ_{-m} of ξ (with respect to f^m), such that $\xi_{-m} \in B(y, \varepsilon/2) \cap \Lambda$. But now, for any prehistory $\hat\xi \in \hat\Lambda$ of ξ, containing ξ_{-m} on the $(m+1)$-th position, the unstable manifold $W^u_\varepsilon(\hat\xi)$ intersects $W^s_\eta(z)$ in a point $\zeta \in \Lambda$ (from the local product structure of basic sets, [6]). Therefore this point ζ will have a preimage ζ_{-m} which is ε-close to y.

We can take m arbitrarily large, hence we obtain that $W^s_\eta(\zeta_{-m}) \subset f^{-m}(W^s_{\varepsilon_1}(z)) \subset f^{-m}(\Lambda) \cap B(\Lambda, \varepsilon_1)$, so $W^s_\eta(\zeta_{-m}) \subset \Lambda$ by the above Topological Fact applied at each inverse iterate of order less or equal than m.

We consider now $\varepsilon \to 0$ and take for each such ε, the points $\xi, \xi_{-m}, \zeta_{-m}$ for $m = m(\varepsilon) \underset{\varepsilon \to 0}{\to} \infty$; thus one sees that $\zeta_{-m} \to y$ when $\varepsilon \to 0$ since $\zeta_{-m} \in B(y, \varepsilon)$. Now, from the continuity of local stable manifolds, we see that $W^s_\eta(\zeta_{-m}) \to W^s_\eta(y)$ when $\varepsilon \to 0$; but we proved that $W^s_\eta(\zeta_{-m}) \subset \Lambda, 0 < \varepsilon < \varepsilon_1$, therefore $W^s_\eta(y) \subset \Lambda$. But y was taken arbitrarily in Λ, so for any point $y \in \Lambda$, there exists a stable manifold $W^s_\eta(y)$ which is contained in Λ (with $\eta > 0$ fixed, and $\eta < \varepsilon_1$), and assume also that $\eta < \varepsilon_0$, where ε_0 is the injectivity constant of f near Λ. Using the above property, we will prove that there exists a neighbourhood V of Λ such that $V \subset\subset f(V)$. Let then two positive numbers $\rho \in (0, \eta)$ and $r' \in (\rho, \eta)$ so that, if $\hat x$ is a prehistory in $\hat\Lambda$, and $y \in B(x, \rho)$, and y_{-1} is the preimage of y close to x_{-1}, then $y_{-1} \in B(x_{-1}, r')$; moreover we assume that $B(z, r') \cap W^s_{\varepsilon_1}(z) \subset W^s_\eta(z), z \in \Lambda$. Due to the fact that $W^s_\eta(x) \subset \Lambda$, we know that y has a preimage y_{-1} such that $d(y_{-1}, W^s_\eta(x_{-1})) \le \lambda \cdot d(y, W^s_\eta(x))$, for some fixed $\lambda \in (0,1)$ independent of x, y, η. If $y \in B(x, \rho)$, then $y_{-1} \in B(x_{-1}, r')$. But recall that the entire $W^s_\eta(x_{-1})$ is contained in Λ, so there must exist a point $x^1 \in W^s_\eta(x_{-1})$ with $d(x^1, y_{-1}) < \rho$; hence there exists a preimage y_{-2} of y_{-1}, with $y_{-2} \in B(x^1_{-1}, r')$, for some preimage x^1_{-1} of x^1, in Λ.

We recall also that $d(y_{-2}, W^s_{r'}(x^1_{-1})) \le \lambda \cdot d(y_{-1}, W^s_\eta(x_{-1}))$. In conclusion, by repeating this procedure, we shall find a sequence of consecutive preimages $(y, y_{-1}, y_{-2}, ...)$, with $d(y_{-n}, \Lambda) < \eta, n \ge 0$. Thus, from the local maximality of Λ, it follows that $y \in W^u_\eta(\hat\Lambda)$. So we proved that there exists a neighbourhood V of Λ such that $\bar V \subset f(V)$, and $V \subset W^u_\eta(\hat\Lambda)$.

□

We will use in the sequel the following Covering Theorem, proved in [12], as a consequence of the classical Besicovitch Theorem:

Theorem (Covering Theorem)**.** *Let A be a bounded set of $\mathbb{R}^m$; assume that A is covered by a family of balls $\{B(x_i, r_i)\}_{i\in I}$ centered at some points x_i of A, where $r_i > 0, i \in I$. Then there exists a cover of A with balls $\{B(x_j, 2r_j)\}_{j\in J}$, where $J \subset I$ and the multiplicity of this cover is bounded by a universal constant $b(m)$ depending only on the dimension m.*

Proposition 2. *(a) Let f be cf-hyperbolic and preimage-transitive on Λ and assume that Λ is not a repellor. Then for any arbitrary given point $x \in \Lambda$, $m_s(W) = 0$, where $W := W^s_r(x) \cap \Lambda$ (for some $r > 0$ small) and m_s is the Lebesgue measure on $W^s_r(x)$.*

(b) In the setting of (a), for any $\kappa \in (0, 1)$, there exists a positive integer $N = N(\kappa)$ and a covering of W with sets of the form $M(C', r), C' \in \Gamma \subset \mathcal{C}_N$ such that

$$\sum_{C'\in\Gamma} m_s(M(C', r) \cap W^s_r(x)) < \kappa \cdot m_s(W^s_r(x)),$$

for any point $x \in \Lambda$ and $r > 0$ small enough; $N(\kappa)$ is independent of x, r.

Same conclusions follow also when f is holomorphic on $\mathbb{P}^2$ and s-hyperbolic on a basic set Λ.

Proof. First, let us notice that the measure $\mathcal{H}^2$ restricted to $W^s_r(x)$ is equivalent with the Lebesgue measure m_s on $W^s_r(x)$.

(a) If Λ is not a repellor, it follows from the last Lemma that there exists $r > 0$ and $\gamma > 0$ such that for any point $z \in \Lambda$, there exists $z' \in W^s_r(z)$ with $d(z', \Lambda) > \gamma$; the same conclusion follows also in the case when f is holomorphic on $\mathbb{P}^2$ and s-hyperbolic on Λ.

