Metric properties and dynamics for conformal maps

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Abstract

In this paper we study some of the dynamical properties of non-invertible hyperbolic conformal maps. Non-invertibility is a very important factor which prevents one from using the same methods as in the diffeomorphism case, and in fact generates new phenomena and examples. We will give in this paper several results about establishing hyperbolicity for skew product maps, estimating Hausdorff dimension by using thermodynamical formalism, and investigating the geometric structure of dynamically interesting sets using equilibrium states.

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1 Introduction

In dynamical systems, one of the most important directions is to study the invariant sets which appear from the iteration of maps (or from a flow). Usually such sets have a very irregular (fractal) structure and are not manifolds, except for special cases.

One is then compelled to investigate the structure of such sets and to study their Hausdorff dimension, upper (lower) box dimension, capacity, measures which are supported on these sets, etc. This can be done with the help of thermodynamical formalism.

There are however important differences between the invertible and non-invertible cases, and simple examples show that the structure of fractal sets obtained from iterations of non-invertible maps is different and in some cases, more difficult to understand than in the diffeomorphism case.

Our goal in Section 2 will be to estimate the Hausdorff dimension of fractal sets obtained from non-invertible hyperbolic maps, and also to give information about the geometric structure of stable/unstable manifolds. We will also give many examples of non-invertible maps.

First, let us give the definition of basic sets.

Definition 1. Let M be a compact Riemannian manifold and $f : M \to M$ be a smooth (for example $\mathcal{C}^1$) map; we will say that a set Λ of M is **f-invariant** if $f(\Lambda) = \Lambda$.

Then we say that Λ is a **basic set** if $f|_\Lambda : \Lambda \to \Lambda$ is topologically transitive and there exists a neighbourhood U of Λ such that $\Lambda = \bigcap\limits_{n=-\infty}^{\infty} f^{-n}(U)$.

More general information about dynamical systems and ergodic theory definitions and basic concepts (like Axiom A, hyperbolicity, transitivity, local stable/unstable manifolds, etc.), can be found for example in [5], [18], [16].

We introduce now an important notion from thermodynamical formalism, which appeared as a generalization of the topological entropy $h_{top}(f)$ of a transformation $f : X \to X$ ([18]).

Definition 2. Let $f : X \to X$ be a continuous map on a compact metric space and $\phi : X \to \mathbb{R}$ continuous. For an integer $n \geq 1$, let $d_n(x, y) := \sup\{d(f^i(x), f^i(y)), i = 0, \dots, n\}$.

We say that a set $E \subset X$ is (n, ε)-**separated** if for all $x, y \in E, x \neq y$, we have $d_n(x, y) > \varepsilon$. Denote also $B_n(x, \varepsilon) := \{z \in X, d_n(x, z) < \varepsilon\}$, which is called the **Bowen ball** for d_n, centered in x and of radius ε.

We say that a set $F \subset X$ is (n, ε)-**spanning** for X if for any $x \in X$, there exists a point $y \in F$ such that $x \in B_n(y, \varepsilon)$.

We shall denote in the sequel the consecutive sum $\phi(x) + \phi(fx) + \ldots + \phi(f^n x)$ by $S_n\phi(x)$ for any $x \in X$.

Definition 3. For a map $f : X \to X$ and a potential $\phi : X \to \mathbb{R}$ as above, let the quantities:

$$P_{span}(\phi, n, \varepsilon) := \overline{\lim_{n\to\infty}} \frac{1}{n} \log \inf\{\sum_{x \in F_n} e^{S_n\phi(x)}, F_n \ (n, \varepsilon) - \text{spanning set for } X\} \text{ and}$$

$$P_{sep}(\phi, n, \varepsilon) := \overline{\lim_{n\to\infty}} \frac{1}{n} \log \sup\{\sum_{x \in E_n} e^{S_n\phi(x)}, E_n \ (n, \varepsilon) - \text{separated set in } X\}$$

Then it can be proved that the limit $\lim\limits_{\varepsilon\to 0} P_{span}(\phi, n, \varepsilon)$ exists and it is equal to $\lim\limits_{\varepsilon\to 0} P_{sep}(\phi, n, \varepsilon)$. Their common value is called the **topological pressure** of ϕ with respect to f on X, denoted by $P(\phi)$, or $P_f(\phi)$ when it is necessary to specify the map f as well.

Observation: When the potential $\phi \equiv 0$, we obtain $P(0) = h_{top}(f)$, namely the topological entropy of f on X.

We note also that the notion of topological pressure has been extended by Mihailescu and Urbanski in a series of papers to several notions of **inverse pressure**, which are necessary for dimension estimates purposes when working with a non-invertible map ([11], [12]).

Let us give now a classical and important result obtained by Manning and McCluskey [7], about the relation between the Hausdorff dimension of intersections between stable/unstable manifolds and basic sets on one hand, and the zeros of topological pressure on the other hand.

