DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **32**, Number **7**, July **2012**

doi:10.3934/dcds.2012.32.xx

pp. **1–XX**

EQUILIBRIUM MEASURES, PREHISTORIES DISTRIBUTIONS AND FRACTAL DIMENSIONS FOR ENDOMORPHISMS

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Abstract. The dynamics of endomorphisms (i.e non-invertible smooth maps) presents many significant differences from that of diffeomorphisms, as well as from the dynamics of expanding maps. There are numerous concrete examples of hyperbolic endomorphisms. Many methods cannot be used here due to overlappings in the fractal set and to the existence of (possibly infinitely) many local unstable manifolds going through the same point. First we will present the general problems and explain how to construct certain useful limit measures for atomic measures supported on various prehistories. These limit measures are in many cases shown to be equal to certain equilibrium measures for Hölder potentials. We obtain thus an analogue of the SRB measure, namely an inverse SRB measure in the case of a hyperbolic repeller, or of an Anosov endomorphism. We study then the 1-sided Bernoullicity (or lack of it) for certain measures invariant to endomorphisms, and give a Classification Theorem for the ergodic and metric types of behaviour of perturbations of a class of maps on their respective basic sets, in terms of the values of the stable dimension. We give also relations between thermodynamic formalism and fractal dimensions (Hausdorff dimension of stable/unstable intersections with basic sets, stable/unstable box dimensions, dimension of the global unstable set for endomorphisms). Applications to certain nonlinear evolution models are also given in the end.

1. **Introduction to the non-invertible case.** The dynamics of hyperbolic diffeomorphisms has answered a great deal of important questions related to the distribution of forward iterates near an attractor (given by the associated SRB measure), in the form of the celebrated results of Sinai, Bowen and Ruelle (see [3], [47], [6], [50], [15], etc.) In the hyperbolic diffeomorphism case one always has Markov partition ([3], [47], etc.) which help in coding the system endowed with some equilibrium measure, with a 2-sided Bernoulli shift. An important tool in proving results for diffeomorphisms is the Birkhoff Ergodic Theorem. If we work with a homeomorphism f which has an inverse f^{-1}, we can find the ergodic distribution of the consecutive preimages of a generic point (with respect to an f-invariant probabilistic measure μ) by using Birkhoff Ergodic Theorem for f^{-1} on a basic set Λ. If the map f is **not invertible** on the basic set Λ, then there is no inverse of f for which to apply Birkhoff Ergodic Theorem.

2000 *Mathematics Subject Classification.* Primary: 37D35, 37A35, 37D20; Secondary: 37C45.
Key words and phrases. Hyperbolic non-invertible maps, equilibrium measures, 1-sided Bernoullicity, attractors/repellers with overlaps, SRB and inverse SRB measures, fractal dimensions of folded sets, complex dynamics, inverse pressure.

Also in the diffeomorphism case, or in the expanding 1-dimensional complex case, thermodynamic formalism was originally applied to obtain formulas for various types of fractal dimensions as in [14], [38], [45], [43], etc.

In this survey we will collect several results of the author or of other researchers, about the case of non-invertible hyperbolic dynamics. Here the methods and results are often completely new, and different from the diffeomorphism case.

By **basic set** for a smooth endomorphism $f : M \to M$ defined on a Riemannian manifold, we understand (as in [9]) a compact f-invariant set Λ such that there exists a neighbourhood U of Λ with $\Lambda = \bigcap\limits_{n \in \mathbb{Z}} f^n(U)$. We can consider then a useful construction, namely that of the **natural extension** (or **inverse limit**), $\hat\Lambda := \{\hat x = (x, x_{-1}, x_{-2}, \ldots), f(x_{-i}) = x_{-i+1}, x_{-i} \in \Lambda, i \geq 1\}$. The infinite sequences of consecutive preimages of type $\hat x$ are called *prehistories* of x, for $x \in \Lambda$. We define the shift homeomorphism $\hat f : \hat\Lambda \to \hat\Lambda$, $\hat f(\hat x) = (f(x), x, x_{-1}, \ldots)$; we have also $\pi : \hat\Lambda \to \Lambda$ the canonical projection on the first coordinate. Let us also make the observation that, if μ is an f-invariant probability measure on Λ, then there exists a unique $\hat f$-invariant probability measure $\hat\mu$ on $\hat\Lambda$ so that $\pi_* \hat\mu = \mu$ (see [42]); it is easy to show that μ is ergodic if and only if its lift $\hat\mu$ is ergodic.

Hyperbolicity for endomorphisms is defined similarly as for diffeomorphisms, with the crucial difference that the unstable spaces/unstable manifolds depend now on whole prehistories; they are then defined as $E^u_{\hat x}, W^u_r(\hat x), \hat x \in \hat\Lambda$ ([39], [44]). In the case of a smooth endomorphism f which is uniformly hyperbolic on the basic set Λ, we shall denote by $\Phi^s(x) := \log |Df_s(x)|, x \in \Lambda$ the stable potential of f, and by $\Phi^u(\hat x) := -\log |Df_u(\hat x)|, \hat x \in \hat\Lambda$ the unstable potential, where again we denoted by $Df_s(x), Df_u(\hat x)$, the restrictions of Df to the stable space E^s_x, respectively to the unstable space $E^u_{\hat x}, \hat x \in \hat\Lambda$. Also we will denote by $\delta^s(x) := HD(W^s_r(x) \cap \Lambda)$ and by $\delta^u(\hat x) := HD(W^u_r(\hat x) \cap \Lambda)$, the **stable dimension** at a point $x \in \Lambda$, respectively the **unstable dimension** at $\hat x \in \hat\Lambda$ (see also [19], [24]). Let us state now the Birkhoff Ergodic Theorem for the inverse $\hat f^{-1}$:

Theorem 1 (Birkhoff Theorem on the natural extension)**.** *Let $\hat f : \hat\Lambda \to \hat\Lambda$ be the lift homeomorphism on the natural extension $\hat\Lambda$ as above and $\hat\mu$ the $\hat f$-invariant lift measure of an f-invariant Borel probability measure μ on Λ. Let also $\phi \in \mathcal{C}(\hat\Lambda, \mathbb{R})$ be a continuous map on $\hat\Lambda$; then for $\hat\mu$ almost all points $\hat x$ from $\hat\Lambda$, we have $\frac{1}{n}(\phi(x) + \ldots + \phi(\hat f^{-n}\hat x)) \underset{n\to\infty}{\to} \bar\phi(\hat x)$, where $\bar\phi$ is $\hat\mu$-integrable, $\bar\phi \circ \hat f = \bar\phi$ on $\hat\Lambda$, and $\int \bar\phi d\hat\mu = \int \phi d\hat\mu$. In particular, if μ is ergodic on Λ, then $\bar\phi$ is constant $\hat\mu$-a.e.*

The problem is that this Theorem does not give us the distribution of all preimages at once, but instead only that of a certain generic prehistory (one at a time); however a point may have uncountably many different prehistories. Thus in the case of non-invertible maps, when there may exist several preimages of any given point from Λ, we need a different approach and different methods for studying the global set of preimages.

In the hyperbolic endomorphism case we also notice that the local unstable manifolds do not form a foliation like in the diffeomorphism setting, but instead they may intersect both inside and outside Λ. Another difficulty is that we do not always have Markov partitions on basic sets like in the diffeomorphism so it is impossible in general to code the system with a Bernoulli shift. One may also have sudden drops in the fractal dimensions caused by self-intersections in the basic set; these dimensions do not have to depend continuously on parameters (unlike for diffeomorphisms); see [31] for a discussion and examples of polynomial maps which become homeomorphisms when restricted to some invariant sets. Thus subtle methods must be devised to overcome all these problems.