If $y \in W^s_r(z)$, for some $z \in \Lambda$, and if $\eta > 0$, denote by $B_s(y, \eta)$ the intersection $B(y, \eta) \cap W^s_r(z)$. Take also some small $\delta = \delta(\gamma) \in (0, r)$ such that $d(B(z', \delta), \Lambda) > \gamma/2$, for all $z \in \Lambda$ and z' as above. Consequently there exists some constant $\beta = \beta(r) \in (0, 1)$, independent of z, z', such that

$$m_s(B_s(z', \delta)) > \beta \cdot m_s(B_s(z, r)), \text{and } B_s(z', \delta) \cap \Lambda = \emptyset \tag{1}$$

We want to prove that $m_s(W^s_r(x) \cap \Lambda) = 0$, for any point $x \in \Lambda$. For this let us take an arbitrary point $y \in \Lambda$, a local stable manifold $W^s_\rho(y)$, and then an arbitrary point $z \in \Lambda$; we know that there exists the point $z' \in W^s_r(z)$ with $B_s(z', \delta) \cap \Lambda = \emptyset$. Assume that δ is the largest number with this property; it must be less or equal to r since $z \in \Lambda$. Then from the maximality of δ it follows that there exists a point w on the boundary of $B_s(z', \delta)$ which is in Λ. Then from the preimage-transitivity of f on Λ, it follows that w has preimages as close as we want to y. So there exists a j-preimage w_{-j} of w (for j large enough) such that there exists a local inverse j-iterate $f^{-j}_{w_{-j}}(B_s(w, 2\delta))$ of $B_s(w, 2\delta)$ which is very close to $W^s_\rho(y)$. When j increases, $f^{-j}_{w_{-j}}(B_s(w, 2\delta)) \cap B(y, \rho)$ becomes as close as we want to $W^s_\rho(y)$. But since w is on the boundary of $B_s(z', \delta)$ which is outside Λ, it follows that there is always a subset of $B_s(w, 2\delta) \setminus \Lambda$ whose Lebesgue measure is larger than a fixed percentage larger than $\frac{1}{4}$, from the area of $B_s(w, 2\delta)$. And then all the preimages of $B_s(w, 2\delta) \setminus \Lambda$

are outside Λ (otherwise their forward iterates would be in Λ, which is f-invariant). So if j is large enough it follows that $f_{w_{-j}}^{-j}(B_s(w, 2\delta)) \cap B(y, \rho)$ has a certain subset outside Λ, whose area is larger than $\frac{1}{4}$ of $m_s(f_{w_{-j}}^{-j}(B_s(w, 2\delta)) \cap B(y, \rho))$. Since this area is in fact a disk sector (we use always the fact that f is conformal on stable manifolds), we obtain that in $W_\rho^s(y) \setminus \Lambda$ there is a subset whose area is larger than $\frac{1}{4} m_s(W_\rho^s(y))$. But since this is happening for all $\rho \in (0, r)$, it follows that y is not a point of Lebesgue density along stable manifolds, and hence there are no Lebesgue density points in $W_r^s(x) \cap \Lambda$. Thus $m_s(W_r^s(x) \cap \Lambda) = 0$, for any $x \in \Lambda$.

(b) Let κ arbitrary in the interval $(0, 1)$, and $r > 0$ fixed, as in (a).

We proved in (a) that $m_s(W_r^s(x) \cap \Lambda) = 0, \forall x \in \Lambda$. Then, from applying inductively the uniform procedure of constructing disk sectors in $W_\rho^s(y) \setminus \Lambda$ detailed in a), we see that for any $\theta \in (0, 1)$ there exists $N = N(\theta) > 1$ independent of $x \in \Lambda$, such that there exists a covering $\tilde{\Gamma}_x \subset \mathcal{C}_N$ of $W_r^s(x) \cap \Lambda$ with

$$m_s(\underset{C \in \tilde{\Gamma}_x}{\cup} M(C, r) \cap W_r^s(x)) < \theta$$

But the sets $M(C, r) \cap W_r^s(x), C \in \tilde{\Gamma}_x$ can be assimilated with disks since f is conformal on stable manifolds. So we can use the previous Covering Theorem and obtain a subcover with multiplicity bounded by a universal constant b. We will denote this subcover by Γ_x; hence we obtain

$$\sum_{C \in \Gamma_x} m_s(W_r^s(x) \cap M(C, r)) \leq b\theta,$$

so the conclusion of b) follows for $\kappa = b\theta$.

□

If f is a smooth function on the real 4-dimensional manifold M, then the **stable potential** $\phi^s(y)$ is computed as $\log |Df_s(y)|, y \in \Lambda$, where $|Df_s|$ represents the norm of the $\mathbb{R}$-linear transformation Df_s between real 2-dimensional vector spaces. Now we prove that, if Λ is not a local repellor, then the Hausdorff dimension of the intersection of any stable manifold with Λ, is strictly less than 2.

Theorem 1. *Let M be a smooth compact Riemannian manifold of real dimension 4 and $f : M \to M$ be a cf-hyperbolic map on a basic set of saddle type Λ which is not a local repellor. Then for any point $x \in \Lambda$, we have $\delta^s(x) \leq t^s(\varepsilon) < 2$, for some $\varepsilon > 0$. In particular this holds also in the case of a holomorphic map $f : \mathbb{P}^2 \to \mathbb{P}^2$ which is s-hyperbolic on a basic set of saddle type Λ.*

Proof. Denote by $W := W_r^s(x) \cap \Lambda$, for a point $x \in \Lambda$. We will denote by m_s the induced Lebesgue measure on a local stable manifold.

We know from the Introduction that $\delta^s(x) \leq t^s(\varepsilon), \varepsilon > 0$ small, where $t^s(\varepsilon)$ is the unique zero of the function $t \to P^-(t\phi^s, \varepsilon)$, with $\phi^s(y) := \log |Df_s(y)|, y \in \Lambda$. Consider a fixed $\varepsilon > 0$ small enough (in particular $\varepsilon < \varepsilon_0$). We will show that $P^-(2\phi^s, \varepsilon) < 0$, which will imply that $t^s(\varepsilon) < 2$.

In order to do this, recall first that $P^-(2\phi^s, \varepsilon)$ can be computed using $P_n^-(2\phi^s, \varepsilon)$ (from the Introduction). But from the Laminated Distortion Lemma we know that there exists a constant $\chi > 0$ such that, if $\omega \in M(C, \varepsilon), C = (y, y_{-1}, ..., y_{-n}) \in \mathcal{C}_n(\Lambda)$, and $(\omega, \omega_{-1}, ..., \omega_{-n})$ is the corresponding prehistory of ω which is ε-shadowed by C, then $\frac{1}{\chi}|Df_s^n(y_{-n})| \leq |Df_s^n(\omega_{-n})| \leq \chi|Df_s^n(y_{-n})|$.