Theorem (Manning, McCluskey)**.** *Let Λ be a basic set for a $\mathcal{C}^1$ Axiom A diffeomorphism of a real surface $f : M \to M$, with a hyperbolic (1,1) splitting of the tangent bundle $T_\Lambda M = E^s \oplus E^u$. Define also the negative function $\Phi^u(x) := -\log |Df|_{E^u_x}|$. Then the Hausdorff dimension of $W^u_r(x) \cap \Lambda$ is given by the unique zero t^u of the pressure function $t \to P_{f|_\Lambda}(t\Phi^u)$.*

Thus, $HD(W^u_r(x) \cap \Lambda)$ is independent of $x \in \Lambda$ and depends continuously on f in the $\mathcal{C}^1$ topology on diffeomorphisms.

The same conclusion applies to $HD(W_r^s(x) \cap \Lambda)$ which is equal to t_0^s, the unique zero of the function $t \to P(t\Phi^s)$, where $\Phi^s : \Lambda \to \mathbb{R}, \Phi^s(x) := \log |Df|_{E_x^s}|, x \in \Lambda$.

Corollary. *For a $\mathcal{C}^1$ open dense subset of the $\mathcal{C}^1$ Axiom A no-cycle diffeomorphisms of a real surface M, each basic set that is not an attractor has $\delta^u < 1$, where $\delta^u := HD(W_r^u(x) \cap \Lambda), x \in \Lambda$ (we saw in the previous theorem that $HD(W_r^u(x) \cap \Lambda)$ does not depend on x).*

Proof. The property is true for a $\mathcal{C}^2$ difeomorphism (Bowen, [3]). Then, by continuity, we still have $\delta^u < 1$ for a $\mathcal{C}^1$-neighbourhood of f.

Thus, the union of these neighbourhoods gives the required $\mathcal{C}^1$-open dense set.

□

Let us note however that there exists an example of a $\mathcal{C}^1$-diffeomorphism of a surface with a horseshoe of positive measure, obtained as a product of sets with positive Lebesgue measure in both stable and unstable manifolds ([2]).

2 Non-invertible dynamics of conformal maps

Our goal now is to extend some of these results to a non-invertible situation. In this case the situation is different due to the number of preimages (which may or may not be constant along the respective basic set). For example in [10] it has been noted that the stable dimension δ^s even for a simple function as $(z, w) \to (z^2 + c, w^2)$, is not equal to the zero of the pressure function $t \to P(t\Phi^s)$. Indeed if the examples are perturbations of this map, the situation becomes even more complicated, as it is not clear how many of the preimages of a point $x \in \Lambda$ still belong to the same basic set Λ.

The hyperbolicity condition is a cornerstone of the theory of dynamical systems, and in particular that of the metric properties of fractal sets; we shall give below the definition of hyperbolicity for non-invertible maps.

Let us consider a smooth (for example $\mathcal{C}^2$) map $f : M \to M$ on a $\mathcal{C}^2$ compact Riemannian manifold M; take also a basic set Λ in M (the definition of basic sets for non-invertible maps remains the same, namely we require f to be topologically transitive on Λ and that there exists a neighbourhood U of Λ such that $\Lambda = \bigcap_{n=-\infty}^{\infty} f^{-n}(U)$).

For a compact invariant set $\Lambda \subset M$, define also the **space of prehistories** (or the **natural extension**) $\hat{\Lambda}$, as the set $\{\hat{x} = (x, x_{-1}, x_{-2}, \ldots), x_{-i} \in \Lambda, f(x_{-i}) = x_{-i+1}, i \geq 0\}$. We endow this set with the structure of a compact metric space by putting the metric $d(\hat{x}, \hat{y}) = \sum_{m \geq 0} \frac{d(x_{-m}, y_{-m})}{2^m}$. On $\hat{\Lambda}$ we have a natural homeomorphism, $\hat{f}(\hat{x}) = (f(x), x, x_{-1}, \ldots), \hat{x} \in \hat{\Lambda}, \hat{f} : \hat{\Lambda} \to \hat{\Lambda}$.

Definition 4. ([16]) We will say that f is **hyperbolic** on a basic set Λ if there exists a continuous invariant splitting of the tangent bundle over $\hat{\Lambda}$, where $T_{\hat{\Lambda}}M := \{(\hat{x}, v), v \in T_x M\}$. Namely we have the splitting

$$T_{\hat{\Lambda}}M = E^s_{\Lambda} \oplus E^u_{\hat{\Lambda}},$$

such that $Df(E^s_x) \subset E^s_{fx}, Df^{-1}(E^u_{\hat{x}}) \subset E^u_{\hat{f}^{-1}\hat{x}}$, and Df is uniformly contracting on E^s_x, respectively uniformly expanding on $E^u_{\hat{x}}$, for every $\hat{x} \in \hat{\Lambda}$.

If f is hyperbolic over Λ, then one can form local stable and unstable prehistories, but the latter will depend in general on full prehistories.

Definition 5. If in the above setting, f is hyperbolic on a basic set Λ, then there exists a positive number r so we can form local stable/unstable manifolds of size r, given by

$$W^s_r(x) = \{y \in M, d(f^k x, f^k y) < r, k \geq 0\}$$

$$W^u_r(\hat{x}) = \{y \in M, \exists \hat{y} = (y, y_{-1}, \ldots) \text{ prehistory of } y, \text{s.t } d(x_{-k}, y_{-k}) < r, k \geq 0\},$$

for all $\hat{x} \in \hat{\Lambda}$.

