Hyperbolic dynamics on folded saddle sets

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We are concerned with the case when f is a smooth endomorphism on a manifold M, having a **basic set with overlaps** $\Lambda \subset M$. We shall assume also that the endomorphism f is **hyperbolic** on Λ; however in this non-invertible case the unstable directions depend on full backwards trajectories $\hat{x} \in \hat{\Lambda}$ ([14], [15], [4]). The map f is not assumed expanding on Λ. Through a given point x from Λ there may pass many (even uncountably many) local unstable manifolds corresponding to different prehistories of x in the natural extension $\hat{\Lambda}$; these unstable manifolds may intersect also outside Λ. In general in this non-invertible case we **do not** have a Markov partition on Λ.

The non-invertible case has different techniques and phenomena than the diffeomorphism or the expanding cases (see [1], [11], [6] and the references there). For instance in this case the stable dimension does *not* vary continuously on perturbation. There are many examples of interesting and/or unexpected dynamical behaviour for endomorphisms, for instance: examples of perturbations of linear toral endomorphisms, for which the unstable manifolds through a given point depend on prehistories (Przytycki, [14]); horseshoes with overlaps (Bothe, [1]); hyperbolic non-expanding fractal repellers with overlaps (Mihailescu,[5]); skew products with overlaps, having in fibers Cantor sets of points with uncountably many unstable directions associated to them (Mihailescu, [4]); examples from higher dimensional complex dynamics, etc.

For equilibrium measures on folded basic sets we have the following approximation with discrete measures supported on those n-preimages remaining in Λ (notice that Λ is *not* totally invariant):

Theorem 1 ([8])**.** *Let $f : M \to M$ be a smooth map (say $\mathcal{C}^2$) on a Riemannian manifold M, so that f is hyperbolic on a saddle basic set Λ; assume also that the critical set $\mathcal{C}_f$ of f does not intersect Λ. Let ϕ be a Hölder continous potential on Λ and μ_ϕ the equilibrium measure of ϕ on Λ. Then* $\forall g \in \mathcal{C}(\Lambda, \mathbb{R})$, $\int_\Lambda | < \frac{1}{n} \sum\limits_{y \in f^{-n}(x)\cap\Lambda} \frac{e^{S_n\phi(y)}}{\sum\limits_{z\in f^{-n}(x)\cap\Lambda} e^{S_n\phi(z)}} \cdot \sum\limits_{i=0}^{n-1} \delta_{f^i y} - \mu_\phi, g > | d\mu_\phi(x) \underset{n\to\infty}{\longrightarrow} 0.$

Let now Λ be a connected hyperbolic *repeller* for a smooth endomorphism $f : M \to M$ defined on a Riemannian manifold M, and assume f has no critical points in Λ. Let V be a neighbourhood of Λ in M and for any $z \in V$ define the measures $\mu_n^z := \frac{1}{n} \sum\limits_{y\in f^{-n}(z)\cap V} \frac{1}{d(f(y))\ldots d(f^n(y))} \sum\limits_{i=1}^{n} \delta_{f^i y}$, where $d(y)$ is the number of f-preimages belonging to V of a point $y \in V$. Then we proved in [5] that there exists an f-invariant measure μ^- on Λ, a neighbourhood V of Λ and a borelian set $A \subset V$ with $m(V \setminus A) = 0$ (where m is the Lebesgue measure on M) and a subsequence $n_k \to \infty$ s.t for any $z \in A$, $\mu_{n_k}^z \underset{k\to\infty}{\to} \mu^-$. The measure μ^- is called the **inverse SRB measure** of the non-invertible hyperbolic repeller. We showed that μ^- is the equilibrium measure of the stable potential

$\Phi^s(x) := \log|\det Df_s(x)|, x \in \Lambda$, with respect to f. The difficulty is that f is non-invertible, hence μ^- is **not** simply the SRB measure for the inverse f^{-1} (the inverse does not even exist). We proved that μ^- is the unique f-invariant measure μ satisfying an inverse Pesin entropy formula; if f is d-to-1 on Λ we have: $h_{\mu^-}(f) = \log d - \sum\limits_{i, \lambda_i(\mu^-)<0} \lambda_i(\mu^-) m_i(\mu^-)$.