In this survey we will discuss the particularities of the dynamics of hyperbolic endomorphisms **from several angles**. First, in **Section 2** we give several classes of examples of non-reversible dynamical systems and showcase some of their differences from the diffeomorphism situation. Then in **Section 3** we present the new problem of inverse SRB measures, which give the distribution of the various preimages of generic points near folded repellors. They appear in some physical non-reversible models, when describing the distribution of past trajectories for Lebesgue almost all points. In **Section 4** we study the relationships between the stable dimension and the preimage counting function on a basic set of saddle type; and in **Section 5** we give connections with the ergodic theory of 1-sided Bernoulli shifts, for various equilibrium measures of the non-invertible system. Finally in **Section 6** we present several results about applications of inverse pressure to dimension estimates of fractal stable slices or of global unstable sets, in the conformal case; in particular this applies to hyperbolic invariant sets for holomorphic maps in higher dimensions. Also the above results on non-invertible fractals can be applied to certain chaotic evolution models from statistical physics, economics, etc.

2. **Examples of non-invertible dynamics.** Let us mention now some literature in the case of endomorphisms f on a basic set Λ, and some examples. In [6] Eckmann and Ruelle studied the relations between attractors, SRB measures, dimension, Lyapunov exponents and entropy, with a view towards applications in physics (chaotic dynamics, turbulence theory, etc.); in the same paper there appears a non-injective example in the plane, due to Ushiki et al., in which the computer picture of the attractor displays folded drapes.

In [7] it was constructed a family of piecewise linear maps which were proved to be homeomorphisms on their respective basic sets, for Lebesgue-almost all parameters. This kind of behaviour appears when there is a **transversality type condition** present for a certain parametrized family (as in [37], [48], [46] or [30]). In [30] we gave actually examples of skew products having iterated function systems in the base, and also several examples from higher dimensional complex dynamics which satisfy the transversality condition; for these examples it is then possible to find the stable dimension (i.e the Hausdorff dimension of the intersection between the basic set Λ and the local stable manifolds) as the zero of the pressure of a certain growth potential on $\hat{\Lambda}$.

In [39], Przytycki studied Anosov endomorphisms and gave also examples of perturbations of hyperbolic toral endomorphisms for which, through a given point $x \in \mathbb{T}^m$ there pass uncountably many local unstable manifolds, corresponding to the different prehistories of x.

Also in the recent paper [16] we considered the dynamics of a family of skew products with overlaps in fibers f_α, which were shown to be far from being homeomorphism and also far from being constant-to-1 maps. For these maps we showed that different prehistories of the same point have different unstable tangent spaces associated to them and we estimated also the angle between such unstable directions. This implies that the associated local unstable manifolds are different too.

In order to present this class, let us fix first a small $\alpha \in (0,1)$; then take the subintervals $I_1^\alpha, I_2^\alpha \subset I = [0,1]$ so that I_1^α is contained in $[\frac{1}{2} - \epsilon(\alpha), \frac{1}{2} + \epsilon(\alpha)]$ and I_2^α is contained in $[1-\alpha-\epsilon(\alpha), 1-\alpha+\epsilon(\alpha)]$, for some small $\epsilon(\alpha) < \alpha^2$. We consider moreover a strictly increasing smooth map $g : I_1^\alpha \cup I_2^\alpha \to I$ such that $g(I_1^\alpha) = g(I_2^\alpha) = I$; assume there exists a large $\beta >> 1$ s. t $\beta^2 > g'(x) > \beta >> 1, x \in I_1^\alpha \cup I_2^\alpha$. Hence there exist subintervals $I_{11}^\alpha, I_{12}^\alpha \subset I_1^\alpha, I_{21}^\alpha, I_{22}^\alpha \subset I_2^\alpha$ such that $g(I_{11}^\alpha) = g(I_{21}^\alpha) = I_1^\alpha$ and $g(I_{12}^\alpha) = g(I_{22}^\alpha) = I_2^\alpha$. Then let $J^\alpha := I_{11}^\alpha \cup I_{12}^\alpha \cup I_{21}^\alpha \cup I_{22}^\alpha$ and $J_*^\alpha := \{x \in J^\alpha, g^i(x) \in J^\alpha, i \geq 0\}$. Now define the endomorphism $f_\alpha : J_*^\alpha \times I \to J_*^\alpha \times I$,

$$f_\alpha(x,y) = (g(x), h_\alpha(x,y)), \text{ with } h_\alpha(x,y) = \begin{cases} \psi_{1,\alpha}(x) + s_{1,\alpha} y, \ x \in I_{11}^\alpha \\ \psi_{2,\alpha}(x) + s_{2,\alpha} y, \ x \in I_{21}^\alpha \\ \psi_{3,\alpha}(x) - s_{3,\alpha} y, \ x \in I_{12}^\alpha \\ s_{4,\alpha} y, \ x \in I_{22}^\alpha, \end{cases} \tag{1}$$

where for some small ε_0, we take $s_{1,\alpha}, s_{2,\alpha}, s_{3,\alpha}, s_{4,\alpha}$ to be positive numbers, ε_0-close to $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ respectively; and $\psi_{1,\alpha}(\cdot), \psi_{2,\alpha}(\cdot), \psi_{3,\alpha}(\cdot)$ are smooth (say $\mathcal{C}^2$) functions on I which are ε_0-close in the $\mathcal{C}^1$-metric, to the linear functions $x \to x$, $x \to 1-x$ and $x \to 1$, respectively. By $|g_1 - g_2|_{\mathcal{C}^1}$ we denote the distance in the $\mathcal{C}^1(I)$-metric between two smooth functions on I, g_1 and g_2. We shall denote also the function $h_\alpha(x,\cdot) : I \to I$ by $h_{x,\alpha}(\cdot)$, for $x \in J_*^\alpha$. Denote by

$$\Lambda(\alpha) := \bigcup_{x \in J_*^\alpha} \bigcap_{n \geq 0} \bigcup_{y \in g^{-n}x \cap J_*^\alpha} h_{y,\alpha}^n(I), \tag{2}$$

where $h_{y,\alpha}^n := h_{f^{n-1}y,\alpha} \circ \ldots \circ h_{y,\alpha}, n \geq 0$. For $x \in J_*^\alpha$ let also: $\Lambda_x(\alpha) := \bigcap_{n \geq 0} \bigcup_{y \in g^{-n}x \cap J_*^\alpha} h_{y,\alpha}^n(I)$, the **fiber** (or **slice**) of the fractal $\Lambda(\alpha)$ over x. Then we have the following:

Theorem 2. *([16]) There exists a function $\vartheta(\alpha) > 0$ defined for all positive small enough numbers α, with $\vartheta(\alpha) \underset{\alpha \to 0}{\to} 0$, such that if f_α is the map defined in (1) whose parameters satisfy:*

$$\max \left\{ |\psi_{1,\alpha}(x) - x|_{\mathcal{C}^1}, |\psi_{2,\alpha}(x) - 1 + x|_{\mathcal{C}^1}, |\psi_{3,\alpha}(x) - 1|_{\mathcal{C}^1}, |s_{i,\alpha} - \frac{1}{2}|, i = 1, \ldots, 4 \right\} < \vartheta(\alpha) \tag{3}$$

then we obtain:

a) For $x \in J_*^\alpha \cap I_1^\alpha$, there exists a Cantor set $F_x(\alpha) \subset \Lambda_x(\alpha)$, s. t every point of $F_x(\alpha)$ has two different f_α-preimages in $\Lambda(\alpha)$. And if $x \in J_*^\alpha \cap I_2^\alpha$, then there exists a Cantor set $F_x(\alpha) \subset \Lambda_x(\alpha)$ s. t every point of $F_x(\alpha)$ has two different f_α^2-preimages in $\Lambda(\alpha)$.

b) f_α is hyperbolic on $\Lambda(\alpha)$.

c) If $\hat{z}, \hat{z}' \in \hat{\Lambda}(\alpha)$ are two different prehistories of an arbitrary point $z \in \Lambda(\alpha)$, then $E_{\hat{z}}^u \neq E_{\hat{z}'}^u$.