Therefore we can write $P_n^-(2\phi^s,\varepsilon) = \omega(\varepsilon)\cdot \inf\{\sum_{C\in\Gamma} m_s(M(C,\varepsilon)), \Gamma \subset \mathcal{C}_n, \Gamma\varepsilon - \text{covering } \Lambda\}$, with $\omega(\varepsilon)$ some positive function of ε and $m_s(M(C,\varepsilon)) := m_s(W^s(y,\varepsilon)\cap M(C,\varepsilon))$. Then we have

$$P^-(2\phi^s,\varepsilon) = \overline{\lim_{n\to\infty}} \frac{\log P_n^-(2\phi^s,\varepsilon)}{n}$$

The idea will be to find a number $\upsilon \in (0,1)$ and a positive integer $N = N(\upsilon)$, such that for $n > N$, we have $P_{n+N}^-(2\phi^s,\varepsilon) \le \upsilon \cdot P_n^-(2\phi^s,\varepsilon)$.

Now, let an arbitrarily small $\varepsilon' > 0$ and find an integer n and a collection $\Gamma \subset \mathcal{C}_n$ such that $P^-(2\phi^s,\varepsilon) \le \varepsilon' + \frac{\log(\sum_{C\in\Gamma} m_s(M(C,\varepsilon))}{n}$, where $C = (y, y_{-1}, ..., y_{-n}) \in \Gamma$. Due to the fact that Λ is not a local repellor for f, there are no local stable manifolds contained in Λ, hence there will exist a positive integer $N = N(\varepsilon)$ such that for any $z \in \Lambda$, we can cover the set $\Lambda \cap W_\varepsilon^s(z)$ with sets of the form $M(C',\varepsilon), C' \in \Gamma_z \subset \mathcal{C}_N$ such that

$$\sum_{C'\in\Gamma_z} m_s(M(C',\varepsilon)\cap W_\varepsilon^s(z)) \le \upsilon \cdot m_s(W_\varepsilon^s(z)),$$

for some $\upsilon \in (0,1)$. The collection Γ_z depends on z, but N is independent of z. Consider now the collection $\Gamma \subset \mathcal{C}_n(\Lambda)$ found above, which ε-covers Λ. For each prehistory $C = (y, y_{-1}, ..., y_{-n}) \in \Gamma$ we can cover the set $\Lambda \cap W_\varepsilon^s(y_{-n})$ with sets of the form $M(C',\varepsilon)$, where $C' \in \Gamma(C) \subset \mathcal{C}_N$, for N found above; this cover $\Gamma(C)$ is in fact the family $\Gamma_{y_{-n}}$, and hence satisfies the condition:

$$\sum_{C'\in\Gamma(C)} m_s(M(C',\varepsilon)\cap W_\varepsilon^s(y_{-n})) \le \upsilon \cdot m_s(W_\varepsilon^s(y_{-n})) \tag{2}$$

Consider now a positive integer n and a prehistory $C \in \mathcal{C}_n(\Lambda), C = (y, y_{-1}, ..., y_{-n})$ like above; assume also that f_*^{-n} is the local inverse iterate of f, which takes y into y_{-n}; then $f_*^{-n}(M(C,\varepsilon)\cap W_\varepsilon^s(y)) \subset W_\varepsilon^s(y_{-n})$. Let us see now what happens to the points in $M(C,\varepsilon)$ after applying f_*^{-n}: the points in $M(C,\varepsilon)\cap W_\varepsilon^s(y)$ are taken by f_*^{-n} into $W_\varepsilon^s(y_{-n})$, while the points outside $W_\varepsilon^s(y)$ will be taken into points which are $(\lambda')^n$-close to $W_\varepsilon^s(y_{-n})$, for some $\lambda' \in (0,1)$ (λ' does not depend on n, y, C). Recall also that we cover each set $W_\varepsilon^s(y_{-n}) \cap \Lambda$ for $C = (y, y_{-1}, ..., y_{-n}) \in \Gamma$, with sets of the form $M(C',\varepsilon), C' \in \Gamma(C)$, where $\Gamma(C) \subset \mathcal{C}_N$. Therefore from the above discussion it follows that, if n is large enough in comparison to N, i.e if $n > n(N)$, then $\bigcup_{C'\in\Gamma(C)} M(C',\varepsilon)$ is an open neighbourhood of $W_\varepsilon^s(y_{-n})\cap\Lambda$, and so it contains the local inverse iterate $f_*^{-n}(M(C,\varepsilon))$. This means that we obtain a cover of Λ with sets of type $M(CC', 2\varepsilon), C \in \Gamma, C' \in \Gamma(C)$, where $\Gamma \subset \mathcal{C}_n(\Lambda)$, $\Gamma(C) \subset \mathcal{C}_N(\Lambda)$, and $n > n(N)$; CC' represents the prehistory obtained by concatenation of C and then C' ([14] for more details on the concatenation procedure). The new collection obtained from these concatenations CC' is called Γ' and we see that $\Gamma' \in \mathcal{C}_{n+N}(\Lambda)$. Then after multiplying by $|Df_s(y_{-n})|^n$ in both sides of (2), we obtain from the fact that f is conformal on stable manifolds that:

$$\sum_{C'\in\Gamma(C)} m_s(M(CC',\varepsilon)) \le \upsilon \cdot m_s(M(C,\varepsilon))$$

So there exists positive integers N and $n(N)$ such that for all $n > n(N)$ we have:

$$P_{n+N}^-(2\phi^s,\varepsilon) \le \upsilon \cdot P_n^-(2\phi^s,\varepsilon)$$

But then $P^-_{n+kN} \leq \upsilon^k \cdot P^-_n(2\phi^s, \varepsilon), k \geq 1$, therefore $\log P^-_{n+kN}(2\phi^s, \varepsilon) \leq k \log \upsilon + \log P^-_n(2\phi^s, \varepsilon)$, hence

$$P^-(2\phi^s, \varepsilon) \leq \frac{\log \upsilon}{N} < 0,$$

The last inequality follows since $\upsilon \in (0,1)$. In conclusion we obtained $t^s(\varepsilon) < 2, \varepsilon > 0$ small. Since from the Introduction, $\delta^s(x) \leq t^s(\varepsilon), x \in \Lambda$, we obtain the announced conclusion, i.e $\delta^s(x) < 2$ for all $x \in \Lambda$.