Theorem 2 (Mihailescu, arxiv.org 2011)**.** *Let f be a hyperbolic toral endomorphism on $\mathbb{T}^m, m \geq 2$ given by an integer-valued matrix A without zero eigenvalues, and let g be a $\mathcal{C}^1$ perturbation of f. Consider μ_g^- the inverse SRB measure of g and μ_g^+ the (usual forward) SRB measure. Then the entropy production $e_g(\mu_g^-)$ of μ_g^- and the entropy production $e_g(\mu_g^+)$ satisfy the following:*

a) $e_g(\mu_g^-) \leq 0$ and $F_g(\mu_g^-) = \log d$. Moreover $e_g(\mu_g^+) \geq 0$.

b) $e_g(\mu_g^-) = 0$ if and only if $|detDg|$ is cohomologous to a constant on $\mathbb{T}^m$.

One can find a measurable partition ξ of Λ, subordinated to the stable manifolds W^s and can define the **lower pointwise stable dimension** of μ as $\underline{\delta}^s(\mu, x, \xi) := \liminf\limits_{r\to 0} \frac{\log \mu_x^\xi(B(x,r))}{\log r}$, where $\{\mu_x^\xi\}_x$ is the system of conditional measures of μ associated to ξ. Similarly define $\overline{\delta}^s(\mu, x, \xi)$.

Theorem 3 ([7])**.** *Let $f : M \to M$ be a smooth endomorphism which is hyperbolic on a saddle basic set Λ and conformal on its stable manifolds. Assume f is d-to-1 on Λ and let $\Phi^s(y) := \log|Df_s(y)|, y \in \Lambda$, δ^s the stable dimension of Λ, and μ_s the equilibrium measure of $\delta^s\Phi^s$. Then the conditional measures $\mu_{s,A}^s$ of μ_s associated to ξ are geometric probabilities, i.e for $(\mu_s)_\xi$-a.a $\pi_\xi(A)$ of Λ/ξ there is a constant $C_A > 0$ s.t*

$$C_A^{-1}\rho^{\delta^s} \leq \mu_{s,A}^s(B(y,\rho)) \leq C_A \rho^{\delta^s}, y \in A \cap \Lambda, 0 < \rho < \frac{r(A)}{2}$$

In particular the lower and upper pointwise stable dimensions of μ_ϕ are equal a.e to δ^s.

For automorphisms Ornstein proved a famous result, namely that two invertible Bernoulli shifts on Lebesgue spaces are isomorphic if and only if they have the same measure theoretic entropy. However as Parry and Walters showed, for measure-preserving *endomorphisms* $f : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$, the entropy alone $h_\mu(f)$ *does not* determine the conjugacy class (see [13]). So the problem of coding for endomorphisms of Lebesgue spaces (in particular for **1-sided Bernoulli shifts**) is subtle and there are no exhaustive classifications (see also [2] and [3]). If f_A is a linear toral endomorphism given by a hyperbolic matrix A with $|\det(A)| > 1$, then its natural extension is by Katznelson, 2-sided Bernoulli. However it does not mean that $(\mathbb{T}^m, f_A, m)$ is 1-sided Bernoulli.

Theorem 4 ([10])**.** *Let f_A be a toral endomorphism on $\mathbb{T}^m, m \geq 2$, given by the integer-valued matrix A, all of whose eigenvalues are strictly larger than 1 in absolute value. Then the endomorphism f_A on the torus $\mathbb{T}^m$ equipped with its Lebesgue (Haar) measure μ_m, is isomorphic to a uniform model 1-sided Bernoulli shift.*

We consider now the general case of equilibrium measures of endomorphisms on folded fractals.

Theorem 5 ([9])**.** *Let f be a smooth hyperbolic endomorphism on a connected basic set Λ; let also ϕ be a Holder continuous potential on Λ and μ_ϕ the unique equilibrium measure of ϕ. If (Λ, f, μ_ϕ)*

is 1-sided Bernoulli, then: a) either f is distance-expanding on Λ; or b) the stable dimension of μ_ϕ is zero, i.e $HD^s(\mu_\phi, x) = 0$ for μ_ϕ-a.e $x \in \Lambda$.

