Research Project PN III-P4-ID-PCE-2016-0823

Dynamics and Differentiable Ergodic Theory

This project is financed by UEFISCDI–Romanian Executive Agency for Higher Education, Research, Development and Innovation.

Brief description of the project:

The period of this project is July 2017–December 2019. It is based at the Institute of Mathematics of the Romanian Academy, in Bucharest.

Our research project will investigate several topics of high current interest in the fields of Dynamical Systems, Differentiable Ergodic Theory, and Analysis on Fractals. Some of the topics to be investigated are dynamics of hyperbolic non-invertible maps, dimensions of invariant measures on fractals, notions of entropy and pressure, ergodic number theory, basic sets of saddle type, chaotic systems. During the project we will publish articles in relevant international journals, we will disseminate our results in international conferences and seminars, and will collaborate with researchers from several countries. The project will contribute also to the formation of future researchers in Mathematics.

Goals of the project:

The main objective of the project is to study various important notions from Dynamical Systems and Ergodic Theory, as well as applications of these. Among the research goals of the project are the dynamics of noninvertible dynamical systems, finite and infinite iterated function systems with overlaps, applications of ergodic theory in number theory, continued fractions, Smale endomorphisms, various types of entropy for invariant probabilistic measures, dimension theory for hyperbolic sets and projection measures, applications of thermodynamic formalism, and relations with fractal geometry, geometric analysis, harmonic analysis and functional analysis.

Project Team:

Eugen Mihailescu – Project Director
Aurelian Gheondea
Radu Munteanu
Florin Boca
Camil Muscalu
Rodica Marineac – Ph.D student

Results and Articles:

A number of papers were elaborated in the project, namely:

(1) Eugen Mihailescu, Mariusz Urbanski, Skew product Smale endomorphisms over countable shifts of finite type, **Ergodic Theory and Dynamical Systems**, 2019, DOI: 10.1017/etds.2019.31 (Article Influence Score 2019 Red Zone), 45 pages.
(2) Eugen Mihailescu, Hyperbolic lifts and estimates for overlap numbers, **Journal of Statistical Physics** (2019), 177, 468484, https://doi.org/10.1007/s10955-019-02373-6. (Article Influence Score 2019 Red Zone).
(3) Eugen Mihailescu, Mariusz Urbanski, Smale endomorphisms over graph-directed Markov systems, 2019, arXiv:1907.13476, submitted for publication at **Ergodic Theory and Dynamical Systems** (Article Influence Score 2019 Red Zone).
(4) A. Gheondea, Symmetries versus conservation laws in dynamical quantum systems: a unifying approach through propagation of fixed points, **Annales Henri Poincaré**, 19 (2018), 1787-1816.(Article Influence Score 2019 Red Zone).
(5) Ay, Serdar; Gheondea, Aurelian, Invariant weakly positive semidefinite kernels with values in topologically ordered *-spaces, **Studia Mathematica**, 248, 2019, 255–294. (Article Influence Score 2019 Yellow Zone).
(6) Boca, Florin P., Linden, Christophe, On Minkowski type question mark functions associated with even or odd continued fractions, **Monatshefte Math**. 187 (2018), no. 1, 35–57. (Article Influence Score 2019 Yellow Zone).
(7) Boca, Florin P., Merriman, Claire, Coding of geodesics on some modular surfaces and applications to odd and even continued fractions, **Indagationes Math**. (N.S.) 29 (2018), no. 5, 1214–1234. (Impact Factor 2019 Yellow Zone).
(8) Eugen Mihailescu, Hyperbolic endomorphisms and overlap numbers, **Institut des Hautes Etudes Scientifiques**, 2018, IHES/M/18/10.
(9) Eugen Mihailescu, Pointwise dimension for a class of measures on limit sets, 2019, arXiv:1908.10050 , submitted for publication.
(10) Aurelian Gheondea, The spectral theorem for locally normal operators, **Opuscula Math**, 38, 5 (2018), 597621.
(11) Radu Munteanu, The flow of weights of some factors arising as fixed point algebras, 2018, arXiv:1811.10846v1.
(12) Petru Cojuhari, Aurelian Gheondea, A comparison of two generalizations of triplets of Hilbert spaces, 2019, ArXiv:1911.03460v1

Conference Organization:

In the period 7–11 October 2019 we organized the International Conference "Ergodic Theory and Related Fields" in this project, at the Institute of Mathematics of the Romanian Academy in Bucharest. The organizers were Eugen Mihailescu (IMAR) and Mariusz Urbański (Univ North Texas). The speakers in the conference were of a very high reputation, from many countries such as USA, UK, France, Poland, Austria, Canada, Portugal, Romania, Brazil, Japan, Hungary, etc. The conference was attended also by the members of the project, Eugen Mihailescu, Florin Boca, Radu Munteanu, Aurelian Gheondea, Rodica Marineac. The topics of the Conference were those of the project, namely differentiable ergodic theory, hyperbolicity for endomorphisms, dimension theory for iterated

function systems, applications to ergodic number theory, holomorphic dynamics, stochastics, relations with operator theory. Postdocs and graduate students from Romania also participated in this conference.