In case M is a complex manifold and $f : M \to M$ is holomorphic, it follows from construction that the local stable /unstable manifolds are complex submanifolds.

There is an abundance of examples of non-invertible hyperbolic maps, from which we will give a few:

Examples:

1) Let f be a non-degenerate holomorphic map, $f : \mathbb{P}^2\mathbb{C} \to \mathbb{P}^2\mathbb{C}$, on the complex projective space of dimension 2. Suppose f can be written in homogeneous coordinates as

$$f[z : w : t] = [P(z,t) : Q(w,t) : t^d],$$

where P, Q are the homogenized versions of polynomials of degree d in one variable. Assume that the critical points of $P(z), Q(w)$ are in the basin of attraction of attracting cycles, i.e P, Q are hyperbolic on their respective Julia sets J_P, J_Q.

The basic sets of unstable index 1 (i.e the basic sets where the local unstable manifolds have complex dimension 1) are, in $t = 1$, the sets of the form $\{\text{periodic sinks of } P\} \times J_Q$ and $J_P \times \{\text{periodic sinks of } Q\}$, and in $t = 0$, the basic set is the Julia set of the restriction of f, namely $f_0 := [P(z,0) : Q(w,0)]$.

In this case, f is hyperbolic on all these basic sets and all unstable manifolds are contained in complex lines. More general hyperbolic holomorphic maps on $\mathbb{P}^2\mathbb{C}$ have been studied in [4], where the authors considered the currents and measures induced on the global unstable sets of these maps.

2) Let now Φ be the Segre map from $\mathbb{P}^1\mathbb{C} \times \mathbb{P}^1\mathbb{C}$ to $\mathbb{P}^2\mathbb{C}$, $\Phi([z_0 : z_1], [w_0 : w_1]) = [z_0w_0 : z_1w_1 : z_0w_1 + z_1w_0]$. Consider $f_0 : \mathbb{P}^1\mathbb{C} \to \mathbb{P}^1\mathbb{C}$ holomorphic map of degree $d \geq 2$. There is then a function f holomorphic on $\mathbb{P}^2$, of degree d, such that $\Phi(f_0, f_0) = f \circ \Phi$.

If f_0 is hyperbolic, then f is also hyperbolic. The basic sets of unstable index 1 are of the form $\Phi(\text{periodic sink} \times J_0)$ and the unstable manifolds are contained in algebraic varieties.

3) Horseshoes with overlappings.

Let $I^{m+1} = I \times I^m$ be the $(m+1)$-dimensional unit cube in $\mathbb{R}^{m+1}, m \geq 1$. We will take $k \geq 2$ mutually disjoint subintervals $I_1, \ldots, I_k$ inside I and a $\mathcal{C}^2$ map f from a neighbourhood of the union

$I_* := I_1 \cup \ldots \cup I_k$ to I, such that the restriction of f to each subinterval I_j is expanding onto I, $j = 1, \ldots, k$.

Now, for an arbitrary $\lambda \in (0,1)$, let a $\mathcal{C}^0$ function $g : I_1 \cup \ldots \cup I_k \to [0, 1-\lambda]^m$, and form the map

$$F_g(x,y) = (f(x), g(x) + \lambda y), (x,y) \in I_* \times I^m$$

When it is not necessary to emphasize g, we will write just F, instead of F_g. Due to the contraction in y and the dilation in x, it can be shown that this map has a hyperbolic structure on a basic set Λ, which is similar to the Smale horseshoe. However for certain choosings of the maps f, g, we may have overlappings of the horseshoe, thus the unstable manifolds may depend on the way of constructing the point in Λ and they may not form a foliation. This fact prevents one from using the same methods as in the diffeomorphism case.

Examples where the map F is not injective on its attractor $A := \bigcap_{n=1}^{\infty} A_n, A_n := F^n(\{(x,y) \in I_* \times I^m, F^n(x,y) \text{ is defined}\}, n \geq 1$, have been given in [1]. Indeed Bothe proved that, if we denote by $\mathcal{G}$ the space of $\mathcal{C}^0$ maps $g : I_* \to [0, 1-\lambda]^m$ (with the $\mathcal{C}^0$ topology), such that the restriction of F_g to $A \cap (I_* \times I^m)$ is not injective, then:

Theorem (Bothe, [1]). *If $m \geq 3$ is odd, and $\lambda > 12k^{-2/(m-1)}$, then $\mathcal{G}$ has interior points.*

The class of examples can be considerably expanded by taking perturbations (for example of the maps listed above). In this direction, we have an important theorem of conjugation between the natural extensions (see for example [17] for the diffeomorphism case, and [13] for the endomorphism case), which permits transferring some properties (but as we will see, not all), from the original map to its perturbations.