The holomorphic case follows similarly.

□

We will show next that the condition that Λ is not a local repellor for f is stable under perturbations, by proving that the stable dimension remains strictly less than 2 for perturbations g of f. By the Conjugacy Theorem for perturbations ([18]; see also [10]), if g is a perturbation of f, then there exists a basic set Λ_g close to Λ so that g is hyperbolic on Λ_g and there exists a Holder continuous homeomorphism $\Phi_g : \hat{\Lambda} \to \hat{\Lambda}_g$ conjugating $\hat{f}$ with $\hat{g}$. However this Theorem alone does not give us the stability of the property that Λ is not a local repellor for f, since it does not control the local stable manifolds of size much smaller than $d_{\mathcal{C}^1}(f, g)$. For the stability issue we need the following:

Theorem 2. *Let f be a cf-hyperbolic map on a basic set Λ which is not a local repellor. Then for any perturbation g close enough to f (in the $\mathcal{C}^2$ topology), the corresponding basic set Λ_g is not a local repellor for g either.*

Proof. First let us notice that the proof of Theorem 1 works if, for some small ε, there are no local stable manifolds of size ε contained in Λ. This is guaranteed if Λ is not a local repellor for f. From this and the fact that f is cf-hyperbolic it follows that $\delta^s(x) \leq t^s(\varepsilon) < 2$ for all $x \in \Lambda$. But if ε is fixed then there exists a number $\rho = \rho(\varepsilon) > 0$ such that if $d_{\mathcal{C}^2}(f, g) < \rho$, then the local stable manifolds of size $\varepsilon/2$ relative to g, $W^s_{\varepsilon/2}(y, g)$ are not contained in Λ_g, for any $y \in \Lambda_g$. Thus we can repeat the proof of Theorem 1 and obtain that $\delta^s(y, g) := HD(W^s_{\varepsilon/2}(y, g) \cap \Lambda_g) < 2$. Therefore if $\delta^s(y, g) < 2, \forall y \in \Lambda_g$, we obtain that there are no local stable manifolds of size less than $\varepsilon/2$ contained in Λ_g, since otherwise $\delta^s(y, g)$ would be equal to 2; hence Λ_g is not a local repellor for g.

□

This Theorem gives us many classes of examples of maps and corresponding basic sets which are not local repellors, by taking perturbations of some known examples. In particular for these perturbations one can apply Theorem 1, 4, 5.

In the remainder of this Section we will prove an additional theorem, showing that in the above setting we have $\overline{dim}_B(W^s_r(x) \cap \Lambda) \leq t^s(\varepsilon)$, where $\overline{dim}_B$ denotes the upper box (Minkowski) dimension ([9]). First let us remind the definition of upper box dimension.

Definition 7. Let A be a non-empty bounded set of $\mathbb{R}^n$. For $0 < \varepsilon < \infty$, denote by $N(A, \varepsilon)$ the smallest number of balls of radius ε necessary to cover A. Then the **upper box dimension** of A is defined as: $\overline{dim}_B(A) := \inf\{s, \limsup_{\varepsilon \to 0} N(A, \varepsilon)\varepsilon^s = 0\}$.

Theorem 3. *Let f a cf-hyperbolic map on a basic set Λ. Then for any point $x \in \Lambda$ we have $\overline{dim}_B(W^s_r(x) \cap \Lambda) \le t^s(\varepsilon)$, for $\varepsilon > 0$ small.*

Proof. Let us fix a point x from Λ. We will use the inverse pressure ([13]) and coverings of $W := W^s_r(x) \cap \Lambda$ with sets of type $\Lambda(C, \varepsilon)$ having the same stable diameter. Let us consider a number $t > t^s(\varepsilon)$. There exists then a positive integer n_0 and a finite family $\Gamma \subset \mathcal{C}_{n_0}$ such that $\Lambda = \underset{C \in \Gamma}{\cup} \Lambda(C, \varepsilon)$ and if n_0 is large enough, then:

$$\sum_{C \in \Gamma} \text{diam}(\Lambda(C, \varepsilon) \cap W^s_r(x))^t \le A \sum_{C \in \Gamma} e^{S^-_{n_0}(t\Phi^s)(C)} < \frac{1}{2} \tag{3}$$

The above inequality follows from the definition of the pressure $P(t\Phi^s, \varepsilon)$ and the fact that $t^s(\varepsilon)$ is the unique zero of $t \to P(t\Phi^s, \varepsilon)$ ([13], [14]). Let us assume now that $\Gamma = \{C_1, \ldots, C_m\}$ and $\delta_i := A \cdot e^{S^-_{n_0}(t\Phi^s)(C_i)}, i = 1, \ldots, m$. We consider all the products $\delta_i \delta_j, i, j \in \{1, \ldots, m\}$. Denote by $\omega(2) := \inf\{\delta_i \delta_j, i, j = 1, \ldots, m\}$. If for some i, j we have $\delta_i \delta_j > \omega(2)$, then let us consider the concatenation

$$\Lambda(C_i C_j, \varepsilon) := \{z \in \Lambda(C_i, \varepsilon), \text{s.t for the preh. } (z, \ldots, z_{-i})\ \varepsilon\text{−shadowed by } C_i, \text{we have } z_{-i} \in \Lambda(C_j, \varepsilon)\}$$

But then if $z \in \Lambda(C_i C_j, \varepsilon)$ with the corresponding prehistory $(z, \ldots, z_{-(n(C_i)+n(C_j))})$ ε-shadowed by $C_i C_j$, it follows that there exists $k \in \{1, \ldots, m\}$ so that $z_{-(n(C_i)+n(C_j))} \in \Lambda(C_k)$. If $\delta_i \delta_j \delta_k \le \omega(2)$ we stop; if not, then we continue this process until we obtain a concatenated prehistory $C_i C_j C_{k_1} \ldots C_{k_q}$ so that $\delta_i \delta_j \ldots \delta_{k_q} \le \omega(2)$ and $\delta_i \delta_j \delta_{k_{q-1}} > \omega(2)$.