Dissemination of Results:

1) Eugen Mihailescu, Pennsylvania State University, Center for Dynamics and Geometry, USA, 2018.
2) Eugen Mihailescu, University of Illinois Analysis Seminar, USA, 2018.
3) Eugen Mihailescu, Dynamics, Measures and Dimension Conference, Poland, 2019.
4) Aurelian Gheondea, Institut Henri Poincare, France, 2017.
5) Aurelian Gheondea, Operator Theory Conference, Univ de Vest, Timisoara, 2018.
6) Eugen Mihailescu, Institut Mittag Leffler, Sweden, 2017.
7) Florin Boca, Ergodic Theory and Related Fields, Bucharest, Romania, 2019.
8) Eugen Mihailescu, Ergodic Theory and Related Fields Conference, Bucharest, 2019.
9) Radu Munteanu, Ergodic Theory and Related Fields Conference, Bucharest, 2019.
10) Aurelian Gheondea, Ergodic Theory and Related Fields Conference, Bucharest, 2019.
11) Eugen Mihailescu, AMS Sectional Meeting, Dynamics on fractals, USA, 2019.
12) Aurelian Gheondea, Conference on Harmonic Analysis and Operator Theory, Turkey, 2019.

Students:

In this project, R. Marineac started the Ph.D program at IMAR under the supervision of Eugen Mihailescu.

Results 2017:

Articles:

• E. Mihailescu, M. Urbański, Skew product Smale endomorphisms over countable shifts of finite type. Submitted for publication 2017. ArXiv:1705.05880

Abstract: We introduce and study skew product Smale endomorphisms over finitely irreducible topological Markov shifts with countable alphabets. This case is very different from the one with finite alphabets, and we develop new methods. We prove that almost all conditional measures of equilibrium states of summable Hölder continuous potentials are dimensionally exact, and their dimension is equal to the ratio of (global) entropy and Lyapunov exponent. We show that the exact dimensionality of conditional measures on fibers implies global exact dimensionality of the original measure. We then study equilibrium states and dimension for skew products over expanding Markov-Rényi transformations, and settle the question of exact dimensionality of such measures. In particular, we obtain the exact dimensionality of such measures with respect to skew products over the continued fractions transformation. We then prove results related to Diophantine approximation, which extend and improve the Doeblin-Lenstra Conjecture on the distribution frequencies

of Diophantine approximation coefficients, for a much larger class of measures. Namely the asymptotic frequencies of Diophantine approximation coefficients are studied very precisely for equilibrium measures of geometric potentials, measures which are singular with respect to the Lebesgue measure.

• S. Ay, A. Gheondea, Invariant weakly positive semidefinite kernels with values in topologically ordered *-spaces, preprint 2017.

Abstract: We consider weakly positive semidefinite kernels valued in ordered *-spaces with or without certain topological properties, and investigate their linearisations (Kolmogorov decompositions) as well as their reproducing kernel spaces. The spaces of realisations are of VE (Vector Euclidean) or VH (Vector Hilbert) type, more precisely, vector spaces that possess gramians (vector valued inner products). The main results refer to the case when the kernels are invariant under certain actions of *-semigroups and show under which conditions *- representations on VE-spaces, or VH-spaces in the topological case, can be obtained. Finally we show that these results unify most of dilation type results for invariant positive semidefinite kernels with operator values.

• F. Boca, C. Linden, On Minkowski type question mark functions associated with even or odd continued fractions, 2017, ArXiv:1705.01238. Paper appeared in Monatshefte Mathematik, 187 (2018), no. 1, 35–57.

Abstract: We study analogues of Minkowskis question mark function $?(x)$ related to continued fractions with even or odd partial quotients. We prove that these functions are Hölder continuous with precise exponents, and that they linearize the appropriate versions of the Gauss and Farey maps.

Research Visits:
Eugen Mihailescu visited Institut Mittag Leffler in Stockholm, Sweden.
J. Schmeling, Lund University, was invited at IMAR to work with E. Mihailescu.
A. Gheondea visited Institut Henri POincaré, France.

Raport Stiintific 2017:

In cadrul etapei unice pe 2017 am studiat si rezolvat obiectivele propuse pentru aceasta etapa, si anume formalismul termodinamic si relatiile sale cu alte domenii. O directie principala a fost in IFS (sisteme iterative de functii) cu generatori finiti sau numarabili, si suprapuneri. In articolul "Skew product Smale endomorphisms over countable shifts of finite type", impreuna cu Mariusz Urbanski (Univ. North TExas, SUA), am studiat endomorfisme de tip produs incrucisat (skew products) peste shift-uri Markov topologice ireductibile cu alfabet numarabil.

Am aratat ca aproape toate masurile conditionale ale masurilor de echilibru pentru potentiali local Hölder continui sunt exact dimensionale, si am demonstrat o formula pentru dimensiunea lor Hausdorff. Am descris apoi formalismul termodinamic pentru skew products de tip Smale peste endomorfisme countable-to-1, si am gasit aplicatii interesante la

masuri pe extensiile naturale ale endomorfismelor. Exact dimensionalitatea masurilor conditionale pe fibre implica exact dimensionalitatea globala a masurii, in multe cazuri.

Am studiat masurile de echilibru si pentru skew products peste sisteme Markov graph-directed, in particular pentru produse skew peste aplicatii de expansiune Markov-Rényi (EMR). Am rezolvat o problema cunoscuta a exact dimensionalitatii globale pentru astfel de masuri. In particular, aceasta se aplica produselor skew peste transformarea Gauss a fractiilor continue, si peste aplicatii parabolice.

Apoi am demonstrat doua rezultate in aproximarea Diofantina, care extind Conjectura Doeblin-Lenstra, la numere x din multimea de masura Lebesgue nula, care nu este controlata de Conjectura Doeblin-Lenstra originala. Deasemenea, am aplicat rezultatele la extensii naturale $\mathcal{T}_\beta$ ale aplicatiilor-beta, pentru $\beta > 1$ arbitrar.