Another large class of endomorphisms consists of holomorphic maps in one dimension or in several dimensions. The one-dimensional case has been studied first, and here were the first applications of thermodynamic formalism to calculate the Hausdorff dimension of rational hyperbolic maps (starting with Bowen, Ruelle). In the higher dimensional case we may have saddle type basic sets obtained as components of the non-wandering set, for Axiom A holomorphic maps on the projective space $\mathbb{P}^k, k \geq 2$; see [8]. As in the diffeomorphism case (for example Henon maps), great importance is given to the set K^-; in the diffeomorphism case this is the set of points having bounded backward iterates, but in the holomorphic endomorphism case on $\mathbb{P}^k$, the set K^- is the complement of the set of points that have all of their preimages converging towards the support of the Green measure. In fact it can be shown in the s-hyperbolic holomorphic endomorphism case, that K^- is the global unstable set $W^u(\hat{S}_1)$ union with the (finite) set of periodic attracting points (see [8]). In [25] we proved that for s-hyperbolic holomorphic endomorphisms on $\mathbb{P}^k$, the set K^- has empty interior. Then in [19] we studied in greater detail the Hausdorff dimension of K^-, and the Hausdorff and upper box dimensions of the stable intersections, with the help of the inverse pressure (see also [32], [29]). Many differences appear from the case of holomorphic automorphisms on Stein manifolds ([26]).

Another type of non-invertible dynamics is given by the family of hyperbolic horseshoes with overlaps introduced by Bothe in [2]. He proved in fact that the set of such non-invertible horseshoes with overlaps has non-empty interior in some sense. As examples of non-invertible systems, we mention also the families of self-similar sets with overlaps studied in [48], and the conformal iterated function systems with overlaps from [33].

In [49], Tsujii studied a class of dynamical systems generated by maps $T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}, T(x, y) = (\ell x, \lambda y + g(x))$, where $\ell \geq 2$ is an integer, $0 < \lambda < 1$ and g is a $\mathcal{C}^2$-function on S^1. Then T is an Anosov endomorphism, which has thus a unique SRB measure μ. Let $\mathcal{D} \subset (0, 1) \times C^2(S^1, \mathbb{R})$ be the set of pairs (λ, f) for which the SRB measure of the corresponding endomorphism T is absolutely continuous with respect to Lebesgue measure on $S^1 \times \mathbb{R}$; also let $\mathcal{D}^\circ$ the interior of $\mathcal{D}$ with respect to the product topology. In [49] it was shown that there exist examples as above for which the SRB measure is totally singular; nevertheless "most" maps T satisfy:

Theorem 3. *([49]) Let $\ell^{-1} < \lambda < 1$. There exists a finite collection of C^∞ functions $\phi_i : S^1 \to \mathbb{R}, i = 1, \dots, m$ s.t for any C^2 function $g \in C^2(S^1, \mathbb{R})$, the subset of $\mathbb{R}^m$*

$$\{(t_1, \dots, t_m) \in \mathbb{R}^m, (\lambda, g(x) + \sum_{i=1}^m t_i \phi_i(x)) \notin \mathcal{D}^\circ\}$$

is a null set with respect to the Lebesgue measure on $\mathbb{R}^m$. Consequently $\mathcal{D}$ contains an open and dense subset of $(\frac{1}{\ell}, 1) \times C^2(S^1, \mathbb{R})$.

In [30] we considered a parametrized family of skew products with overlaps in their fibers $F : X \times V \to X \times V, F(x, y) = (f(x), h(x, y))$, where $f : X \to X$ is an expanding map on a compact metric space X, and $h(x, \cdot) : V \to V$ (denoted also by h_x) is a contraction on an open convex set $V \subset \mathbb{R}^m$; the map h_x is assumed to depend continuously on $x \in X$. The basic set here is $\Lambda := \underset{x \in X}{\cup} \overset{\infty}{\underset{n=0}{\cap}} \underset{z \in f^{-n}x}{\cup} h_z^n(\bar{V})$, where $h_z^n := h_{f^{n-1}z} \circ \ldots \circ h_z, n \geq 1, z \in X$. We studied then the conditional measures of equilibrium states induced on fibers and their relation to the stable dimension of fibers. We employed a transversality type condition in order to show that for Lebesgue almost all parameters, the stable dimension of the fibers is given by the unique solution of a Bowen type equation on $\hat{\Lambda}$. Several examples where these results can be applied were given in [30], among which some iterated function systems and examples from higher dimensional complex dynamics.

3. **SRB, and inverse SRB measures for endomorphisms.** Axiom A endomorphisms and the SRB (Sinai, Ruelle, Bowen) measures for such non-invertible maps, have been studied in [40]. In [13], Liu established a Pesin entropy formula in

the case of an absolutely continuous invariant measure for an endomorphism. And in [41], Qian and Zhu studied a notion of SRB measures in the non-uniform setting of invariant measures for smooth endomorphisms. They showed the following Pesin type entropy formula (see also the entropy formula for diffeomorphisms established earlier by Ledrappier and Young in [11]):

Theorem 4 (Pesin's entropy formula for endomorphisms, [41])**.** *Let $f : M \to M$ be a $\mathcal{C}^2$ endomorphism having an f-invariant probability borelian measure μ, so that $\log |detDf| \in L^1(M, \mu)$. Then the entropy formula*

$$h_\mu(f) = \int_M \sum_{\lambda_i > 0} \lambda_i(x) m_i(x) d\mu(x)$$

holds if and only if μ has the SRB property, i.e for every measurable partition η of the natural extension $\hat M$ subordinate to the unstable manifolds of (f, μ), we have that, for $\hat\mu$-a. e $\hat x \in \hat M$, $\pi(\hat\mu^\eta_{\hat x}) << m_{\hat x}$; here $m_{\hat x}$ is the Lebesgue measure induced on the local unstable manifold $W^u_{r(\hat x)}(\hat x)$, λ_i are the Lyapunov exponents of the measure μ and m_i their respective multiplicities.

Another research direction is the study of fractal dimensions, among which the Hausdorff dimension of the intersections between local stable/unstable manifolds and the basic set Λ, the Hausdorff dimension of Λ, the upper and lower box dimensions, the dimensions of certain invariant measures, etc. Ruelle and Bowen were the first to use thermodynamical formalism in order to find formulas for the Hausdorff dimension of dynamically significant fractal sets. In the case of hyperbolic rational maps f (i.e f is expanding on its Julia set $J(f)$), Ruelle proved that the Hausdorff dimension of the respective Julia set is equal to the unique zero of the pressure $t \to P(t\Phi^u), \Phi^u(y) := -\log |Df(y)|, y \in J(f)$ (see [45]). For the case of hyperbolic diffeomorphisms on surfaces, Manning and McCluskey ([14]) proved that the stable dimension on a basic set Λ (i.e the Hausdorff dimension of the intersection between the local stable manifolds and Λ) is equal to the unique zero of the pressure $t \to P(t\Phi^s)$, where $\Phi^s(y) := \log |Df_s(y)|, y \in \Lambda$, where we denote $Df_s(y)$ the derivative of f restricted to the stable tangent space, $Df|_{E^s_y}, y \in \Lambda$. And similarly for the unstable dimension.

In [43] (see also [15]) it was studied the distribution of the preimages for expanding maps; the main method was the use of Perron-Frobenius operators, and the fact that the diameters of the images of small balls by local inverse iterates decrease exponentially. However for **non-invertible non-expanding maps**, this useful property does no longer hold.