Denote by $\mathcal{I}(i, j) := \{(k_1, \ldots, k_q), q \ge 1, s.t\ \delta_i \delta_j \delta_{k_1} \ldots \delta_{k_q} \le \omega(2), \text{ but } \delta_i \delta_j \ldots \delta_{k_{q-1}} > \omega(2)\}$; then we have $\Lambda(C_i C_j) = \underset{(k_1, \ldots, k_q) \in \mathcal{I}(i,j)}{\cup} \Lambda(C_i C_j C_{k_1} \ldots C_{k_q})$. Therefore $\Lambda = \underset{1 \le i,j \le m}{\cup} \Lambda(C_i C_j) = \underset{1 \le i,j \le m}{\cup} \underset{(k_1, \ldots, k_q) \in \mathcal{I}(i,j)}{\cup} \Lambda(C_i C_j C_{k_1} \ldots C_{k_q})$. On the other hand it is clear that $\text{diam}(\Lambda(C_i C_j C_{k_1} \ldots C_{k_q}) \cap W^s_r(x))^t = \delta_i \delta_j \ldots \delta_{k_q} \approx \omega(2)$.

Hence we covered $\Lambda \cap W^s_r(x)$ with sets of comparable diameter, and by the same procedure we can cover $\Lambda \cap W^s_r(x)$ with sets of type $\Lambda(C_{i_1} \ldots C_{i_n} C_{k_1} \ldots C_{k_l})$ for $1 \le i_1, \ldots, i_n \le m, (k_1, \ldots, k_l) \in \mathcal{I}(i_1, \ldots, i_n), n \ge 2$. And if we denote by $\omega(n) := \inf\{\delta_{j_1} \ldots \delta_{j_n}, 1 \le j_1, \ldots, j_n \le m\}$, then

$$\text{diam}(\Lambda(C_{i_1} \ldots C_{i_n} C_{k_1} \ldots C_{k_l}) \cap W^s_r(x))^t \in (\omega(n)\chi_s, \omega(n)\chi_s^{-1}), \text{for } (k_1, \ldots, k_l) \in \mathcal{I}(i_1, \ldots, i_n)$$

Thus we obtained a cover of $W^s_r(x) \cap \Lambda$ with sets of comparable diameter (i.e the ratios of diameters of any two sets from this cover are bounded below and above by some positive universal constants). On the other hand we have $\lim_{n \to \infty} \omega(n) = 0$. So we can use this cover for estimating $\overline{dim}_B(W)$. Denote by $\mathcal{U}_n$ this cover with the sets $W^s_r(x) \cap \Lambda(C_{i_1} \ldots C_{i_n} C_{k_1} \ldots C_{k_l}), i_1, \ldots, i_n \in \{1, \ldots, m\}, k_1, \ldots, k_l \in \mathcal{I}(i_1, \ldots, i_n), n \ge 2$. Now:

$$\begin{aligned} \sum_{U \in \mathcal{U}_n} \text{diam}(U)^t &\le \sum_{1 \le i_1, \ldots, i_n \le m} \ \sum_{k_1, \ldots, k_l \in \mathcal{I}(i_1, \ldots, i_n)} \delta_{i_1} \ldots \delta_{i_n} \delta_{k_1} \ldots \delta_{k_l} \le \\ &\le \sum_{p \ge 1} (\delta_1 + \ldots + \delta_m)^p < \sum_{p \ge 1} \frac{1}{2^p} < \infty \end{aligned} \tag{4}$$

Therefore from 4 we conclude that $\overline{dim}_B(W^s_r(x) \cap \Lambda) \leq t^s(\varepsilon)$ for any ε small enough.

□

By combining Theorem 1 with Theorem 3 we obtain the following Corollary showing that the upper box dimension of the intersection between the basic set Λ and the local stable manifolds is also strictly less than 2 if Λ is not a local repellor.

Corollary 1. *Let us take a smooth function $f : M \to M$ on a Riemannian manifold of real dimension 4 and assume that f is cf-hyperbolic on a basic set Λ which is not local repellor. Then the stable upper box dimension is strictly less than 2 on Λ, i.e $\overline{dim}_B(W^s_r(x) \cap \Lambda) < 2, x \in \Lambda$.*

3 Applications to the Lebesgue measure and Hausdorff dimension of the set $W^u(\hat{\Lambda})$.

In this section we study the global unstable set $W^u(\hat{\Lambda})$ of a basic set Λ for a cf-hyperbolic map $f : M \to M$ on a Riemannian manifold of real dimension 4. Since we work with non-invertible maps on Λ, the unstable manifolds are not uniquely determined by their base points, but instead depend on prehistories, $W^u_r(\hat{x}), \hat{x} \in \hat{\Lambda}$. So through a given point $x \in \Lambda$ there may pass several (possibly infinitely many) local unstable manifolds.

In [11] we showed that for a holomorphic s-hyperbolic map on the complex projective space $\mathbb{P}^2$, the interior of $W^u(\hat{\Lambda})$ is empty for any basic set of saddle type Λ.

However it remains the question whether the global unstable set $W^u(\hat{\Lambda})$ has zero volume and even if its volume is zero, whether its Hausdorff dimension is strictly less than 4. The situation is complicated also by the possible complicated foldings of Λ and by the fact that different points in Λ may have different number of preimages belonging to Λ.

We will show below that, if Λ is not a local repellor and if the system $f|_\Lambda$ intuitively contracts volume, then $HD(W^u(\hat{\Lambda})) < 4$.

Then we will obtain an estimate for the Hausdorff dimension of $W^u(\hat{\Lambda})$ when Λ is not a local repellor, by using the Holder estimates for the distances between local unstable manifolds in the non-invertible case ([10]). We will also give in the end some examples for which one can conclude that the volume (4-dimensional Lebesgue measure) of the global unstable set is strictly less than 4.