Notiunea de spatiu Smale pe care o folosim, desi inspirata initial de cea a lui Ruelle, este diferita. Mai intai dezvoltam un formalism termodinamic complet pentru shift-uri bilaterale, apoi pentru endomorfisme Smale care sunt conforme pe fibre, am demonstrat exact dimensionalitatea masurilor de echilibrupe fibre si am gasit formule pentru dimensiune. Exact dimensionalitatea este o proprietate importanta, deoarece implica faptul ca mai multe cantitati dimensionale sunt egale (dimensiunea Hausdorff, dimensiunea punctuala, dimensiunea Minkowski, etc.)

Apoi am studiat produse de tip skew generale peste endomorfisme numarabile-la-1. Cazul principal in care aplicam aceste rezultate este la extensii naturale de endomorfisme, care in multe cazuri, in special din teorie ergodica a numerelor, pot fi privite ca produse skew peste anumite endomorfisme.

Am definit endomorfisme Smale de tip produs skew modelate pe subshift-uri de tip finit cu alfabet numarabil, si am specificat cateva clase seminificative. Am aratat ca daca un endomorfism Smale de tip produs skew este continuu si de tip compact, atunci exista o bijectie intre masurile invariante pentru dinamica simbolica, si acelea pentru endomorfismul Smale. Presupunand ca endomorfismul Smale este Holder continuu, am demonstrat existenta si unicitatea masurilor de echilibru pentru potentiali local Holder continui.

Apoi spre exemplu am aplicat rezultatele obtinute pentru a da o generalizare a Conjecturii Doeblin-Lenstra din aproximarea Diofantina, obtinand comportamentul statistic al frecventelor asimptotice ale aproximantilor diofantini in raport cu unele masuri singulare.

Theorem 1. *Fie $F : X \times Y \to X \times Y$ un endomorfism generalizat Smale de tip skew produs conform. Fie $\phi : J(X) \to \mathbb{R}$ un potential a.i:*

$$\psi := \phi \circ p_J \circ \hat{\pi} : E_A \to \mathbb{R}$$

e local Hölder continu si sumabil. Presupunem ca aplicatia de coding $p : E_A^+ \to X$ este $\mu_\phi \circ p_1^{-1}$–injectiva.

Atunci pentru $\mu_\phi \circ p_1^{-1}$-a.e $x \in X$, masura conditionala μ_ϕ^x este exact dimensionala pe J_x, si in plus

$$\lim_{r\to 0} \frac{\log \mu_\phi^x(B(y,r))}{\log r} = \frac{\mathrm{h}_{\mu_\phi}(\mathrm{F})}{\chi_{\mu_\phi}(F)} = \mathrm{HD}(\mu_\phi^x),$$

pentru μ_ϕ^x-a.e $y \in J_x$; asadar, echivalent, pentru μ_ϕ-a.e $(x, y) \in J(X)$.

Ca o consecinta, am obtinut urmatorul:

Corollary 2. *Fie $F : X \times Y \to X \times Y$ un endomorfism generalizat Smale de tip produs skew. Fie $\phi : J(X) \to \mathbb{R}$ un potential local Hölder continuu, a.i*

$$\sum_{e \in E} \exp(\sup(\phi|_{\pi([e]) \times Y})) < \infty.$$

Sa presupunem si ca aplicatia coding $p : E_A^+ \to X$ este $\mu_\phi \circ p_1^{-1}$–injectiva.

Atunci pentru $\mu_\phi \circ p_1^{-1}$-a.e $x \in X$, masura conditionala μ_ϕ^x este exact dimensionala pe J_x, si

$$\lim_{r \to 0} \frac{\log \mu_\phi^x(B(y,r))}{\log r} = \frac{\mathrm{h}_{\mu_\phi}(\mathrm{F})}{\chi_{\mu_\phi}(F)} = \mathrm{HD}(\mu_\phi^x),$$

pentru μ_ϕ^x-a.e $y \in J_x$; deci, echivalent pentru μ_ϕ-a.e $(x, y) \in J(X)$.

O problema importanta studiata in cadrul proiectului se refera si la proprietatile dinamice si metrice ale fractiilor continue, care se pot obtine ca si cazuri particulare de sisteme iterative numarabile conforme. Am studiat astfel masurile de echilibru pentru potentiali geometrici pentru aplicatia Gauss T si extensia sa naturala.

Deasemenea in cadrul proiectului in anul 2017, s-au studiat unele legaturi intre dinamica si analiza armonica (masuri de convolutie infinita), cat si nuclee semidefinite slab positive in spatii ordonate, cu sau fara anumite proprietati topologice. Principalele rezultate se refera la cazul cand nucleele sunt invariante la actiuni ale unor *-semigrupuri. Deasemenea am aratat ca aceste rezultate unifica mai multe rezultate de dilatare pentru nuclee invariante pozitiv semidefinite cu valori operatori, si rezultate recente despre aplicatii pozitiv semidefinite pe *-semigrupuri.

Results 2018:

Articles:

• Eugen Mihailescu, Hyperbolic endomorphisms and overlap numbers, Institut des Hautes Études Sciéntifiques, 2018, IHES/M/18/10.

Abstract: Hyperbolic endomorphisms and overlap numbers are studied on lifts of invariant sets. We prove an estimate on the box dimension of an invariant measure on the limit set, by using the overlap number of μ . Then we compute topological overlap numbers in several concrete cases. Topological overlap numbers are then used in dimension estimates.

• Aurelian Gheondea, Symmetries versus conservation laws in dynamical quantum systems: a unifying approach through propagation of fixed points, Annales Henri Poincaré, vol 19 (2018), 1787-1816.

Abstract: We unify recent Noether-type theorems on the equivalence of symmetries with conservation laws for dynamical systems of Markov processes, of quantum operations, and of

quantum stochastic maps, by means of some abstract results on propagation of fixed points for completely positive maps on C^* -algebras. We extend most of the existing results with characterisations in terms of dual infinitesimal generators of the corresponding strongly continuous one-parameter semigroups. By means of an ergodic theorem for dynamical systems of completely positive maps on von Neumann algebras, we show the consistency of the condition on the standard deviation for dynamical systems of quantum operations.