In a series of papers, namely [22], [23], [20] we initiated a study of an analogue of the SRB measure for endomorphisms, but this time involving the various consecutive preimages of points. As noticed before, due to the non-invertibility of f, we cannot apply the case of the forward iterates, and the problem is difficult and subtle. New methods were developed, involving estimates of the equilibrium measures on pieces of neighbourhoods of unstable manifolds (corresponding to various prehistories), inverse pressure, non-Bernoullicity of some measures, combinatorial arguments, estimates of the lifts of measures on certain borelian sets from the natural extension, consideration of families of certain appropriate conditional measures, etc. A priori there may exist preimages of points from Λ which do not remain in Λ, as Λ is a fractal set, **not necessarily totally invariant**. First let us specify what we understand by repellor for an endomorphism:

Definition 5. Let f be a smooth (say $\mathcal{C}^2$) endomorphism on a Riemannian manifold M and let Λ be a basic set for f. We say that Λ is a **repellor** for f if the critical set of f does not intersect Λ and if there exists a neighbourhood U of Λ such that $\bar{U} \subset f(U)$.

We can prove the following result for the number of preimages remaining in the repellor:

Proposition 6. *In the setting of Definition 5, if Λ is a repellor for f, then $f^{-1}\Lambda \cap U = \Lambda$. If moreover Λ is assumed to be connected, the number of f-preimages that a point has in Λ is constant.*

Proof. Consider a point $x \in \Lambda$, and y be an f-preimage of x from U. Then $f^n y \in \Lambda, n \geq 1$. From Definition 5, since Λ is assumed to be a repellor, the point y has a preimage y_{-1} in U; then y_{-1} has a preimage y_{-2} from U, and so on. Thus y has a full prehistory belonging to U and also its forward orbit belongs to U, hence $y \in \Lambda$ since Λ is a basic set. So $f^{-1}\Lambda \cap U = \Lambda$.

We prove now the second part of the statement. Let a point $x \in \Lambda$ and assume that it has d f-preimages in Λ, denoted $x_1, \ldots, x_d$. Consider also another point $y \in \Lambda$ close to x. If y is close enough to x and since $\mathcal{C}_f \cap \Lambda = \emptyset$, it follows that y also has exactly d f-preimages in U, denoted by $y_1, \ldots, y_d$. Since from the first part we know that $f^{-1}\Lambda \cap U = \Lambda$, we obtain then $y_1, \ldots, y_d \in \Lambda$. In conclusion the number of f-preimages in Λ of a point is locally constant. If Λ is assumed to be connected, then the number of preimages belonging to Λ of any point from Λ, must be constant. □

The importance of the fact that all points in Λ have a constant number of preimages remaining in Λ, is given by the following Theorem, proved in [20]:

Theorem 7. *([20]) Consider Λ to be a connected hyperbolic repellor for the smooth endomorphism $f : M \to M$; let us assume that the constant number of f-preimages belonging to Λ of any point from Λ is equal to d. Then $P(\Phi^s - \log d) = 0$.*

Using this, we showed in [20] that in the case of a hyperbolic repellor **which is not necessarily expanding**, we can obtain the distribution of consecutive preimages as an **inverse SRB measure**; the important role of the inverse SRB measure is played here by the equilibrium measure of the stable potential. The methods of proof are however different from the diffeomorphism case, and involve a careful study of the types of behaviours of consecutive sums along various prehistories.

Theorem 8. *([20]) Let Λ be a connected hyperbolic repellor for a smooth endomorphism $f : M \to M$. There exists a neighbourhood V of Λ, $V \subset U$ such that if we denote by*

$$\mu_n^z := \frac{1}{n} \sum_{y \in f^{-n} z \cap U} \frac{1}{d(f(y)) \cdot \ldots \cdot d(f^n(y))} \sum_{i=1}^{n} \delta_{f^i y}, z \in V$$

where $d(y)$ is the number of f-preimages belonging to U of a point $y \in V$, then for any continuous function $g \in \mathcal{C}(U, \mathbb{R})$ we have

$$\int_V |\mu_n^z(g) - \mu_s(g)| dm(z) \underset{n \to \infty}{\to} 0,$$

where μ_s is the equilibrium measure of the stable potential $\Phi^s(x) := \log |det(Df_s(x))|$, $x \in \Lambda$.

Corollary 9. *In the same setting as in Theorem 8, it follows that there exists a borelian set $A \subset V$ with $m(V \setminus A) = 0$ and a subsequence $(n_k)_k$, such that for any point $z \in A$, we have the following weak convergence of measures on U*

$$\mu^z_{n_k} \underset{k\to\infty}{\to} \mu_s$$

Moreover we proved in [20] that a property like the one satisfied by usual SRB measures in regard to their conditional measures on unstable manifolds, is verified now by the inverse SRB measure, but on local stable manifolds:

Theorem 10. *([20]) Let Λ be a connected hyperbolic repellor for a smooth endomorphism $f : M \to M$ on a Riemannian manifold M; assume that f is d-to-1 on Λ. Then there exists a unique f-invariant probability measure μ^- on Λ satisfying an inverse Pesin entropy formula:*

$$h_{\mu^-}(f) = \log d - \int_\Lambda \sum_{i,\lambda_i(x)<0} \lambda_i(x) m_i(x) d\mu^-(x)$$

In addition the measure μ^- has absolutely continuous conditional measures on local stable manifolds.

Connected hyperbolic repellors are very useful as examples since, at perturbations, they preserve the property of having a constant number of preimages remaining in the repellor, for any point (Proposition 6 above). Also, their hyperbolicity and connectedness are preserved by perturbations. Therefore one can construct examples like the one below, from [20]:

Example. Let us take $F : \mathbb{PC}^1 \times \mathbb{T}^2 \to \mathbb{PC}^1 \times \mathbb{T}^2$ given by:

$F([z_0 : z_1], (x, y)) = ([z_0^2 : z_1^2], f_A(x, y))$, where f_A is the toral endomorphism induced by the matrix $A = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$. Then F has a connected hyperbolic repellor $\Lambda := S^1 \times \mathbb{T}^2$. Consider the following perturbation of F, $F_\varepsilon : \mathbb{PC}^1 \times \mathbb{T}^2 \to \mathbb{PC}^1 \times \mathbb{T}^2$ given by:

$$\begin{aligned} &F_\varepsilon([z_0 : z_1], (x, y)) \\ &= \Big([z_0^2 + \varepsilon z_1^2 \cdot e^{2\pi i(2x+y)} : z_1^2], (2x + y + \varepsilon sin(2\pi(x + y)), 2x + 2y + \varepsilon cos^2(4\pi x))\Big) \end{aligned}$$

Then F_ε is well defined as a smooth endomorphism on $\mathbb{PC}^1 \times \mathbb{T}^2$ and it is a $\mathcal{C}^1$ perturbation of F. It follows from the discussion above that F_ε has a connected hyperbolic repellor Λ_ε (on which F_ε has both stable as well as unstable directions), and that Λ_ε is close to Λ. However Λ_ε is different from Λ, and it has a complicated fractal structure with self-intersections; its projection on the second coordinate is $\mathbb{T}^2$. On Λ_ε we can apply Theorem 8 to get a physical measure μ^-_ε for the local inverse iterates of F_ε. This physical measure μ^-_ε is the equilibrium measure of the non-constant stable potential

$$\Phi^s_\varepsilon([z_0 : z_1], (x, y)) := \log|\det(DF_\varepsilon)_s([z_0 : z_1], (x, y))|, \text{for } ([z_0 : z_1], (x, y)) \in \Lambda_\varepsilon$$

$\square$

The distribution of preimages for expanding maps is given by equilibrium measures (see [43]). However if the basic set Λ is of **saddle type**, the problem is very different and needs new methods for the proof. We lack the fact that the local inverse iterates act as contractions on small balls; in fact they are dilations in the stable directions in backward time, and this is changing completely the situation and the ideas of proof. In [22] we solved the above problem of the weighted preimage distribution with a Holder continuous weight ϕ, along a **general hyperbolic basic set** (i. e **not** necessarily a repellor, and **not** necessarily for an expanding map):