Let us mention also that for Henon **diffeomorphisms** $g(z, w) = (w, p(w) - az)$ with p a monic polynomial of degree $d \geq 2$ and $a \neq 0$, Bedford and Smillie ([1]) proved that $K^-(g) = W^u(K(g))$, where $K^-(g) = \{x \in \mathbb{C}^2, (g^{-n}(x))_n$ is bounded in $\mathbb{C}^2\}$ and $K(g) := \{x \in \mathbb{C}^2, (g^{\pm n}(x))_n$ is bounded in $\mathbb{C}^2\}$. They proved that, if g is hyperbolic on its Julia set, it follows that for $|a| \leq 1$ the interior of $W^u(K(g))$ is empty, and if $|a| > 1$, then $Int(W^u(K(g))) = \bigcup_{i=1}^{m} B(p_i)$, where $B(p_i)$ are repelling basins for some repelling periodic points $p_1, \ldots, p_m$. This has some similarity with our result mentioned above for the dissipative case, since $|a|$ represents the Jacobian of the Henon map g. Before proceeding to the theorems in this section, let us give a Lemma which will be used in the sequel, and whose proof can be found in Mattila's book [9].

Theorem (Frostman Lemma)**.** *Let B be a Borel set in $\mathbb{R}^n$. Then $\mathcal{H}^s(B) > 0$ if and only if there exists a Radon measure μ with compact support contained in B, with $0 < \mu(\mathbb{R}^n) < \infty$ and satisfying $\mu(B(x,r)) \le r^s$ for any $x \in \mathbb{R}^n$ and $r > 0$. Moreover we can find μ so that $\mu(B) \ge c \cdot \mathcal{H}^s_\infty(B)$, where $c > 0$ is a constant depending only on n.*

We shall prove now that, under a derivative condition implying that the contraction is stronger than the dilation near Λ, the set $W^u(\hat\Lambda)$ has Hausdorff dimension strictly smaller than 4. Notice that Theorem 4 will complement well the main theorem in [11], which says that the interior of K^- is empty. Indeed, if $f : \mathbb{P}^2 \to \mathbb{P}^2$ is holomorphic and s-hyperbolic, then $K^- = W^u(\hat S_1) \cup S_0$, where S_0 is just a finite set of attracting periodic points, and S_1 is the set of points from the non-wandering set with (complex) unstable index 1 (so $\Lambda \subset S_1$). We recall that in the case of non-invertible maps the unstable manifolds do not realize a lamination near Λ, and that through every point x of Λ there may pass uncountably many local unstable manifolds, which makes their union, i.e $W^u_r(\hat\Lambda)$ hard to control **outside** its intersection with Λ.

Theorem 4. *Let M be a compact Riemannian manifold of real dimension 4, and $f : M \to M$ be a smooth cf-hyperbolic map on a basic set of saddle type Λ, which is not a local repellor. Assume also that the following condition on derivatives is satisfied:*

$$\sup_{\hat\xi \in \hat\Lambda} |Df_u(\hat\xi)| \cdot |Df_s(\xi)| < 1 \tag{5}$$

Then $HD(W^u(\hat\Lambda)) < 4$.

The same conclusion holds if $f : \mathbb{P}^2 \to \mathbb{P}^2$ is a holomorphic map which is s-hyperbolic on a basic set of saddle type Λ and satisfies (5).

Proof. We suppose for the begining that $HD(W^u(\hat\Lambda)) = 4$ and will obtain from here a contradiction. If $HD(W^u(\hat\Lambda)) = 4$, then $\mathcal{H}^\sigma(W^u(\hat\Lambda)) = \infty, \forall \sigma < 4$. We can find then a subset of $W^u(\hat\Lambda)$ with Hausdorff dimension 4, and if it is not close enough to Λ, then we can take backward iterates until we get a set $\tilde\Delta_0$ close to Λ (for example so close that f can be approximated well with Df, and moreover $|Df_s| > 0$); the condition $HD(\tilde\Delta_0) = 4$ is preserved by taking backward iterates.

Then we construct inductively a sequence of Borel sets $\tilde\Delta_n$ such that $d(\tilde\Delta_n, \Lambda) \to 0$ when $n \to \infty$, and $f(\tilde\Delta_{n+1}) = \tilde\Delta_n, n \ge 1$. Let also $\delta_0 > 0$ be a small number so that we can apply the Mean Value Inequality for f on balls of diameter δ_0.

We shall estimate $\mathcal{H}^\sigma_\infty(\tilde\Delta_{n+1})$. Without loss of generality we can assume that $\tilde\Delta_{n+1}$ is covered with sets $E_i, i \in I$, which are cubes with side equal to $r_i, i \in I$. Then $\mathcal{H}^\sigma_\infty(\tilde\Delta_{n+1}) = \inf\{\sum_{i\in I} r_i^\sigma, \tilde\Delta_{n+1} \subset \bigcup_i E_i\}$. If there exists some i with $r_i > \delta_0$, then $\mathcal{H}^\sigma_\infty(\tilde\Delta_{n+1}) \ge \delta_0^\sigma$.

We notice also that, if $(E_i)_{i\in I}$ cover $\tilde\Delta_{n+1}$, then $(fE_i)_{i\in I}$ will cover $\tilde\Delta_n$. Now, fE_i will have its side in the stable direction of length $(|Df_s(\xi_i)| + \eta(n))r_i$, and the "unstable side" of length $(|Df_u(\hat\xi_i')| + \eta(n))r_i$, where $\eta(n) > 0$ is a small positive number which converges towards 0 when $n \to \infty$, and where $\xi_i, \xi_i' \in E_i$ and $\hat\xi_i'$ is an arbitrary prehistory of ξ_i'. So, $f(E_i)$ is approximately a box with a smaller side $(|Df_s(\xi_i)| + \eta(n))r_i$, and a larger side $(|Df_u(\hat\xi_i')| + \eta(n))r_i$. Assume also that n is large enough such that $|Df_s(\xi_i)| + \eta(n) < |Df_u(\hat\xi_i')| + \eta(n), i \in I$.

Then the set $f(E_i)$ can be covered with m_i^2 cubes with side $(|Df_s(\xi_i)| + \eta(n)) \cdot r_i$, where m_i is a positive integer satisfying $m_i(|Df_s(\xi_i)| + \eta(n)) \cdot r_i \geq (|Df_u(\hat{\xi}_i')| + \eta(n)) \cdot r_i \geq (m_i - 1)(|Df_s(\xi_i)| + \eta(n)) \cdot r_i, i \in I$.