• Eugen Mihailescu, Pointwise dimension of projection measures and dynamics over limit set, arXiv:1908.10050, 2018, submitted for publication.

Abstract: We study the dynamics of an associated skew product and the pointwise dimension of projected measures on limit sets for conformal iterated systems with overlaps. We employ the hyperbolic structure of the associated skew product to study several families of conditional measures. This is used then to prove that the pointwise dimension of projected measures of equilibrium measures, is given by a formula involving the folding entropy and Lyapunov exponents. In particular this formula relates the projectional entropy with the folding entropy. We also obtain the exact dimensionality of a non-canonical projection measure on the limit set.

• Boca, Florin P., Merriman, Claire, Coding of geodesics on some modular surfaces and applications to odd and even continued fractions. Indagationes Math. (N.S.) 29 (2018), no. 5, 1214–1234.

Abstract: The connection between geodesics on the modular surface $PSL(2,\mathbb{Z})\setminus\mathbb{H}$ and regular continued fractions, established by Series, is extended to a connection between geodesics on $\Gamma\setminus\mathbb{H}$ and odd and grotesque continued fractions, where $\Gamma\equiv\mathbb{Z}_3\star\mathbb{Z}_3$ is the index two subgroup of $PSL(2,\mathbb{Z})$ generated by the order three matricial elements , and having an ideal quadrilateral as fundamental domain. A similar connection between geodesics on $\Theta\setminus\mathbb{H}$ and even continued fractions is discussed in our framework, where Θ denotes the Theta subgroup of $PSL(2,\mathbb{Z})$ generated by two matrices.

• Eugen Mihailescu, Ergodic lifts and overlap numbers, arXiv:1808.01674. Submitted for publication 2018.

Abstract: We study skew product lifts and overlap numbers for equilibrium measures μ_ψ of Hölder continuous potentials ψ on such lifts. We find computable formulas and estimates for the overlap numbers in several concrete significant cases of systems with overlaps. In particular we obtain iterated systems which are asymptotically irrational-to-1 and absolutely continuous on their limit sets. Then we look into the general structure of the Rokhlin conditional measures of μ_ψ with respect to different fiber partitions associated to the lift Φ, and find relations between them. Moreover we prove an estimate on the box dimension of a certain associated invariant measure ν_ψ on the limit set Λ by using the overlap number of μ_ψ.

• Aurelian Gheondea, The spectral theorem for locally normal operators, Opuscula Math. 38, no. 5 (2018), 597-621.

Abstract: We prove the spectral theorem for locally normal operators in terms of a locally spectral measure. In order to do this, we first obtain some characterisations of local projections and we single out and investigate the concept of a locally spectral measure.

• Radu Munteanu, The flow of weights of some factors arising as fixed point algebras, 2018, arXiv:1811.10846v1.

Abstract: In this paper we study the associated flow of some factors arising as fixed point algebras under product type actions. We compute their associated flow and show that under certain conditions these flows are approximately transitive.

Research Visits:

E. Mihailescu was invited for research collaboration and gave a talk at Pennsylvania State University, USA, in April 2018. He also visited for research collaboration and gave a talk at University of Illinois, USA in April 2018.

A.Gheondea visited Univ Vest, Timisoara.

Raport Stiintific 2018:

In cadrul etapei unice 2018, echipa proiectului a studiat mai multe directii de cercetare actuale pe plan international, si a obtinut rezultate importante. Aceste directii au cuprins diverse notiuni de entropie (folding entropy, entropie de masura, entropie topologica, entropie conditionala), dinamica (non) conforma, dimensiuni fractale pentru proiectii de masuri, liftari ergodice si numere de suprapunere in raport cu masuri de echilibru, legaturi cu teoria ergodica a numerelor, masuri conditionale pe fibre, fractii continue pare sau impare, caturi partiale de fractii continue, sisteme dinamice ale unor procese Markov sau ale unor aplicatii stocastice, curenti de ponderi.

Una din directiile din 2018 a fost in studiul liftarilor ergodice la endomorfisme skew-product ale unor sisteme iterative de functii cu suprapuneri. Presupunem ca avem un sistem $\mathcal{S} = \{\phi_1, \ldots, \phi_m\}$ de functii conforme injective definite pe un deschis marginit $V \subset \mathbb{R}^d$, a.i $\mathcal{S}$ este cu suprapuneri. Notam cu Λ multimea sa limita, si deci avem intersectii nevide de tipul

$$\phi_i(\Lambda) \cap \phi_j(\Lambda) \neq \emptyset$$

Atunci asociem sistemului $\mathcal{S}$ endomorfismul $\Phi : \Sigma_I^+ \times \Lambda \to \Sigma_I^+ \times \Lambda$, definit prin

$$\Phi(\omega, x) = (\sigma\omega, \phi_{\omega_1}(x))$$

Luam si un potential Hölder $\psi : \Sigma_I^+ \to \mathbb{R}$, care din cauza structurii hiperbolice a lui Φ, are o unica masura de echilibru μ_ψ pe $\Sigma_I^+ \times \Lambda$. Am aratat in [9] ca aceasta masura este si Gibbs si ca are doua proiectii masuri probabilistice pe Λ,

$$\nu_1 := \pi_{1*}\mu_\psi, \text{ si } \nu_2 := \pi_{2*}\mu_\psi$$

In [16] am introdus notiunea de *numere de suprapunere* $o(\mathcal{S}, \mu_\psi)$ pentru o masura de echilibru μ_ψ pentru un potential Hölder pe $\Sigma_I^+ \times \Lambda$. Acest numar de suprapunere este un