Theorem 11. *([22]) Let $f : M \to M$ be a smooth (say $\mathcal{C}^2$) map on a Riemannian manifold M, which is hyperbolic and finite-to-one on a basic set Λ so that $\mathcal{C}_f \cap \Lambda = \emptyset$. Assume that ϕ is a Holder continuous potential on Λ and that μ_ϕ is the equilibrium measure of ϕ on Λ. Then*

$$\int_\Lambda | < \frac{1}{n} \sum_{y \in f^{-n}x \cap \Lambda} \frac{e^{S_n\phi(y)}}{\sum\limits_{z \in f^{-n}x \cap \Lambda} e^{S_n\phi(z)}} \cdot \sum_{i=0}^{n-1} \delta_{f^i y} - \mu_\phi, g > | d\mu_\phi(x) \underset{n\to\infty}{\to} 0, \forall g \in \mathcal{C}(\Lambda, \mathbb{R})$$

The proof of this Theorem is difficult and is based on a careful study of the measure μ_ϕ of various pieces of Bowen balls, and of iterates of Bowen balls; one has to estimate the measure of the set of n-preimages y_{-n} behaving badly, i.e on which the consecutive averages $\frac{\phi(y)+\ldots+\phi(y_{-n})}{n}$ oscillate more than some positive ε, from their median value $\int \phi d\mu_\phi$. As a Corollary, we obtained in [22] the following result giving the weak convergence of the above atomic measures along the same subsequence, for all points in a set of full μ_ϕ-measure, in the case of a basic set of saddle type Λ and a smooth non-invertible map f :

Corollary 12. *([22]) In the same setting as in Theorem 11, for any Holder potential ϕ, it follows that there exists a subset $E \subset \Lambda$, with $\mu_\phi(E) = 1$ and an infinite subsequence $(n_k)_k$ such that for any $z \in E$ we have the weak convergence of measures*

$$\mu^z_{n_k} \underset{k\to\infty}{\to} \mu_\phi$$

Corollary 13. *([22]) Assume that $f : M \to M$ is an Anosov endomorphism without critical points on a Riemannian manifold. Let also ϕ a Holder continuous potential on M and μ_ϕ the equilibrium measure of ϕ. Then*

$$\int_M | < \frac{1}{n} \sum_{y \in f^{-n}x} \frac{e^{S_n\phi(y)}}{\sum\limits_{z \in f^{-n}x} e^{S_n\phi(z)}} \cdot \sum_{i=0}^{n-1} \delta_{f^i y} - \mu_\phi, g > | d\mu_\phi(x) \underset{n\to\infty}{\to} 0, \forall g \in \mathcal{C}(\Lambda, \mathbb{R})$$

In particular, if μ_0 is the measure of maximal entropy, it follows that for μ_0-almost all points $x \in \Lambda$, $\frac{1}{n} \sum\limits_{y \in f^{-n}x} \frac{\sum_{i=0}^{n-1} \delta_{f^i y}}{Card(f^{-n}x)} \underset{n\to\infty}{\to} \mu_0$

By applying some results from [12] and [41] we obtain sufficient conditions when the usual SRB measure is equal to our inverse SRB measure (see [20], [22]):

Corollary 14. *Let $f : M \to M$ be an Anosov endomorphism, $\phi : \Lambda \to \mathbb{R}$ a Holder potential and assume that the equilibrium measure μ_ϕ is absolutely continuous with respect to the Lebesgue measure on M. Then the measure μ_ϕ with this property is unique, it is an SRB measure and it also satisfies an inverse SRB condition in the sense that there exists a set E of full Lebesgue measure in M and a sequence $(n_k)_k$ such that $\mu^z_{n_k} \underset{k}{\to} \mu_\phi, z \in E$.*

An example of an Anosov endomorphism, so that each point has a constant number of preimages, is that of a hyperbolic toral endomorphism $f_A : \mathbb{T}^m \to \mathbb{T}^m, m \geq 2$, given by an integer-valued matrix A whose eigenvalues λ_i all have absolute values different from 1. Each point of $\mathbb{T}^m$ has exactly $|\det A|$ preimages in $\mathbb{T}^m$. Then for any Holder continuous potential ϕ on $\mathbb{T}^m$, we can apply the Corollary 13 in order to obtain the **weighted distribution of all n-preimages on** $\mathbb{T}^m$, asymptotically converging to the equilibrium measure μ_ϕ, when $n \to \infty$. In particular, if $\phi \equiv 0$, we obtain the distribution of the atomic measures supported on preimages of order smaller than n, towards the measure of maximal entropy (i. e towards the Lebesgue measure on $\mathbb{T}^m$). We notice also that Corollary 13 applies for Anosov endomorphisms on infranilmanifolds (see for example [15] for definitions). Moreover, Theorem 11 applies also to basic sets of saddle type which are not necessarily Anosov, like the examples from [30] recalled in Section 2.

4. **Fractal dimensions and the preimage counting function.** We talk now about the relations between ergodic theory/thermodynamic formalism and the fractal dimensions of dynamical significance in the case of smooth hyperbolic endomorphisms (i.e smooth hyperbolic non-invertible maps) on basic sets. This case is very different from the diffeomorphism case and new methods are needed for the proofs. Also new type of phenomena appear, which did not appear in the diffeomorphism case. Firstly the usual Bowen formula does not always work in this case, for the Hausdorff dimension of the intersection between a local stable manifold and a basic set Λ (see [31], [29]). Also the unstable dimension has to be computed on the natural extension $\hat\Lambda$, but in this case we do have a Bowen type formula, namely $\delta^u(\hat x) = t^u$, where t^u is the unique zero of the pressure function

$$t \to P_{\hat f}(t\Phi^u), \Phi^u(\hat y) := -\log|Df_u(\hat y)|, \hat y \in \hat\Lambda,$$

as it was shown in [24]. However we **do not have** such a formula for the stable dimension since, by contrast with the unstable manifolds, the stable manifolds do not lift bi-Lipschitz homeomorphically to the natural extension. This is caused by the fact that the local unstable manifolds depend on their whole prehistories (see [39], [16] for examples with different unstable spaces corresponding to different prehistories), and thus do not form a nice foliation like in the diffeomorphism case; so a holonomy map on local stable manifolds (in the sense of [38]) cannot be defined in general.

In [29] and [28] we gave the following answer to the difficult question of establishing a Bowen formula for the stable dimension in the non-invertible case; in this setting the problem presents many qualitative differences from the classical diffeomorphism or expanding cases (see also [31]). It was stated originally for holomorphic maps on $\mathbb{P}^2$, but the same proof can be extended as below:

Theorem 15. *([29], [28]) Consider a smooth endomorphism $f : M \to M$ which is hyperbolic on a basic set of saddle type Λ such that f is conformal on local stable manifolds and there are no critical points in Λ. Assume also that there exists a continuous function ω on Λ such that for any point $z \in \Lambda$, we have $d(z) \le \omega(z)$. Then $\delta^s(x) \ge t_\omega$, for any $x \in \Lambda$, where t_ω is the unique zero of the pressure function $t \to P(t\Phi^s - \log \omega)$.*

In case the preimage counting function is constant on Λ and equal to d, we obtain that for any $x \in \Lambda$ the stable dimension $\delta^s(x) = t^s_d$, where t^s_d is the unique zero of the pressure function $t \to P(t\Phi^s - \log d)$, and $\Phi^s(y) := \log|Df_s(y)|, y \in \Lambda$.

The proof of this Theorem uses some new ideas related to the inverse pressure and to the concatenation of tubular unstable neighbourhoods. Moreover we proved in [21] that the **conditional measures** induced by the equilibrium measure μ_s of $\delta^s\Phi^s$

on local stable manifolds (corresponding to a measurable partition whose existence we showed in [21]), are geometric probabilities of exponent δ^s; this was done by estimating the measure μ_s of different parts of a generic $f^n(B_n(z,\varepsilon))$.