Theorem (Conjugation Theorem). *If $f : M \to M$ is $\mathcal{C}^2$ smooth on a compact $\mathcal{C}^2$ Riemanninan manifold M, and f is hyperbolic on a basic set Λ, and if g is a perturbation of f (in the $\mathcal{C}^2$ topology on the space of smooth maps), then g will have also a basic set Λ_g, and there exists a homeomorphism $\hat{\Phi}(g) : \hat{\Lambda} \to \hat{\Lambda}_g$, such that $\hat{g} \circ \hat{\Phi}(g) = \hat{\Phi}(g) \circ \hat{f}$.*

We will consider in the sequel a large class of maps, namely **skew products with overlaps**. The main idea is the following:

Consider an integer $k \geq 2$ and k mutually disjoint subintervals in the unit interval I; take also a $\mathcal{C}^2$ map f from a neighbourhood of $I_* := I_1 \cup \ldots I_k$ to I. Assume that f is expanding, i.e $|f| > 1$ on I_*, and $f(I_j) = I, j = 1, \ldots, k$.

Then consider also a $\mathcal{C}^2$ map $g : I_* \times I \to I$, which is contracting on vertical lines, i.e $\partial_y g(x,y) < 1, (x,y) \in I_* \times I$, where $\partial_y g$ represents the partial derivative of g with respect to the second coordinate $y \in \mathbb{R}$. Now construct the skew product:

$$F(x,y) = (f(x), g(x,y)), (x,y) \in I_* \times I$$

Denote by $I_\infty := \{x \in I, f^n x \in I_*, n \geq 0\}$. Also, we will write g_x for the map $y \to g(x,y)$ with g a map as above.

Denote by Λ_x the set of points obtained as the intersections of compact sets of decreasing diameters, $\bigcap_{n\geq 0} g_{x_{-1}} \circ \ldots \circ g_{x_{-n}}(I), x \in I_\infty$, where we consider these intersections for all prehistories $\hat{x} = (x, x_{-1}, \ldots)$ of x from I_*. We obtain thus Λ_x as a compact set, as can be easily checked, due to the fact that the natural extension $\hat{I}_*$ of the restriction $f|_{I_*} : I_* \to I_*$, is compact.

Then let $\Lambda := \bigcup_{x\in I_\infty} \Lambda_x$. We will denote Λ also by $\Lambda(F)$ when it will be necessary to emphasize the relationship between F and Λ.

Let us notice that although it may appear at a first sight that F is expanding horizontally, the calculation on derivative shows this to be false. Indeed, we have the derivative of F,

$$DF(x,y) = \begin{pmatrix} f'(x) & 0 \\ \partial_x g(x,y) & \partial_y g(x,y) \end{pmatrix},$$

where $\partial_x g(x,y)$ represents the partial derivative of g with respect to x at the point (x,y).

So, for a vector $\bar{w} = (0, v) \in \mathbb{R} \times \mathbb{R}$, we get $DF(x,y) \cdot \bar{w} = \begin{pmatrix} 0 \\ \partial_y g(x,y) v \end{pmatrix}$, hence the vector space $\{(0,v) \in \mathbb{R} \times \mathbb{R}\}$ is invariant and due to the uniform contraction of g on vertical lines, we have that DF is contracting on vertical lines, which hence represent the stable tangent subspaces.

However, if we consider the horizontal vector $\bar{\omega} = (\zeta, 0) \in \mathbb{R} \times \mathbb{R}$, then $DF(x,y) \cdot \bar{\omega} = \begin{pmatrix} f'(x) \cdot \zeta \\ \partial_x g(x,y) \cdot \zeta \end{pmatrix}$, so the horizontal line $\{(\zeta, 0), \zeta \in \mathbb{R}, 0 \in \mathbb{R}\}$ is not invariated by DF, and thus the unstable spaces do not always have to be equal to this line.

In order to prove hyperbolicity we will use a generalization of a theorem of Newhouse ([14]); this generalization treats the non-invertible case and it is proved similarly to the Newhouse result.

Theorem 1. *Let $f : M \to M$ smooth, not necessarily invertible, suppose that Λ is a compact f-invariant set in M and let $\hat{\Lambda}$ the natural extension of the map $f|_\Lambda : \Lambda \to \Lambda$. Assume also that there exists a field of cones in the tangent space, $\mathcal{C} = \{C_{\hat{z}}\}_{\hat{z}\in\hat{\Lambda}}$, so that the dimension of the core linear space of $C_{\hat{z}}$ is constant on $\hat{\Lambda}$ (the cone field $\mathcal{C}$ is not necessarily assumed to be Df-invariant).*

Let us say that a function f is ***expanding and co-expanding on the cone field*** *$\mathcal{C}$, if, given the notations:*

$$m_{\mathcal{C},\hat{z}}(f) := \inf_{v\in \mathcal{C}_{\hat{z}}\setminus\{0\}} \frac{|Df_z v|}{|v|}, \text{ and}$$

$$m'_{\mathcal{C},\hat{z}}(f) := \inf_{v\notin C_{\hat{f}\hat{z}}} \frac{|Df_{fz}^{-1} v|}{|v|}, \hat{z} \in \hat{\Lambda},$$

we have, by definition, that $\inf_{\hat{z}\in\hat{\Lambda}} m_{\mathcal{C},\hat{z}}(f^N) > 1$, and $\inf_{\hat{z}\in\hat{\Lambda}} m'_{\mathcal{C},\hat{z}}(f^N) > 1$.