Thus we obtain the estimate:

$$\mathcal{H}_\infty^\sigma(\tilde{\Delta}_n) \leq \sum_{i\in I} m_i^2 \cdot (|Df_s(\xi_i)| + \eta(n))^\sigma \cdot r_i^\sigma \leq \sum_{i \in I} r_i^\sigma (1 + \frac{|Df_u(\hat{\xi}_i')| + \eta(n)}{|Df_s(\xi_i)| + \eta(n)})^2 \cdot (|Df_s(\xi_i)| + \eta(n))^\sigma \quad (6)$$

But we can consider a finite iterate of f instead of f; assume this iterate is f^p for some p large enough. The basic set Λ remains the same, the stable/unstable local manifolds remain the same as before. But for p large enough we will have $1 + \frac{|D(f^p)_u(x)|}{|D(f^p)_s(x)|} < 2\frac{|D(f^p)_u(x)|}{|D(f^p)_s(x)|}, x \in \Lambda$. Now recall that $d(\xi_i, \xi_i') < 3r_i, i \in I$. Hence there exists a small $\delta_1 \in (0, \delta_0)$ such that if $r_i < \delta_1, i \in I$, and n is sufficiently large (equivalently $\eta(n)$ sufficiently small), then condition (5) implies:

$$(|Df_s(\xi_i)| + \eta(n))^\sigma (1 + \frac{|Df_u(\hat{\xi}_i')| + \eta(n)}{|Df_s(\xi_i)| + \eta(n)})^2 < 2^2,$$

for σ very close to 4, i.e $\sigma \in (\sigma_0, 4)$ $(0 < \sigma_0 := \sigma(p) < 4$ being independent of $n)$. Thus, for σ very close to 4, we will obtain

$$\mathcal{H}_\infty^\sigma(\tilde{\Delta}_n) \leq \sum_{i \in I} r_i^\sigma,$$

in case $r_i < \delta_1, i \in I$. So in this case (i.e if $r_i < \delta_1, i \in I$), we got $\mathcal{H}_\infty^\sigma(\tilde{\Delta}_n) \leq \mathcal{H}_\infty^\sigma(\tilde{\Delta}_{n+1})$. Therefore in general $\mathcal{H}_\infty^\sigma(\tilde{\Delta}_n) \geq \min\{\delta_1^\sigma, \mathcal{H}_\infty^\sigma(\tilde{\Delta}_0)\}, n \geq 1, \sigma \in (\sigma_0, 4)$. This means that there exists some number $\beta_0 > 0$ such that $\mathcal{H}_\infty^\sigma(\tilde{\Delta}_n) > \beta_0 > 0$, for $n \geq 1$ and $\sigma \in (\sigma_0, 4)$.

Since $\mathcal{H}^\sigma(\tilde{\Delta}_n) = \infty, n \geq 1$, we can apply Frostman Lemma, to get that for each $n \geq 1$, there exists a Radon measure μ_n on $\tilde{\Delta}_n$ with $\mu_n(\tilde{\Delta}_n) \geq c \cdot \mathcal{H}_\infty^\sigma(\tilde{\Delta}_n) > c \cdot \beta_0 > \beta_0' > 0$, (where c, β_0, β_0' are constants which do not depend on n). We also have that $\mu_n(B(y,r)) \leq r^\sigma, y \in M, r > 0, n \geq 1$. The measure μ_n is compactly supported inside the Borel set $\tilde{\Delta}_n$.

But, since $d(\tilde{\Delta}_n, \Lambda) \to 0$, as $n \to \infty$, we see that there exists $R > 1$ large enough such that for each n, $\tilde{\Delta}_n \subset B(y_0, R)$, for some $y_0 \in \Lambda$. Hence $\mu_n(\tilde{\Delta}_n) \leq R^4, n \geq 1$, so by a classical theorem in functional analysis, there exists a convergent subsequence of $(\mu_n)_n$. For brevity, we will denote this convergent subsequence also by $(\mu_n)_n$, and denote its limit by μ. We see also that, due to the fact that $d(\text{supp } \mu_n, \Lambda) \to 0$ when $n \to \infty$, it follows that $\text{supp } \mu \subset \Lambda$. But, since $\Lambda \subset B(y_0, R)$, it follows that $\mu(\Lambda) \leq R^4 < \infty$; on the other hand, $\mu(\overline{B(y_0, R)}) \geq \overline{\lim_n} \mu_n(\overline{B(y_0, R)}) > \beta_0' > 0$, so $0 < \mu < \infty$. Notice also that for all $y \in M$ and all $r > 0$, the properties of the limit μ (Theorem 1.24 of [9]), imply that $\mu(B(y,r)) \leq \underline{\lim}_n \mu_n(B(y,r)) \leq r^\sigma$.

In conclusion, μ is a Radon measure supported inside Λ, with $0 < \mu < \infty$ and such that $\mu(B(y,r)) \leq r^\sigma, y \in M, r > 0$. Frostman's Lemma implies then that $\mathcal{H}^\sigma(\Lambda) > 0$, for $\sigma \in (\sigma_0, 4)$.

But recall that we showed in Section 2 that $\delta^s(x) = HD(W_r^s(x) \cap \Lambda) \leq t^s(\varepsilon) < 2$, for all $x \in \Lambda$. Since Λ can be laminated locally with intersections of type $W_r^s(x) \cap \Lambda$, we conclude that there exists $\sigma_1 \leq 2 + t^s(\varepsilon) < 4$ with $\mathcal{H}^\sigma(\Lambda) = 0, \forall \sigma \in (\sigma_1, 4)$. This leads then to a contradiction with the previous conclusion, and hence $HD(W^u(\hat{\Lambda})) < 4$.

□

Next we will use Holder estimates from [10] in order to prove a Theorem about the Hausdorff dimension of $W^u(\hat\Lambda)$ by taking in consideration also the number of preimages of points in Λ. This condition can be verified on a number of examples.

Theorem 5. *Let M be a compact Riemannian manifold of real dimension 4, and $f : M \to M$ be a smooth cf-hyperbolic map on a basic set of saddle type Λ, which is not a local repellor. Let us denote by $\chi_s := \inf_\Lambda |Df_s|, \lambda_s := \sup_\Lambda |Df_s|$ and $\sup\limits_{\hat x \in \hat\Lambda} |Df_s(x)| \cdot |(Df|_{E^u(\hat x)})^{-1}| =: \tau$. Suppose that every point from Λ has at most d f-preimages and at least d' f-preimages in Λ. If the condition:*

$$2\inf\{1, \frac{-\log\tau}{|\log\chi_s|}\} - \frac{\log d}{|\log\chi_s|} \geq \frac{h_{top}(f|_\Lambda) - \log d'}{|\log\lambda_s|}$$

is satisfied, then $HD(W^u(\hat\Lambda)\cap\Delta) < 2$ for any disk Δ transversal to the unstable directions. Moreover we obtain $HD(W^u(\hat\Lambda)) < 4$.