Theorem 16. *([21]) Let f, Λ be as in Theorem 15. Assume moreover that f is d-to-1 on Λ. Let δ^s the stable dimension of Λ, and μ_s be the equilibrium measure of the potential $\delta^s\Phi^s$ on Λ. Then the conditional measures of μ_s, associated to the partition ξ subordinated to the local stable manifolds, are geometric probabilities of exponent δ^s.*

A surprising, and useful consequence of Theorem 15 is the following:

Corollary 17. *([28]) Assume that f is c-hyperbolic on a basic set Λ and that the preimage counting function $d(\cdot)$ reaches a maximum value of d on Λ. If there exists a point $x \in \Lambda$ such that $\delta^s(x) = t_d$, where t_d is the unique zero of the pressure function $t \to P(t\Phi^s - \log d)$, then $d(y) = d, \forall y \in \Lambda$. And hence the stable dimension at every point of Λ is equal to t_d.*

So the last Corollary says that, if the stable dimension attains its lowest possible value at a certain point $x \in \Lambda$, then the preimage counting function is constant on Λ; and hence, by Theorem 15 the stable dimension must be constant on Λ (this is not always known for general endomorphisms).

Theorem 15 and Corollary 17 can be applied to **c-hyperbolic skew products with overlaps in fibers**, and with a finite iterated function system (IFS) in the base. Let us consider a finite union of compact sets $X_1, \ldots, X_m$ in an open set $S \subset \mathbb{R}^l$ and denote by $X := X_1 \cup \ldots \cup X_m$. Consider also a continuous expanding topologically transitive function $f : X \to X$. Assume that f is injective on each X_i and that $f(X_i) = X(i,1) \cup \ldots \cup X(i,m_i), i = 1, \ldots, m$, where $X(i,j)$ are sets from the same collection $\{X_1, \ldots, X_m\}$. The source-model for this is the case of an expanding map $f : I_1 \cup \ldots \cup I_m \to I_1 \cup \ldots \cup I_m$, with $I_1, \ldots, I_m$ compact subintervals in $[0,1]$, such that $f(I_j)$ is a union of some of the same subintervals, i.e $f(I_j) = I(j,1) \cup \ldots \cup I(j,m_j), j = 1, \ldots, m$. Take the functions $g(x,y) : X \times \tilde{W} \to X \times \tilde{W}$, with $\tilde{W} \subset \mathbb{R}^k$ a neighbourhood of the closure of an open set W, such that g is smooth (say $\mathcal{C}^2$) in (x,y), and such that for every $x \in X$, the function $g(x,\cdot) : W \to W$ is contracting uniformly in x, and it is injective and conformal. We shall denote the function $g(x,\cdot)$ also by g_x; due to the contraction, $g_x(\bar{W})$ is strictly contained in V. We define then the f-invariant set $X^* := \{y \in X, f^j y \in X, j \geq 0\}$ and for each $x \in X^*$, let us consider the fiber $\Lambda_x := \bigcap\limits_{n\geq 0} \bigcup\limits_{z \in f|_{X^*}^{-n}(x)} g_{f^n z} \circ \ldots \circ g_z(\bar{W})$. Then consider the compact set: $\Lambda := \bigcup\limits_{x \in X^*} \Lambda_x$, (see for example [30] for a similar type of skew products). It can be seen that $F(x,y) := (f(x), g(x,y))$ defined on $X^* \times W$ is hyperbolic on its basic set Λ. In this case we can apply the above Theorem 15. Indeed for each j with $1 \leq j \leq m$, we know that a point $z \in X^* \cap X_j$ has at most q_j preimages in Λ, where q_j is the number of subsets $X_i, 1 \leq i \leq m$ such that $f(X_i) \supset X_j$. Then we have that the preimage counting function associated to F and Λ is smaller or equal than a locally constant function ω given by $\omega(x,y) := q_j$ if $x \in X_j, 1 \leq j \leq m$. However points in $\Lambda \cap (\{x\} \times W)$ may have strictly less than q_j F-preimages in Λ for $x \in X_j \cap X^*$. Hence we obtain the following:

Corollary 18. *([28]) Let a c-hyperbolic skew-product pair (F, Λ) as above. Then the stable dimension of Λ, i.e the Hausdorff dimension of the fibers $\Lambda_x, x \in X^*$, is* *larger or equal than the unique zero of the pressure function* $t \to P(t\Phi^s - \log \omega)$, *where* $\omega|_{(X^* \cap X_j) \times W} = q_j, 1 \le j \le m$.

Another Corollary, obtained in [28], and giving a relationship between the number of preimages remaining in Λ and the stable dimension is the following:

Corollary 19. *([28]) In the setting of Corollary 18, if* $f(I_j)$ *contains all subintervals* $I_1, \dots, I_m, 1 \le j \le m$, *it follows that* F *is* m*-to-1 on* Λ *if and only if* $\exists z \in \Lambda$ *with* $\delta^s(z) = 0$. *In this case we obtain* $\delta^s(y) = 0, \forall y \in \Lambda$.

The stable dimension has also some interesting applications to a phenomenon of **geometric rigidity** for basic saddle sets for smooth endomorphisms. Indeed we proved in [23] that, if the stable dimension attains its lowest value (zero) at some point, then the preimage counting function is constant on Λ and actually Λ is contained in a finite number of unstable manifolds (like in the case of the map $(z, w) \to (z^2 + c, w^2)$ on its basic set $\Lambda = \{p_c\} \times S^1$, where p_c is the unique fixed attracting point of the map $z \to z^2 + c$, for small $|c| \ne 0$). The precise result proved in [23] is this:

Theorem 20. *([23]) Let* $f : M \to M$ *be a smooth endomorphism which is hyperbolic on a basic set* Λ *with* $C_f \cap \Lambda = \emptyset$ *and such that* f *is conformal on local stable manifolds. Assume that* d *is the maximum possible value of* $d(\cdot)$ *on* Λ, *and that there exists a point* $x \in \Lambda$ *where* $\delta^s(x) = t_d = 0$. *Then it follows that* $d(\cdot) \equiv d$ *on* Λ *and there exist a finite number of unstable manifolds whose union contains* Λ. *In particular if* Λ *is connected, there exists an unstable manifold containing* Λ, *and* $f|_\Lambda$ *is expanding.*

5. **1-sided Bernoullicity and stable dimension.** An important general problem in ergodic theory is to see whether the system (Λ, f, μ) is **1-sided (or 2-sided) Bernoulli** for a certain f-invariant measure on the basic set Λ (see [4], [15], [35], [42], etc.). Again, the problem for endomorphisms (i.e for smooth non-invertible maps) is very different than the one for diffeomorphisms, and presents many particularities. For example while we know that entropy gives a classification of 2-sided Bernoulli automorphisms (by work of Ornstein), no such classification exists for *endomorphisms of Lebesgue spaces*; in fact Parry and Walters proved:

Theorem 21 ([35])**.** *There are non-isomorphic exact endomorphisms* S, T *of a Lebesgue space* $(X, \mathcal{B}, \mu)$ *so that* $S^2 = T^2$ *(hence* S, T *have the same entropy w.r.t* μ*),* $S^{-n}\mathcal{B} = T^{-n}\mathcal{B}, n \ge 0$ *and s.t the Jacobians of* S *and* T *w.r.t* μ *are equal.*

However a lot of work has been done recently in order to establish whether a certain endomorphism on a Lebesgue space together with a certain measure, is a 1-sided Bernoulli endomorphism (see for instance [4]). The problem is complex and must be attacked on a case-by-case approach.

In [4], Bruin and Hawkins gave several criteria for maps to be 1-sided Bernoulli; in fact for interval/circle maps, there exist rigidity type results for 1-sided Bernoulli maps:

Theorem 22. *([4]) Let* $T : I \to I$ *be a piecewise* C^2 n*-to-1 interval map preserving a probability measure* μ *equivalent to Lebesgue measure* m *s.t the Radon-Nikodym derivative* $g = \frac{d\mu}{dm}$ *is continuous and bounded away from 0. Then* T *is 1-sided Bernoulli on* $(I, \mathcal{B}, \mu)$ *if and only if* T *is* C^1*-conjugate to a map* $S : I \to I$ *whose graph consists of* n *linear pieces with slopes* $\pm\frac{1}{p_i}$ *s.t* $h_\mu(T) = -\sum_{i=1}^{n} p_i \log p_i$.