Assume that there exists an integer $N \geq 1$ such that f^N is expanding and co-expanding on $\mathcal{C}$; then it follows that f is hyperbolic on Λ.

We can prove consequently the following theorem of hyperbolicity for our skew product:

Theorem 2. *In the above setting, i.e with $f : I_* \to I$ expanding and $g : I_* \times I \to [0, 1-\lambda]$ contracting in the second coordinate over the invariant set Λ defined above (for $m = 1$), we have that $F(x,y) = (f(x), g(x,y))$ is uniformly hyperbolic on Λ.*

Proof. We have defined the set $\Lambda = \Lambda(F)$ as the union of the fibers $\Lambda_x, x \in I_\infty$ and Λ_x as the set of all points of the form $\bigcap\limits_{n\geq 0} g_{x_{-1}} \circ \ldots \circ g_{x_{-n}}(I)$, for all prehistories $\hat{x} = (x, x_{-1}, \ldots, x_{-n}, \ldots), x_{-i} \in I_\infty$.

Let a continuous positive function γ defined on $\hat{\Lambda}$ and the cone $C^u_{\hat{z}} := \{(v,w) \in \mathbb{R}^2, |w| \leq \gamma(\hat{z}) \cdot |v|\}, z = (x,y) \in \Lambda, \hat{z} \in \hat{\Lambda}$. The dimension of the core real linear space of this cone is 1.

Our cone field will be then $\mathcal{C}^u = \{C^u_{\hat{z}}\}_{\hat{z}\in\hat{\Lambda}}$.

We have $DF_z(v,w) = \begin{pmatrix} f'(x)\cdot v \\ \partial_x g(z)\cdot v + \partial_y g(z)\cdot w \end{pmatrix}$. So, in order to have an F-expanding field of cones, it is enough to take

$$|f'(x)|^2 > 1 + \gamma^2(\hat{z}), z \in \Lambda$$

If we assume $|f'(x)| > \beta > 1, x \in X$, then it would be enough to have

$$0 < \gamma(\hat{z}) < \sqrt{\beta^2 - 1}, \text{ or, } 0 < \gamma(\hat{z}) < \sqrt{\beta^{2N} - 1}, \tag{1}$$

where the second inequality is needed if we work with f^N instead of f. So in this last case, f^N is expanding on the cone field $\mathcal{C}^u$.

Now we estimate the co-expansion coefficient.

If $N \geq 1$ is an integer and if $(v,w) \notin C^u_{\hat{F}^N\hat{z}}$, then $|w| > \gamma(\hat{F}^N\hat{z})\cdot|v|$. Denote also $F^N z = (f^N(x), g(f^{N-1}x, g_{N-1}(x,y)))$, where we assumed that $F^{N-1}z = (f^{N-1}(x), g_{N-1}(x,y))$. So

$$\begin{aligned}
&\partial_x g_N(x,y) = \partial_x g(f^{N-1}x, g_{N-1}(x,y))\cdot\partial_x f^{N-1}(x) + \partial_y g(f^{N-1}x, g_{N-1}(x,y))\cdot\partial_x g_{N-1}(x,y) = \\
&= \partial_x g(f^{N-1}x, g_{N-1}(x,y))\cdot\partial_x f^{N-1}(x) + \partial_y g(f^{N-1}x, g_{N-1}(x,y))\cdot\partial_x g(f^{N-2}x, g_{N-2}(x,y))\cdot\partial_x f^{N-2}(x) + \\
&\partial_y g(f^{N-1}x, g_{N-1}(x,y))\cdot\partial_y g(f^{N-2}x, g_{N-2}(x,y))\cdot\partial_x g_{N-2}(x,y)
\end{aligned} \tag{2}$$

Denote by $K := \sup\limits_{\Lambda} |\partial_x g|$ and $K' := K \cdot \frac{1}{1-\delta/\beta}$, where $\delta \in (0,1)$ is such that $|\partial_y g| < \delta < 1$ on Λ.

Hence by induction in (2) we will obtain

$$|\partial_x g_N(x,y)| \leq K\cdot|(f^{N-1})'x| + \delta K\cdot|(f^{N-2})'x| + \ldots \leq K'\cdot|(f^{N-1})'x| \tag{3}$$

But $D(F^N)^{-1}_{F^N z}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \frac{v}{(f^N)'(x)} \\ \frac{-\partial_x g_N(z) v}{(f^N)'(x)\cdot\partial_y g_N(z)} + \frac{w}{\partial_y g_N(z)} \end{pmatrix}$.

Hence $||D(F^N)^{-1}_{F^N z}\begin{pmatrix} v \\ w \end{pmatrix}||^2 \geq \frac{v^2}{|(f^N)'(x)|^2}\left(1 + \frac{|\partial_x g_N|^2(z)}{|\partial_y g_N|^2(z)}\right) + \frac{w^2}{|\partial_y g_N|^2(z)}\cdot\left(1 - \frac{2|\partial_x g_N(z)|}{|(f^N)'(x)\cdot\gamma(\hat{F}^N\hat{z})|}\right)$, for any $N \geq 1$.