Proof. Corollary 2 from [10] can be extended easily to the setting of cf-hyperbolic maps. Obviously $\tau < 1$.

Let us assume that $\Delta \subset B(x,r)$ for some $x \in \Lambda$ and $r > 0$, and that $\Delta \cap W^u(\hat\Lambda) = \bigcup\limits_{y\in W^s_r(x)\cap\Lambda, \hat y\in\hat\Lambda} (\Delta\cap W^u_r(\hat y))$. Let us cover now the set $W^s_r(x)\cap\Lambda$ with disks U_i of radius $\delta_i, i \in I$, such that

$$\sum_{i\in I}\delta_i^\eta < 1, \tag{7}$$

where η is arbitrarily larger than the Hausdorff dimension of $W^s_r(x)\cap\Lambda$.

We want now to cover $\Delta\cap W^u(\hat\Lambda)$ with disks centered at $W^u_r(\hat y)\cap\Delta$ for $y \in U_i, i \in I$. Here we take into consideration the dependence of the distance between two unstable manifolds going through the same point, with respect to the distance between their corresponding prehistories. Indeed we have from Corollary 2 of [10] that $d(W^u_r(\hat x), W^u_r(\hat y)) \leq Cd_K(\hat x,\hat y)^\theta$, where C is a positive constant, $\theta \in (0, 1]$, and $K > 1$ satisfies the relationship $\tau\cdot K^\theta < 1$.

Let us consider a certain prehistory $\hat y = (y, y_{-1}, \ldots,) \in \hat\Lambda, y \in W^s_r(x)\cap\Lambda$. We will assume that

$$K\cdot\chi_s > 1$$

Also let us assume that n_i is the first positive integer n so that $\frac{1}{K^n} < \delta_i$. Without loss of generality we can assume that $\frac{1}{K^{n_i}} = \delta_i$. Then if we consider the tubular set $M(D, r)$, where $D = (x, \ldots, x_{-n_i})$ and if $(y, \ldots, y_{-n_i})$ is the prehistory of y r-shadowed by D, we have that $d(x_{-j}, y_{-j}) \leq d(x, y)\chi_s^{-j}, j = 0, \ldots, n_i$. But $d_K(\hat x,\hat y) \leq d(x, y)(1 + \frac{\chi_s^{-1}}{K} + \ldots + \frac{\chi_s^{-n+1}}{K^{n-1}}) + \frac{M}{K^n} \leq C\delta_i$, if $\frac{1}{K^{n_i}} = \delta_i, i \in I$; therefore we obtain:

$$\text{diam}(M(D, r)\cap\Delta) < Cd_K(\hat x,\hat y)^\theta < C\delta_i^\theta, \tag{8}$$

for a possibly different constant $C > 0$ and all $i \in I$. But there exist at most d^{n_i} such sets of type $M(D, r)\cap\Delta$. Consequently we can cover $\Delta\cap W^u(\hat\Lambda)$ with sets of type $M(D, r)\cap\Delta$, where there are at most d^{n_i} such sets for each $i \in I$. But recall that we assumed $\delta_i = \frac{1}{K^{n_i}}, i \in I$, so $d^{n_i} = \delta_i^{-\frac{\log d}{\log K}}$. Also from above we can take the constant K arbitrarily larger than $\frac{1}{\chi_s}$.

In conclusion we cover the set $\Delta \cap W^u(\hat{\Lambda})$ with d^{n_i} sets of radius $\delta_i + \delta_i^\theta$, for each $i \in I$.

We want to show that there exists some $\rho < 2$ (ρ close to 2), so that $\sum\limits_{i \in I} d^{n_i}(\delta_i + \delta_i^\theta)^\rho < \infty$.This would imply that $HD(\Delta \cap W^u(\hat{\Lambda}) \leq \rho < 2$. For this it would be enough to show that $\sum\limits_{i \in I} d^{n_i}\delta_i^{\rho\theta} < \infty$, since $\theta \leq 1$. But now we have that $d^{n_i} = \delta_i^{\frac{\log d}{-\log K}}, \theta = \frac{\log \tau}{-\log K} \leq 1$. If $\frac{\log \tau}{-\log K} > 1$, then we will take $\theta = 1$. So $d^{n_i}\delta_i^{\theta\rho} = \delta_i^{\rho\theta - \frac{\log d}{|\log \chi_s|}}, i \in I$.

Now we recall from (7) that for any $\eta > HD(W_r^s(x) \cap \Lambda)$, $\sum\limits_{i \in I} \delta_i^\eta < 1$.

Then we shall use an estimate of the stable dimension from [14]; if $f|_\Lambda$ is at least d'-to-1 over Λ, then $HD(W_r^s(x) \cap \Lambda) \leq t_{d'}$, where $t_{d'}$ is the unique zero of the pressure function $t \to P(t\Phi^s - \log d')$. Thus we obtain that $HD(W_r^s(x) \cap \Lambda) \leq \frac{h_{top}(f|_\Lambda) - \log d'}{|\log \lambda_s|}$. Therefore it is enough to have

$$2 \inf\{1, \frac{-\log \tau}{|\log \chi_s|}\} - \frac{\log d}{|\log \chi_s|} > \frac{h_{top}(f|_\Lambda) - \log d'}{|\log \lambda_s|}$$

This implies that $HD(\Delta \cap W^u(\hat{\Lambda})) < 2$, for any disk Δ transversal to the unstable directions.

Then from Fubini Theorem it follows that $HD(W^u(\hat{\Lambda})) < 4$, since we can take the disks Δ to be parallel to $E^s(x)$.

□

Corollary 2. *The conditions in Theorem 4 are satisfied for perturbations g of the holomorphic map $(z, w) \to (z^2 + c, w^2)$, for small $|c|$. Thus $HD(W^u(\hat{\Lambda}_g)) < 4$ for the respective basic set Λ_g of g which is close to $\{p_0(c)\} \times S^1$ (where $p_0(c)$ is the fixed attracting point of $z \to z^2 + c$).*

The conditions in Theorem 5 or in Theorem 4 can be verified also for many skew products with overlaps of the type studied in [15], and for their perturbations.

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