numar asimptotic mediu al n-preimaginilor μ_ψ-generice in Λ (deoarece punctele din Λ se pot acoperi de multe ori cu imaginile $\phi_{i_1\dots i_m}(\Lambda)$). Mai precis, am notat

$$\Delta_n((\omega,x),\tau,\mu_\psi)=\{(\eta_1,\dots,\eta_n)\in I^n,\exists y\in\Lambda,\phi_{\omega_n\dots\omega_1}(x)=\phi_{\eta_n\dots\eta_1}(y),\ |\frac{S_n\psi(\eta,y)}{n}-\int_{\Sigma_I^+\times\Lambda}\psi\,d\mu_\psi|<\tau\},$$

unde $(\omega,x)\in\Sigma_I^+\times\Lambda$, iar $S_n\psi(\eta,y)$ este suma consecutiva a lui ψ in raport cu Φ. Notam deasemenea cu

$$b_n((\omega,x),\tau,\mu_\psi):=Card\Delta_n((\omega,x),\tau,\mu_\psi)$$

Atunci in [16] am aratat ca urmatoarea limita exista si defineste *numarul de suprapunere* al lui μ_ψ,

$$o(\mathcal{S},\mu_\psi)=\lim_{\tau\to 0}\lim_{n\to\infty}\frac{1}{n}\int_{\Sigma_I^+\times\Lambda}\log b_n((\omega,x),\tau,\mu_\psi)\ d\mu_\psi(\omega,x)$$

Deasemenea am aratat ca $o(\mathcal{S},\mu_\psi)$ este legata de *folding entropy* $F_\Phi(\mu_\psi)$. Aceasta notiune a fost introdusa de Ruelle si e definita ca entropia conditionala $H_{\mu_\psi}(\epsilon|\Phi^{-1}\epsilon)$.

In [8] am demonstrat urmatoarea teorema ce face legaturi intre diversele masuri conditionale pe fibre (de exp [26]) si numarul de suprapunere.

Theorem 3 ([8]). *Fie $\Phi:\Sigma_I^+\times\Lambda\to\Sigma_I^+\times\Lambda$ endomorfismul lift de mai sus, si ψ un potential Hölder continuu pe $\Sigma_I^+\times\Lambda$ cu masura de echilibru $\mu:=\mu_\psi$. Atunci numarul de suprapunere $o(\mathcal{S},\mu)$ al lui μ_ψ este determinat de familiile de masuri conditionale asociate $(\mu_\omega)_\omega,(\mu_\omega^+)_\omega$ prin formula:*

$$\log o(\mathcal{S},\mu)=-\sum_{i\in I}\int_{\Sigma_I^+\times\Lambda}\frac{\mu_\omega^+(i)}{\sum_{j\in I}\mu_\omega^+(j)\cdot\lim_{A_2\to x}\frac{\mu_{j\omega}(\phi_j^{-1}\phi_iA_2)}{\mu_{i\omega}(A_2)}}\cdot\log(\frac{\mu_\omega^+(i)}{\sum_{j\in I}\mu_\omega^+(j)\cdot\lim_{A_2\to x}\frac{\mu_{j\omega}(\phi_j^{-1}\phi_iA_2)}{\mu_{i\omega}(A_2)}})\,d\mu(\omega,x)$$

Intr-o teorema din [8] am estimat dimensiunea box inferioara a masurii proiectie $\nu_\psi=\pi_{2*}\mu_\psi$ cu ajutorul numerelor de suprapunere. Astfel se poate estima numeric dimesiunea fractala in multe cazuri concrete, studiate in [8], [9]. Mai mult am construit multimea Z pe care se atinge dimensiunea box inferioara. Dimensiunea box este importanta in studiul masurilor invariante (de exp [2], [23], [11], [13]).

Theorem 4 ([8]). *Consideram sistemul conform $\mathcal{S}=\{\phi_i,i\in I\}$ cu multime limita Λ, si potentialul Hölder $\psi:\Sigma_I^+\times\Lambda\to\mathbb{R}$, cu unica sa masura de echilibru μ_ψ, si fie $\nu_\psi:=\pi_{2*}\mu_\psi$. Atunci,*

$$\underline{dim}_B(\nu_\psi)\le\frac{h_\sigma(\pi_{1*}(\mu_\psi))-\log o(\mathcal{S},\mu_\psi))}{\chi(\mu_\psi)}$$

Am gasit estimari ale numarului de suprapunere si ale dimensiunii box inferioare, pentru anumite cazuri de partial overlaps in [8].

Theorem 5 ([8]). *Presupunem ca exista familii $\mathcal{F}_1,\dots,\mathcal{F}_s\subset I^p$ de p-tuple si $k_1,\dots,k_s\ge 1$ astfel incat, pentru orice $1\le j\le s$ si orice $(i_{j1},\dots,i_{jp})\in\mathcal{F}_j$ exista un k_j-tuple $(j_1,\dots,j_{k_j})\in I^{k_j}$, cu*

$$\phi_{i_{j1}\dots i_{jp}j_1\dots j_{k_j}}(\Lambda)\subset\bigcap_{(\ell_1,\dots,\ell_p)\in\mathcal{F}_j}\phi_{\ell_1\dots\ell_p}(\Lambda)$$

Atunci daca $N(\mathcal{F}_j) := Card\mathcal{F}_j,\ 1 \leq j \leq s$, *obtinem:*

$$\underline{dim}_B(\pi_{2*}\mu_{max}) = \underline{dim}_B(\pi_*\mu^+_{max}) \leq \frac{p \cdot h_\sigma(\mu^+_{max}) - \frac{N(\mathcal{F}_1)\log N(\mathcal{F}_1)}{m^{p+k_1}} - \ldots - \frac{N(\mathcal{F}_s)\log N(\mathcal{F}_s)}{m^{p+k_s}}}{p \cdot \chi(\mu_{max})}$$

Deasemenea in [8], [9] am studiat numarul de suprapunere topologic (adica numarul de suprapunere al masurii de entropie maximala) in cazul unor exemple de sisteme iterative de functii cu suprapuneri partiale.