But then, since $|\partial_x g_N(z)| \leq K'\cdot|(f^{N-1})'(x)|$, and K' depends only on f, g, there exists N sufficiently large such that $\left|\frac{2K'\cdot\frac{1}{f'(f^{N-1}x)}}{\gamma(\hat{F}^N(\hat{z}))}\right| \leq \frac{2K'}{\beta\cdot\sqrt{\beta^{2N}-1}} < \frac{1}{2}$, if we take the map γ to be constant and close to $\sqrt{\beta^{2N}-1}$ (although smaller than $\sqrt{\beta^{2N}-1}$).

Therefore

$$||D(F^N)^{-1}_{F^N z}||^2 \geq \frac{|w|^2}{2|\partial_y g_N(z)|^2} \tag{4}$$

Recall that $|w| > \gamma(\hat{F}^N\hat{z})\cdot|v| > |v|$, so we get

$$\frac{|w|^2}{2|\partial_y g_N(z)|^2} \geq \frac{|w|^2}{2\delta^{2N}} > \frac{|v|^2+|w|^2}{\delta}$$

Thus, $\inf_{\hat{z}\in\hat{\Lambda}} m'_{\mathcal{C}^u,\hat{z}}(F^N) > 1$ for some large integer N. This implies finally that F^N is both expanding and co-expanding on the cone field $\mathcal{C}^u$ over Λ, so the skew-product F is hyperbolic according to Theorem 1.

□

Let us consider now the structure of the fiber sets $\Lambda_x, x \in I_\infty$. Any point $x \in I_\infty$ has an uncountable number of prehistories $\hat{x} = (x, x_{-1}, \dots) \in I_\infty$, due to the condition $f(I_j) = I, j = 1, \dots, k$. Also, if x_{-1}, x'_{-1} are different preimages of x, we may have overlappings in fibers, i.e $g_{x_{-1}}(I) \cap g_{x'_{-1}}(I) \neq \emptyset$, in which case the map F may be non injective on Λ and the unstable spaces depend in general on prehistories.

In order to better understand the overlapping mentioned above, we will define a notion first introduced by Newhouse [15].

Let F be a **Cantor set**, i. e a compact, perfect and nowhere dense subset of $\mathbb{R}$.

Assume that $\mathbb{R} \setminus F = \bigcup_{I=-2}^{\infty} U_i$, U_i open bounded intervals for $i \geq 0$, with possibly U_{-2}, U_{-1} unbounded intervals. Denote by F_0 the smallest closed interval containing F.

Let now $F_i := F_0 \setminus \bigcup_{j<i} U_j$; then we obtain F as a decreasing intersection $F = \bigcap_{i=0}^{\infty} F_i$. The family $\{F_i\}_{i\geq 0}$ is called a **defining sequence** for the Cantor set F; as can be easily noticed, defining sequences are not unique for a given set F. The subinterval U_i divides the connected component of F_i which contains U_i, into 2 subintervals, denoted by C_{il}, C_{ir}.

Definition 6. In the above setting, let $\tau(\{F_i\}_i) := \inf_{i\geq 0} \min\{\frac{l(C_{il})}{l(U_i)}, \frac{l(C_{ir})}{l(U_i)}\}$, which is called the thickness relative to the defining sequence $\{F_i\}_i$ of F.

Then $\tau(F) := \sup\{\tau(\{F_i\}), \{F_i\}_{i\geq 0}$ defining sequence for $F\}$, and $\tau(F)$ is called the **thickness** of F.

Cantor sets are compact but nowhere dense, so it is an interesting and important problem to find out conditions assuring that the intersection of two Cantor sets F and G, is not empty.

Theorem (Newhouse [15])**.** *Let F, G Cantor sets with $\tau(F)\cdot\tau(G) > 1$, and none of F, G is contained in a gap of the other. Then $F \cap G \neq \emptyset$.*

Later, R. Kraft gave a more complete characterization of intersections of Cantor sets, in relation to their thicknesses:

Theorem (Kraft, [6])**.** *Let C, C' be a pair of interleaved Cantor sets (i.e neither of them is contained in the closure of a gap of the other). Denote their thicknesses by τ, τ' respectively.*

Assume that $\tau \cdot \tau' > 1$ and that $(\tau, \tau') \in \{(\tau, \tau'), \tau > \frac{\tau'^2+3\tau'+1}{\tau'^2}$ or $\tau' > \frac{\tau^2+3\tau+1}{\tau^2}\} \cap \{(\tau, \tau'), \tau > \frac{(1+2\tau')^2}{(\tau')^3}$ or $\tau' > \frac{(1+2\tau)^2}{\tau^3}\}$.

In this case, $C \cap C'$ contains a Cantor set itself, hence in particular $C \cap C'$ is uncountable.