Theorem 6 ([9])**.** *a) Let f hyperbolic on the saddle set Λ s.t $\mathcal{C}_f \cap \Lambda = \emptyset$. If the system (Λ, f, μ_0) given by the measure of maximal entropy is 1-sided Bernoulli, then f is distance expanding on Λ.*

b) Assume f is an expanding endomorphism on Λ. If μ_ϕ is the equilibrium measure of the Hölder potential ϕ and if (Λ, f, μ_ϕ) is 1-sided Bernoulli, then $\mu_\phi = \mu_0$, where μ_0 is the unique measure of maximal entropy for f on Λ.

The family of skew products with variable overlaps of Cantor sets in fibers given in [4] are hyperbolic (see also [12]) and strongly non-invertible on their respective basic sets, and on the other hand they are not constant-to-one; one can apply the above results for them too.

There exist also relations between the **stable dimension** and the **geometry** of the fractal Λ.

Theorem 7 ([11])**.** *Let $f : M \to M$ be a smooth endomorphism which is hyperbolic on a basic set Λ with $C_f \cap \Lambda = \emptyset$ and such that f is conformal on local stable manifolds. Assume that d is the maximum possible value of the preimage counting function $d(\cdot)$ on Λ, and that $\exists x \in \Lambda$ with $\delta^s(x) := HD(W^s_r(x) \cap \Lambda) = t_d = 0$. Then it follows that $d(\cdot) \equiv d$ on Λ and $\delta^s(y) = 0,\ y \in \Lambda$.*

Theorem 8 ([6])**.** *In the setting of Theorem 7 if d is the maximum possible value of $d(\cdot)$ on Λ and if $\delta^s = t_d = 0$, then there exist finitely many global unstable manifolds that contain Λ, and $f|_\Lambda$ is expanding. In particular if Λ is connected, there exists one global unstable manifold containing Λ.*

References

[1] H. G. Bothe, Shift spaces and attractors in non-invertible horseshoes, Fund. Math., **152**, 1997, 267-289.

[2] H. Bruin and J. Hawkins, Rigidity of smooth 1-sided Bernoulli endomorphisms, New York J. Math., **15**, 2009, 1-33.

[3] C. Hoffman and D. Rudolph, Uniform endomorphisms which are isomorphic to a Bernoulli shift, Annals of Math., **156**, 2002, 79-101.

[4] E. Mihailescu, Unstable directions and fractal dimensions for a family of skew products with overlaps, Math. Zeitschrift, **269**, 2011, 733-750.

[5] E. Mihailescu, Physical measures for multivalued inverse iterates near hyperbolic repellors, J. Stat. Physics, **139**, 2010, 800-819.

[6] E. Mihailescu, Local geometry and dynamical behavior on folded basic sets, J. Stat. Physics **142**, 2011, 154-167.

[7] E. Mihailescu, On a class of stable conditional measures, Ergod. Th. and Dynam. Syst., **31**, 2011, 1499-1515.

[8] E. Mihailescu, Approximations for Gibbs states of arbitrary Holder potentials on hyperbolic folded sets, Discrete and Cont. Dynam. Syst., **32**, 2012, 961-975.

[9] E. Mihailescu, On some coding and mixing properties for a class of chaotic systems, Monatshefte Math., online 2011, DOI 10.1007/s00605-011-0347-8.

[10] E. Mihailescu, Higher dimensional expanding maps and toral extensions, to appear Proceed. Amer. Math. Soc., 2012.

[11] E. Mihailescu and M. Urbański, Relations between stable dimension and the preimage counting function on basic sets with overlaps, Bull. London Math. Soc., **42**, 2010, 15-27.

[12] S. Newhouse, Cone fields, domination and hyperbolicity, Modern Dynamical Systems and Applications. M. Brin, B. Hasselblatt and Y. Pesin (eds.), Cambridge Univ. Press, 2004.

[13] W. Parry and P. Walters, Endomorphisms of a Lebesgue space, Bull. AMS **78**, 1972, 272-276.

[14] F. Przytycki, Anosov endomorphisms, Studia Math., **58**, 1976, 249-285.

[15] D. Ruelle, Elements of differentiable dynamics and bifurcation theory, Academic Press, 1989.

[16] L. S Young, What are SRB measures and which dynamical systems have them?, J. Stat. Physics, **108**, 2002, 733-754.