In articolul [10] am demonstrat o teorema ce arata ca daca $\tilde{\psi}$ este un potential Hölder pe spatiul shift Σ_I^+, atunci o masura de proiectie asociata ν_2 e exact dimensionala ([23], [30]).

O teorema stabilita in [7] in cadrul proiectului este si urmatoarea:

Theorem 6 ([7]). *Fie* $\mathcal{M}$ *o algebra von Neumann si* $\Phi = \{\Phi_t\}_{t\geq 0}$ *un semigrup* w^**-continuu de aplicatii complet pozitive pe* $\mathcal{M}$. *Atunci:*

i) exista o aplicatie idempotenta $\Psi : \mathcal{M} \to \mathcal{M}$ *astfel incat multimea punctelor fixe* $\mathcal{M}^\Phi$ *este imaginea lui* Ψ.

ii) $\mathcal{M}$ *este stabila la multiplicare daca si numai daca* Ψ *este o conditional expectation, si daca si numai daca* $\mathcal{M}^\Phi$ *este o algebra von Neumann.*

Se fac astfel legaturi intre teoria ergodica si sistemele dinamice cu mecanica stocastica.

Deasemenea in [5] s-au studiat analoage ale functiei lui Minkowski ?(x), legate de fractiile continue cu caturi partiale pare sau impare. Importanta in acest context este si aplicatia Farey $F : [0,1] \to [0,1]$ care are o masura invarianta infinita $\frac{dx}{x(1-x)}$.
S-a introdus o functie analoaga functiei question mark a lui Minkowksi, Q_E, si s-a demonstrat ca:

Theorem 7 ([5]). $Q_E(\cdot)$ *este Hölder continua cu exponent* $\frac{\log 3}{2\log(1+\sqrt{2})}$.

Iar in articolul [4], s-au studiat aplicatii de codare a geodezicelor pe unele suprafete modulare ([29]), cu aplicatii in special la fractiile continue pare sau impare.

In articolul [21] s-a studiat curentul de ponderi asociat unor factori ce apar ca algebre de puncte fixe la actiunile de tip produs pe factori ITPFI. S-a aratat si ca, in anumite conditii, acesti curenti sunt aproximativ tranzitivi.

Results 2019:

Conference Organization:

In the period 7–11 October 2019 we organized the International Conference "Ergodic Theory and Related Fields" in this project, at the Institute of Mathematics of the Romanian Academy in Bucharest. The organizers were Eugen Mihailescu and Mariusz Urbański. The speakers in the conference were of a very high reputation, from many countries such as USA, UK, France, Canada, Romania, Poland, Austria, Portugal, Brazil, Japan, etc. The topics of the Conference were those of the project, namely differentiable ergodic theory,

hyperbolicity for endomorphisms, dimension theory for iterated function systems, applications to ergodic number theory, holomorphic dynamics, stochastics, relations with operator theory. The conference was attended also by the members of the grant. Graduate students from Romania participated too in this conference.

Articles:

• Eugen Mihailescu and Mariusz Urbanski, Skew product Smale endomorphisms over countable shifts of finite type, Ergodic Theory and Dynamical Systems, 2019, DOI: 10.1017/etds.2019.31.

Abstract: We introduce and study skew product Smale endomorphisms over finitely irreducible shifts with countable alphabets. This case is different from the one with finite alphabets and we develop new methods. In the conformal context we prove that almost all conditional measures of equilibrium states of summable Hölder continuous potentials are exact dimensional and their dimension is equal to the ratio of (global) entropy and Lyapunov exponent. We show that the exact dimensionality of conditional measures on fibers implies global exact dimensionality of the original measure. We then study equilibrium states for skew products over expanding Markov-Rényi transformations and settle the question of exact dimensionality of such measures. We apply our results to skew products over the continued fraction transformation. This allows us to extend and improve the DoeblinLenstra conjecture on Diophantine approximation coefficients to a larger class of measures and irrational numbers.

• Eugen Mihailescu, Hyperbolic lifts and estimates for overlap numbers, Journal of Statistical Physics (2019), 177, 468-484, https://doi.org/10.1007/s10955-019-02373-6.

Abstract: We first compute topological overlap numbers in several concrete cases. In particular is obtained a class of Bernoulli convolutions systems which asymptotically are irrational-to-1 on their limit sets. Next, for general conformal iterated function systems and for a class of potentials ψ, we prove an estimate on the box dimension of a measure ν_ψ on the limit set, associated to an equilibrium measure μ_ψ on the respective hyperbolic lift. Also we study the structure of the families of conditional measures of μ_ψ with respect to various fiber partitions, and find relations between them.

• Ay, Serdar; Gheondea, Aurelian, Invariant weakly positive semidefinite kernels with values in topologically ordered *-spaces, Studia Mathematica, 248, 2019, 255–294.