Now for the skew product $F(x,y) = (f(x), g(x,y))$ with $k = 2$ for example, we will have that a point $x \in I_\infty$ has two preimages, x_{-1}, x'_{-1}, and hence in the fiber over x we will have two subintervals, namely $g_{x_{-1}}(I)$ and $g_{x'_{-1}}(I)$, which can be made to intersect if their lengths are big enough. But the fibers $\Lambda_{x_{-1}}, \Lambda_{x'_{-1}}$ are themselves Cantor sets (unless they contain whole subintervals, in which case the situation is even better). So we obtain the fiber Λ_x as a union of images of fibers $\Lambda_{x_{-1}}$ and $\Lambda_{x'_{-1}}$ (one can assume that all maps g_ζ are injective from I to I, for $\zeta \in I_\infty$). It is then enough to find Cantor sets $\Lambda_{x_{-1}}, \Lambda_{x'_{-1}}$ of thicknesses τ, τ' respectively such that the inequality conditions in Kraft's Theorem above are satisfied. This will give uncountably many points in Λ_x with two F-preimages in Λ.

In Theorem 2 we showed that the skew product $F(x,y) = (f(x), g(x,y))$ is uniformly hyperbolic on the invariant set $\Lambda = \bigcup_{x \in I_\infty} \Lambda_x$, where Λ_x was formed by taking intersections of iterates of I along various prehistories of x.

We also noticed that the unstable space is not necessarily equal to the horizontal line $\{(v,0), v \in \mathbb{R}\}$. Thus we have a more intricate structure for the unstable spaces (and for the local unstable manifolds). What can we say in general about these local unstable manifolds? The paper [13] treated in detail the properties of unstable manifolds of a conformal hyperbolic map. One of the theorems proved there was:

Theorem (Mihailescu, [13])**.** *Let $f : M \to M$ be smooth, conformal on its local unstable manifolds and consider Λ a basic set of saddle type (i.e with both stable and unstable directions in the tangent bundle) so that $h_{top}(f|_\Lambda) \neq 0$; let also a small positive number r such that all local unstable manifolds $W^u_r(\hat{x}), \hat{x} \in \hat{\Lambda}$ are defined. Then the unstable dimension $\delta^u(\hat{x}, r) := HD(W^u_r(\hat{x}) \cap \Lambda)$ is equal to the unique zero t^u of the pressure function $t \to P_{\hat{f}|_{\hat{\Lambda}}}(t\Phi^u), \Phi^u(\hat{y}) := -\log |Df|_{E^u_{\hat{y}}}|, \hat{y} \in \hat{\Lambda}$.*

Hence the unstable manifolds of the skew product F, over Λ, have many intersections with the Cantor set Λ, since it follows from the previous Theorem that δ^u is nonzero (indeed since $h_{top}(f|_\Lambda) \neq 0$, we will obtain $t^u \neq 0$ and then use the equality from the Theorem giving $t^u = \delta^u$).

Also in [13] it is proved that for a perturbation G of F, the conjugation map $\Phi(G)$ from Theorem 2 is Holder continuous when restricted to local unstable manifolds of F.

There exist also geometric measures on the intersections between local unstable manifolds of F and its basic set Λ, which implies that the Hausdorff dimension of these sets is equal to their (common) upper box dimension (for a proof, we refer to [13]).

Theorem (Mihailescu, [13])**.** *In the same setting as in the previous Theorem, there exists a geometric measure on $W^u_r(\hat{x}) \cap \Lambda$, of exponent t^u, and thus the unstable dimension is equal to the upper (and lower) box dimension of $W^u_r(\hat{x}) \cap \Lambda$.*

This geometric measure is obtained by considering first the equilibrium measure ([5]) of the Holder potential $t^u\Phi^u$ on $\hat{\Lambda}$, then projecting on the local unstable set (in the Smale space structure) $V^u_r(\hat{x})$ in $\hat{\Lambda}$, and finally projecting again on $W^u_r(\hat{x}) \cap \Lambda$.

These Theorems apply clearly for our hyperbolic skew product F.

However we cannot say in general that the stable dimension of a point in Λ is given by a Bowen type equation in the same way as the unstable dimension was given above; see [10] for counter-examples.

In fact the stable dimension behaves very differently and quite surprisingly for non-invertible maps as opposed to diffeomorphisms.

While in the case of diffeomorphisms, the stable and unstable dimensions over a hyperbolic set behave real analytically with respect to parameters (i.e for perturbations of the original map), for endomorphisms this situation is not true anymore. In [10], we gave an example of a large class of non-invertible functions for which some of their perturbations are homeomorphisms on their respective basic sets, namely the holomorphic maps on $\mathbb{C}^2$, $f_\varepsilon(z,w) = (z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2, w^2)$, with $b \neq 0, c \neq 0, |c|$ small and ε sufficiently small; f_ε are perturbations of $f_0(z,w) = (z^2 + c, w^2)$ and they are homeomorphisms when restricted to their basic sets Λ_ε obtained from the Conjugation Theorem (Λ_ε is close to $\{p_c\} \times S^1$, where p_c is the fixed attracting point of $z \to z^2 + c$). These examples show that the stable dimension of hyperbolic basic sets of endomorphisms does not even vary continuously, since for f_0 the stable dimension is zero, while for the homeomorphism f_ε, the stable dimension is larger than a fixed positive constant independent of ε, as shown in [10].