Corollary 23. *([4]) Let $T : S^1 \to S^1$ be an expanding C^2 degree $n \geq 2$ circle map with $T(0) = 0$; then T is 1-sided Bernoulli if and only if it is C^1-conjugate to $z \to z^n$.*

Also in [4] there were obtained several examples of non-Bernoulli n-to-1 maps (see also [5]).

Theorem 24. *([18]) Let f_A be a toral endomorphism on $\mathbb{T}^m, m \geq 2$, given by the integer-valued matrix A, all of whose eigenvalues are strictly larger than 1 in absolute value. Then the endomorphism f_A on the torus $\mathbb{T}^m$ equipped with its Lebesgue (Haar) measure μ_m, is isomorphic to a 1-sided Bernoulli shift.*

More generally in [18] we proved that **expanding** endomorphisms on tori are *1-sided Bernoulli* with respect to their associated measures of maximal entropy. We also studied in [18] certain toral extensions of these endomorphisms; for general theory of group extensions and relations to Livsic theory see [36].

In [23] we proved the following result, this time for expanding maps on general basic sets:

Theorem 25. *([23]) Assume that Λ is a hyperbolic basic set for a smooth endomorphism f, such that $f|_\Lambda$ is d-to-1, $t_d = 0$ and $f|_\Lambda$ is expanding. Then (Λ, f, μ_0) is 1-sided Bernoulli, where μ_0 is the unique measure of maximal entropy.*

However, if the system has stable directions and if the stable dimension is positive, we obtained the following result of non-Bernoullicity:

Theorem 26. *([23]) Let f be a smooth endomorphism, which is hyperbolic on a basic set Λ, such that $\Lambda \cap C_f = \emptyset$ and f is conformal on stable manifolds. Assume that there exists a point $x \in \Lambda$ with $\delta^s(x) > 0$, and denote by μ_s the equilibrium measure of the potential $\delta^s(x) \cdot \Phi^s(\cdot)$. Then the measure preserving system (Λ, f, μ_s) cannot be 1-sided Bernoulli.*

In [23] we gave a **Classification Theorem** for the possible types of ergodic and metric behaviour that may appear when perturbing a certain class of holomorphic maps in $\mathbb{C}^2$:

Theorem 27. *([23]) Let us consider the holomorphic map $f(z, w) = (z^2 + c, w^2)$ for small $|c| \neq 0$. Let also f_ε be a holomorphic perturbation of f and Λ_ε the corresponding basic set of f_ε, close to the set $\Lambda := \{p_c\} \times S^1$, where p_c is the fixed attracting point of $z \to z^2 + c$. Then we may have exactly one of the following possibilities:*

a) There exists a point $x \in \Lambda_\varepsilon$ where $\delta^s(x) = 0$. Then there exists a manifold W such that $\Lambda_\varepsilon \subset W$, $f_\varepsilon|_{\Lambda_\varepsilon}$ is expanding and $f_\varepsilon|_{\Lambda_\varepsilon}$ is 2-to-1. In this case the stable dimension is 0 at any point from Λ_ε, and the measure preserving system $(\Lambda_\varepsilon, f_\varepsilon, \mu_{0,\varepsilon})$ is 1-sided Bernoulli (where $\mu_{0,\varepsilon}$ is the unique measure of maximal entropy for $f_\varepsilon|_{\Lambda_\varepsilon}$).

b) There exists a point $x \in \Lambda_\varepsilon$ with $0 < \delta^s(x) < 2$. Then the stable dimension is positive at any point of Λ_ε, and the measure preserving system $(\Lambda_\varepsilon, f_\varepsilon, \mu_{s,\varepsilon})$ cannot be 1-sided Bernoulli (where $\mu_{s,\varepsilon}$ is the equilibrium measure of the potential $\delta^s(x)\Phi^s_\varepsilon$). We have two subcases:

b1) $f_\varepsilon|_{\Lambda_\varepsilon}$ is a homeomorphism, and in this case the measure preserving system $(\Lambda_\varepsilon, f_\varepsilon, \mu_\phi)$ is 2-sided Bernoulli for any Holder continuous potential ϕ, where μ_ϕ is the equilibrium measure of ϕ.

b2) there exist both points with one f_ε-preimage in Λ_ε and points with two f_ε-preimages in Λ_ε, but the set of points with one f_ε-preimage in Λ_ε has non-empty interior.

For example the map $f_\varepsilon(z,w) = (z^2+\varepsilon w^4, w^2)$ has a basic set Λ_ε close to $\{0\}\times S^1$. Moreover its basic set is contained in the submanifold

$$W := \{(z,w) \in \mathbb{C}^2, z = \alpha \cdot w^2),$$

where $\alpha = \frac{1-\sqrt{1-4\varepsilon}}{2}$. In fact we have in this case $f(W) = W = f^{-1}(W)$. In this case f_ε is expanding and 2-to-1 on Λ_ε, and the stable dimension is everywhere equal to 0.

In the case when the number of preimages varies along Λ, for example, if $d(\cdot)$ takes two values on Λ, namely $d(x) = d_1, x \in \Lambda_1$ and $d(y) = d_2, y \in \Lambda_2$, and if $\mathring{\Lambda}_1 \neq \emptyset$, we may take a continuous function $\omega(\cdot)$ so that $\omega \equiv d_2$ on a neighbourhood V_2 of Λ_2, $\omega \equiv d_1$ on some open set V_1 with $\bar{V}_1 \subset \mathring{\Lambda}_1$ and $d_1 \leq \omega(x) \leq d_2$ for other points $x \in \Lambda$. Then from Theorem 15 (see [28]), we know that $t_{d_1} \geq \delta^s(x) \geq t_\omega$, where t_ω is the unique zero of the pressure function $t \to P(t\Phi^s - \log \omega)$. □

Relations between 1-sided Bernoullicity and the pointwise dimension of an arbitrary equilibrium measure were given in 2011 in [17]:

Theorem 28. *([17]) Let f be a smooth hyperbolic endomorphism on a connected basic set Λ; let also ϕ be a Holder continuous potential on Λ and μ_ϕ the unique equilibrium measure of ϕ. Then if the measure-preserving system (Λ, f, μ_ϕ) is 1-sided Bernoulli, it follows that either f is distance-expanding on Λ, or the stable dimension of μ_ϕ is zero, i.e $HD^s(\mu_\phi, x) = 0$ for μ_ϕ-a.e $x \in \Lambda$.*

6. **Applications of inverse pressure to certain metric properties.** We will finish with several results from [19] about the Hausdorff dimension of the global unstable set $W^u(\hat{\Lambda})$ for a basic set Λ with **self-intersections**, which **is not** a repellor.

Definition 29. Let a smooth ($\mathcal{C}^2$) map on a Riemannian manifold M, $f : M \to M$, and assume that f is hyperbolic on a basic set Λ. Then we say that Λ is a **local repellor** for f if there exist local stable manifolds of f contained in Λ.

For diffeomorphisms, a basic set Λ is said to be a repellor if there exists a neighbourhood U of Λ such that $f(U) \supset \bar{U}$. However, for endomorphisms this condition alone does not guarantee a priori that all of the local stable manifolds are not contained in Λ. This may happen because of the subtle structure of foldings and overlappings for endomorphisms, which may take a point outside Λ into a point from Λ. If we want to have equivalence between our Definition 29 and the fact that there exists a neighbourhood U of Λ with $\bar{U} \subset f(U)$, then we have to assume in addition that $f|_\Lambda$ is open on Λ for example. The main advantage of a set which is **not** a local repellor is that this property is stable under perturbation, thus giving a whole class of examples.