Abstract: We consider weakly positive semidefinite kernels valued in ordered *-spaces with or without certain topological properties, and investigate their linearisations (Kolmogorov decompositions) as well as their reproducing kernel spaces. The spaces of realisations are of VE (Vector Euclidean) or VH (Vector Hilbert) type, more precisely, vector spaces that possess gramians (vector valued inner products). The main results refer to the case when the kernels are invariant under certain actions of *-semigroups and show under which conditions *- representations on VE-spaces, or VH-spaces in the topological case, can be obtained. Finally we show that these results unify most of dilation type results for invariant positive semidefinite kernels with operator values.

• Eugen Mihailescu, Mariusz Urbanski, Smale endomorphisms over graph-directed Markov systems, arXiv:1907.13476, 2019, submitted for publication.

Abstract: In this paper we study Smale skew product endomorphisms (introduced in [21]) now over countable graph directed Markov systems, and we prove the exact dimensionality of conditional measures in fibers, and then the global exact dimensionality of the equilibrium measure itself. In particular, this applies to natural extensions of graph directed Markov systems, and to skew products over parabolic systems. This comprises large classes of examples. In the end, we apply these results to obtain a general formula for the Hausdorff (and pointwise) dimension of equilibrium measures with respect to induced maps of natural extensions $\mathcal{T}_\beta$ of β-maps, for arbitrary $\beta > 1$.

• P. Cojuhari, A. Gheondea, A comparison of two generalizations of triplets of Hilbert spaces, 2019, ArXiv:1911.03460v1.

Abstract: We compare the concept of triplet of closely embedded Hilbert spaces with that of generalised triplet of Hilbert spaces in the sense of Berezanskii by showing when they coincide, when they are different, and when starting from one of them one can naturally produce the other one that essentially or fully coincides.

• Eugen Mihailescu, Radu Munteanu, Inverse measure-theoretic entropy for endomorphisms, preprint 2019.

Abstract: We introduce a new pointwise notion of inverse measure theoretic entropy for an arbitrary measure μ preserved by a measurable endomorphism f. We show that this inverse entropy $h_f^-(\mu)$ is related to the folding entropy $F_f(\mu)$ and to the usual forward entropy. We compute the inverse entropy in certain concrete examples. Also we can compute the inverse entropy for the SRB measure of fat baker's transformations.

Raport Stiintific Final 2019:

• In cadrul etapei unice **2019**, am continuat cercetarea pe directiile grantului, cat si pe unele directii noi. De exemplu una dintre directii a constituit-o dimensiunea punctuala pentru proiectii non-standard de masuri pe multimi limita ale unor sisteme de contractii conforme cu suprapuneri.

Fie asadar un sistem finit de contractii conforme injective $\mathcal{S} = \{\phi_j, j \in I\}$, cu multimea limita Λ. Daca μ este o masura σ-invarianta pe spatiul shift Σ_I^+, atunci Feng si Hu ([6]) au aratat ca proiectia canonica $\pi_*\mu$ pe Λ este exact dimensionala si are dimensiunea Hausdorff egala cu $\frac{h_\pi(\mu)}{\chi(\mu)}$. Entropia de proiectie $h_\pi(\mu)$ este diferenta unor masuri conditionale si in general este dificil de calculat/estimat. In cazul nostru, am studiat insa o alta proiectie, si anume am luat o masura de echilibru μ_ψ pentru sistemul dinamic hiperbolic asociat $\Phi : \Sigma_I^+ \times \Lambda \to \Sigma_I^+ \times \Lambda$ si un potential Hölder, si am luat proiectia $\nu_\psi := \pi_{2*}\mu_\psi$ pe Λ.

Am demonstrat in articolul din grant [10] urmatoarea:

Theorem 8 ([10]). *Fie S un sistem iterativ de contractii conforme si injective, si ψ un potential Hölder continuu pe Σ_I^+, cu masura de echilibru μ_ψ in raport cu transformarea Φ. Notam cu $\nu_\psi := \pi_{2*}\mu_\psi$. Atunci pentru ν_ψ-a.e $x \in \Lambda$,*

$$\delta(\nu_\psi)(x) = \frac{F_\Phi(\mu_\psi) - h_\sigma(\pi_{1*}\mu_\psi)}{\chi_s(\mu_\psi)}$$

In particular, masura ν_ψ este exact dimensionala pe Λ.

Demonstratia acestei teoreme este dificila si foloseste Lema de Densitate Borel, estimari folosing Teorema Birkhoff, si descompuneri ale masurii dupa preistorii punctelor generice.

Un caz important este cel al masurilor auto-conforme, adica π_1-proiectii de masuri Bernoulli pe Λ. Pentru sistemul iterativ conform $\mathcal{S}$ si vectorul probabilistic $\mathbf{p} = (p_1, \ldots, p_{|I|})$, fie $\nu_\mathbf{p}$ masura Bernoulli asociata pe Σ_I^+. Notam cu $\mu_\mathbf{p}$ lift-ul lui $\nu_\mathbf{p}$ la $\Sigma_I^+ \times \Lambda$. Pentru masuri auto-conforme am aratat in [16] ca masurile proiectii $\nu_{1,\mathbf{p}}$ and $\nu_{2,\mathbf{p}}$ sunt egale. Deci dimensiunile $\delta(\nu_{1,\mathbf{p}}) = \delta(\nu_{2,\mathbf{p}})$. Obtinem deci o formula pentru entropia de proiectie $h_\pi(\sigma, \nu_\mathbf{p})$ a masurii $\nu_{1,\mathbf{p}}$ in functie de numarul sau de suprapunere.