Still we can use the results in [11], [12], in particular the properties of inverse pressure P^- to prove the following:

Theorem 3. *Let a hyperbolic skew product $F(x,y) = (f(x), g(x,y)) : I_* \times I \to I^2$ as above (where $I_* = I_1 \cup \ldots \cup I_k$), with f expanding on I_* and g_x contracting uniformly on I, for all $x \in I_*$; consider also its basic set $\Lambda(F)$. Then for any $x \in \Lambda(F)$, we have $t^s(k') \leq HD(W^s_r(x) \cap \Lambda(F)) \leq t^s(k'')$, where $t^s(j)$ is the unique zero of the pressure function $t \to P_{F|_\Lambda}(t\Phi^s - \log j), j = 1, \ldots, k$ and each point $z \in \Lambda$ has at least k'' F-preimages in Λ and at most k' F-preimages in Λ.*

Also, we have $HD(W^s_r(x) \cap \Lambda(F)) \leq t^-_s$, where t^-_s is the unique zero of the inverse pressure function $t \to P^-(t\Phi^s)$.

Let us now consider briefly also the case of holomorphic maps f on the complex projective space $\mathbb{P}^2$. One distinguishes the class of **s-hyperbolic** maps ([4]), namely those which have Axiom A, and satisfy the following conditions:

(i) $f^{-1}(S_2) = S_2$, where S_i is defined as the subset of points with complex unstable index i inside the nonwandering set $\Omega(f), i = 0, 1, 2$;

(ii) there exists an algebraic variety A of dimension 1 such that $A \cap S_1 = \emptyset$;

(iii) there exists a neighbourhood U of S_1 such that $f^{-1}(S_1) \cap U = S_1$.

In particular such a map is conformal on both its stable and unstable manifolds, and we can replace (ii) by the fact that Λ does not intersect the critical set $\mathcal{C}_f$ of f.

Condition (iii) above can be replaced by the fact that the function $d(x) := Card\{y \in \Lambda, f(y) = x\}$ is constant on Λ.

Denote $K^- := S_0 \cup W^u(\hat{S}_1)$, where $W^u(\hat{S}_1)$ is the union of all iterates of local unstable manifolds for all prehistories in $\hat{S}_1$; so K^- is the union of the set S_0 of periodic attracting points (S_0 being finite) and the global unstable set $W^u(\hat{S}_1)$. Then we have :

Theorem (Fornaess, Sibony [4]). *If f is s-hyperbolic, then the complement of K^- in $\mathbb{P}^2$ is a domain of holomorphy, hence K^- is connected.*

Moreover the interior of this set K^- is empty as was showed by:

Theorem (Mihailescu [9]). *If f is s-hyperbolic, then the interior of K^- is empty.*

In fact, estimates of the Hausdorff dimension of K^- have been given in [8], and it was proved that under mild conditions, not only the interior of K^- is empty, but also $HD(K^-) < 4$. The structure of K^- is related to the fractal structure and metric properties of $W^s_r(x) \cap S_1$, since due to s-hyperbolicity, Ω has local product structure ([5]).

As far as the structure of $W^u_r(\hat{x}) \cap S_1$ is concerned, one may apply again the theorems from [13] to obtain equality between $HD(W^u_r(\hat{x}) \cap S_1)$ and the zero t^u of the pressure function $t \to P_{\hat{f}|_{\hat{S}_1}}(t\Phi^u)$; the unstable dimension varies real analytically with respect to parameters.

Also, since f is holomorphic, we can give a precise form of the Holder exponent of the restriction of the conjugation map $\Phi(g)$ to $W^u_r(\hat{x}) \cap S_1$ (according to Theorem 5 from [13]), for a holomorphic perturbation g of f. Namely $\Phi(g)|_{W^u_r(\hat{x}) \cap \Lambda}$ has Holder exponent $\alpha(g) = \frac{\log(\lambda_u - \theta d_{B(S_1,2r)}(f,g))}{\log \lambda_u}$, where $\lambda_u := \inf_{\hat{x} \in \hat{\Lambda}} |Df_u(\hat{x})|$ and θ is a positive constant independent of g.

In the end, let us notice that condition (ii) in the definition of an s-hyperbolic map implies that the number of preimages in S_1, $d(\cdot) : S_1 \to \mathbb{N}$, is constant on connected components of S_1. This means that we can apply a theorem from [12] in order to obtain a formula also for the stable dimension.

Theorem 4. *Assume $f : \mathbb{P}^2\mathbb{C} \to \mathbb{P}^2\mathbb{C}$ is a holomorphic nondegenerate s-hyperbolic function, and suppose that Λ is a connected f-invariant component of S_1. Then there exists $r > 0$ giving the uniform size of local stable manifolds, and $HD(W^s_r(x) \cap \Lambda) = t^s(d')$, where d' is the number of f-preimages that any given point from Λ has in Λ, and $t^s(d')$ is the unique zero of the pressure function $t \to P_{f|_\Lambda}(t\Phi^s - \log d')$.*

There will exist again a geometric measure obtained by projections of an equilibrium measure supported on the intersection $W^u_r(\hat{x}) \cap S_1$.

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