A **cf-hyperbolic map** is by definition, a map which is hyperbolic on a basic set Λ, conformal on local stable manifolds of real dimension 2 and without critical points in Λ (see [19]).

Theorem 30. *([19]) Let f be a cf-hyperbolic map on a basic set Λ which is not a local repellor. Then for any perturbation g close enough to f (in the $\mathcal{C}^2$ topology), the corresponding basic set Λ_g is not a local repellor for g either.*

If $f : \mathbb{P}^2 \to \mathbb{P}^2$ is holomorphic on the 2-dimensional projective space and s-hyperbolic on a basic set of saddle type Λ, it follows that Λ cannot be a local repellor for f (see [19], using some results of Takeuchi about domains of holomorphy). In [19] we focused mainly on basic sets which **are not local repellors**, i.e which do not have any local stable manifold contained inside Λ. For this type of basic sets we can show (by using arguments related to the inverse pressure) that the stable dimension is always strictly less than 2.

We denote by $t^s(\varepsilon)$ the unique zero of the **inverse pressure function** restricted to covers of Λ of mesh less than ε, namely $t \to P^-(t\Phi^s, \varepsilon)$ (see [32], [29], [19] for definition/properties of the inverse pressure; also [34] for a different notion of preimage entropy). Then $t^s(\varepsilon)$ gives the following bound for the upper box dimension of the stable slice:

Theorem 31. *([19]) Let M be a smooth compact Riemannian manifold of real dimension 4 and $f : M \to M$ be a cf-hyperbolic map on a basic set of saddle type Λ which is not a local repellor. Then for any point $x \in \Lambda$, we have $\delta^s(x) = HD(W^s_r(x) \cap \Lambda) \le \overline{dim}_B(W^s_r(x) \cap \Lambda) \le t^s(\varepsilon) < 2$, for some $\varepsilon > 0$. In particular this holds also in the case of a holomorphic map $f : \mathbb{P}^2 \to \mathbb{P}^2$ which is s-hyperbolic on a basic set of saddle type Λ.*

We do not know yet whether, for endomorphisms, the stable dimension and the stable upper box dimension coincide always. We proved however this fact in the case when the preimage counting function is constant on Λ. For basic sets which are not local repellors we showed in [19] that the Hausdorff dimension of the global unstable set must be less than 4 (if we are in $\mathbb{R}^4$). In the case when f is a holomorphic map on $\mathbb{P}^2$ we obtain thus an estimate for the Hausdorff dimension of the global unstable set (which is equal, up to finitely many points, to the set K^- of points whose prehistories do not converge towards the support of the Green measure); notice that K^- is the analogue of the set of points with bounded backward iterates from the Henon case.

Theorem 32. *([19]) Let M be a compact Riemannian manifold of real dimension 4, and $f : M \to M$ be a smooth cf-hyperbolic map on a basic set of saddle type Λ, which is not a local repellor. Assume also that the following condition on derivatives is satisfied:*

$$\sup_{\hat\xi \in \hat\Lambda} |Df_u(\hat\xi)| \cdot |Df_s(\xi)| < 1 \tag{4}$$

Then $HD(W^u(\hat\Lambda)) < 4$. The same conclusion holds if $f : \mathbb{P}^2 \to \mathbb{P}^2$ is a holomorphic map which is s-hyperbolic on a basic set of saddle type Λ and satisfies (4).

Finally in [19] we proved also the following Theorem about the relation between the stable dimension over Λ and the bounds on the preimage counting function; the conditions of the Theorem can be verified on a number of examples.

Theorem 33. *([19]) Let M be a compact Riemannian manifold of real dimension 4, and $f : M \to M$ be a smooth cf-hyperbolic map on a basic set of saddle type Λ, which is not a local repellor. Let us denote by $\chi_s := \inf_\Lambda |Df_s|, \lambda_s := \sup_\Lambda |Df_s|$ and $\sup\limits_{\hat x \in \hat\Lambda} |Df_s(x)| \cdot |(Df|_{E^u(\hat x)})^{-1}| =: \tau$. Suppose that every point from Λ has at most d f-preimages and at least d' f-preimages in Λ. If the condition:*

$$2\inf\{1, \frac{-\log\tau}{|\log\chi_s|}\} - \frac{\log d}{|\log\chi_s|} \ge \frac{h_{top}(f|_\Lambda) - \log d'}{|\log\lambda_s|}$$

is satisfied, then $HD(W^u(\hat{\Lambda}) \cap \Delta) < 2$ *for any disk* Δ *transversal to the unstable directions. Moreover we obtain* $HD(W^u(\hat{\Lambda})) < 4$.

Applications. The above results can be applied for instance also to problems from economics; there exist several overlapping generations models, or cobweb models with adaptive adjustments, which present non-invertibility character plus some form of hyperbolicity (see [10], [27], etc.) In this setting the complexity of the invariant fractal sets presents significance for the economic problem, and one can apply the above results to investigate their metric (various types of dimension, fractal geometry, etc.) and ergodic properties (entropy, invariant measures, distributions of recurrent values, mixing, etc.)

We can apply Theorem 15 to the invariant fractal sets Λ obtained in chaotic 2-dimensional overlapping generations models or in certain hedging models, in order to obtain estimates on the dimension δ^s of the sets of points which stay close to Λ in forward time; and find the dimension of the local unstable intersections by using [24]. For the applications mentioned above, the inverse limits (natural extension spaces) are important and one can study the equilibrium measures of various utility functions on inverse limits; such utility functions are Hölder continuous with respect to the canonical metric on $\hat{\Lambda}$ (see [27]). A *utility function* on $\hat{\Lambda}$, $W : \hat{\Lambda} \to \mathbb{R}$ is given by $W(\hat{x}) = \sum\limits_{i \geq 0} \beta^i U(x_{-i})$, for $\beta \in (0, 1)$ and

a) in case $\Lambda \subset (0, \infty)$ we have $U(x) := \frac{\min\{1,x\}^{1-\sigma}}{1-\sigma} + \frac{(2-\min\{1,x\})^{1-\gamma}}{1-\gamma}, \ x \in \Lambda$, with $\sigma > 0, \gamma > 0$.

b) or in case $\Lambda \subset (0,1) \times (0,1)$, we have $U(x, y) := \frac{x^{1-\sigma}}{1-\sigma} + \frac{y^{1-\gamma}}{1-\gamma}, \ (x, y) \in \Lambda$, with $\sigma > 0, \gamma > 0$.

The equilibrium measures of such functions are important as they give maximum average utility value, while at the same time keeping the system as under control as possible in the long run, thus improving the predictability of the model (see the Variational Principle, [9]; the entropy gives here a measure of the disorder in the system). Equilibrium measures can be approximated using the known past trajectories as in Theorem 11 and its Corollaries. Moreover in these models the SRB measures for the associated endomorphisms give the distributions of forward iterates near attractors (if they exist), while the inverse SRB measures give the distributions of all local inverse iterates near repellors (if they exist); see also [6]. Both the SRB and the inverse SRB measures are particular examples of equilibrium states of Hölder potentials (see [40], [20]).

In practice it is important to work with models which are not changed drastically by small perturbations. In our case by considering basic sets Λ for non-invertible smooth maps f given in the above systems, we can apply Theorem 30 to assure the non-repellor character of perturbations of Λ and also to have conjugacies on $\hat{\Lambda}$, thus preserving the dynamical properties.

Acknowledgments. This paper is supported by the Sectorial Operational Programme Human Resources Development (SOP HRD), financed from the European Social Fund and by the Romanian Government under the contract number SOP HRD/89/1.5/S/62988.

The author acknowledges also the travel support from the conference "Dynamical Systems II" Denton 2009 in order to attend it.

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Received December 2009; revised August 2010 and January 2012.

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