Theorem 9 ([10]). *In cadrul de mai sus, fie $\nu_{\boldsymbol{p}}$ o masura Bernoulli pe Σ_I^+, si fie $\nu_{1,\boldsymbol{p}}, \nu_{2,\boldsymbol{p}}$ masurile proiectii pe Λ, asociate proiectiilor π_1, π_2. Atunci $\nu_{1,\boldsymbol{p}} = \nu_{2,\boldsymbol{p}}$, si pt $\nu_{1,\boldsymbol{p}}$-a.e $x \in \Lambda$,*

$$\delta(\nu_{1,\boldsymbol{p}})(x) = \delta(\nu_{2,\boldsymbol{p}})(x) = \frac{\exp(o(\mathcal{S}, \hat{\mu}_{\boldsymbol{p}})) - h_\sigma(\nu_{\boldsymbol{p}})}{\chi_s(\hat{\mu}_{\boldsymbol{p}})}$$

Mai mult, entropia de proiectie a lui $\nu_{\boldsymbol{p}}$ este determinata de,

$$h_\pi(\sigma, \nu_{\boldsymbol{p}}) = h_\sigma(\nu_{\boldsymbol{p}}) - \exp(o(\mathcal{S}, \hat{\mu}_{\boldsymbol{p}}))$$

In particular entropia de proiectie a masurii μ_0 de entropie maxima pe Σ_I^+ se obtine ca:

$$h_\pi(\sigma, \mu_0) = \log|I| - \exp(o(\mathcal{S}))$$

Tot in cadrul grantului in 2019 am calculat/estimat in articolul [9] numarul de suprapunere topologic in cazul unui sistem de convolutie Bernoulli $\mathcal{S}_\lambda$ cu λ^{-1} un numar Garsia, sau un numar Pisot.

Theorem 10 ([9]). *Fie $\lambda \in (\frac{1}{2}, 1)$ un numar cu $\frac{1}{\lambda}$ un numar Garsia. Atunci numarul topologic de suprapunere al masurii de convolutie Bernoulli μ_λ este egal cu 2λ, adica*

$$o(\mathcal{S}_\lambda) = 2\lambda$$

In particular, pentru numere Garsia de forma $2^{\frac{1}{n}}, n \geq 2$, avem ca $o(\mathcal{S}_{2^{-\frac{1}{n}}}) = 2^{1-\frac{1}{n}}$, pentru orice $n \geq 2$. Deci aceste sisteme sunt asimptotic irationale-la-1.

In articolul [19] am studiat in 2019 endomorphism de tip Smale skew product this paper we study Smale skew product endomorphisms (introduced in [21]) peste sisteme graf-directionate Markov numarabile, si am demonstrat exact dimensionalitatea masurilor conditionale in fibre, si apoi exact dimensionalitatea globala a masurii de echilibru initiale. In particular, aceasta se aplica la extensii naturale a sistemelor graf-directionate Markov numarabile, cat si endomorphismelor de tip skew product peste sisteme parabolice.

Aceasta cuprinde clase largi de exemple. Deasemenea am aplicat aceste rezultate pentru a obtine o formula generala pentru dimensiunea Hausdorff (si punctuala) a masurilor de echilibru in raport cu aplicatii induse de extensii naturale $\mathcal{T}_\beta$ de β-transformari cu $\beta > 1$ arbitrar.

Un caz interesant studiat in [19], caruia ii putem aplica rezultatele obtinute, este cel al dezvoltarii in fractii continue inverse (backward continued fractions). orice numar irational $x \in [0,1]$ are o unica dezvoltare de forma

$$x = \frac{1}{\omega_1 - \frac{1}{\omega_2 - \frac{1}{\omega_3 - \frac{1}{\cdots}}}},$$

unde $\omega_1, \omega_2, \ldots$ sunt intregi ≥ 2. Aplicatia corespunzatoare aplicatiei Gauss de la fractiile continue este aplicatia Rényi $V : [0,1) \longrightarrow [0,1)$,

$$V(x) := \begin{cases} \left\{\frac{1}{1-x}\right\} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Sistemul de fractii continue inverse e dat de

$$\phi_i : [0,1] \to [0,1], \quad \phi_i(x) = \frac{1}{i-x}, \ i \geq 2. \tag{0.1}$$

Deci ϕ_2 are un punct fix neutru in 1, si e contractiva in orice alt punct. Toate celelalte aplicatii ϕ_j sunt de contractie pe $[0,1]$, pentru $j \geq 3$. Pentru acest tip de fractii continue inverse masura invarianta absolut continua este $\frac{dx}{x}$.

Sistemul de mai sus satisface conditiile pentru un sistem parabolic. Putem deci aplica rezultatele noastre pentru un endomorfism conform de tip skew product Hölder $F : J_{\mathcal{S}^*} \times Y \longrightarrow J_{\mathcal{S}^*} \times Y$ peste f, pentru a obtine exact dimensionalitatea masurilor conditionale ale masurii de echilibru μ_ϕ pentru orice potential Hölder continuu sumabil ϕ.

In articolul din grant [20] am introdus o notiune noua de entropie inversa a unei masuri probabilistice μ, invariata de un endomorfism masurabil Borel $f : \Lambda \to \Lambda$ pe un spatiu metric compact Λ.

Am gasit o formula generala, care arata ca aceasta entropie inversa $h_f^-(\mu)$ este legata de entropia de infasurare $F_f(\mu)$ (introdusa de Ruelle) si de entropia uzuala $h_f(\mu)$. Deasemenea $h_f^-(\mu)$ satisface multe dintre proprietatile entropiei uzuala, dar este si diferita. Spre exemplu, pentru orice aplicatie f de dilatare, $h_f^-(\mu) = 0$ pentru orice masura f-invarianata μ. Am gasit mai multe exemple concrete, in care putem calcula entropia inversa. Deasemenea, am introdus si o notiune de entropie inversa topologica $h^-(f)$, si am aratat ca $h^-(f) \geq h_f^-(\mu)$ pentru orice masura f-invarianta ergodica μ pe Λ.

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