

Emergent algebras

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Abstract

Inspired from research subjects in sub-riemannian geometry and metric geometry, we propose uniform idempotent right quasigroups and emergent algebras as an alternative to differentiable algebras.

Idempotent right quasigroups (irqs) are related with racks and quandles, which appear in knot theory (the axioms of a irq correspond to the first two Reidemeister moves). To any uniform idempotent right quasigroup can be associated an approximate differential calculus, with Pansu differential calculus in sub-riemannian geometry as an example.

An emergent algebra \mathcal{A} over a uniform idempotent right quasigroup X is a collection of operations such that each operation emerges from X , meaning that it can be realized as a combination of the operations of the uniform irq X , possibly by taking limits which are uniform with respect to a set of parameters.

Two applications are considered: we prove a bijection between contractible groups and distributive uniform irqs (uniform quandles) and that some symmetric spaces in the sense of Loos may be seen as uniform quasigroups with a distributivity property.

Keywords: differential structure; idempotent right quasigroups; contractible groups; Carnot groups; symmetric spaces

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1 Introduction

A differentiable algebraic structure, or differentiable algebra, is an algebra (set of operations \mathcal{A}) over a manifold X with the property that all the operations of the algebra are differentiable with respect to the manifold structure of X . Let us denote by \mathcal{D} the differential structure of the manifold X . With these notations a differentiable algebra can be seen as a pair $(\mathcal{A}, \mathcal{D})$, with the properties:

- \mathcal{A} contains the algebraic information, namely the collection of statements which can be formulated using algebraic operations, as well as algebraic relations (like for example "the operation $*$ is associative", or "the operation $*$ is commutative", and so on),
- \mathcal{D} contains differential structure informations, that is the information needed in order to formulate the statement "the function f is differentiable",
- \mathcal{A} is compatible with \mathcal{D} in the sense that any operation from \mathcal{A} is differentiable in the sense given by \mathcal{D} .

In this paper we want to extend the notion of differentiable algebra, based on the following observation: it may happen that some of the algebraic information encoded in \mathcal{A} "emerges" from statements from \mathcal{D} . We shall give precise examples of this phenomenon in the paper, but our basic example is the following: the algebraic structure of a vector space of the tangent space at a point $x \in X$ (when X is a manifold) is deduced from the differential structure \mathcal{D} . Indeed, there are several ways to define tangent spaces of a manifold and all are based on the differential structure of the manifold. For example tangent vectors may be seen as equivalence classes of differentiable curves passing through a point, or as derivations in the algebra of differentiable functions of the manifold. Eventually it is proved that tangent vectors at a point of the manifold form a vector space. Thus from differential type information an algebraic structure "emerges".

The word "emerge" is therefore used in relation with the fact that while statements in \mathcal{A} may be seen as constructs in universal algebra, statements in \mathcal{D} (whatever \mathcal{D} may exactly mean) may be obtain from the richer structure of the "differential" world, where we can prove algebraic statements by "passing to the limit uniformly with respect to a set of parameters". Thus, by carefully choosing the \mathcal{D} , we may obtain at least parts of \mathcal{A} .

We shall choose, as a minimal replacement of a differential structure \mathcal{D} on X , the structure of "uniform idempotent right quasigroup", definition 5.1. Then a generalization of a differentiable algebra over X is a algebra \mathcal{A} over X such that all operations and relations from \mathcal{A} can be constructed or deduced from combinations of operations in the uniform idempotent right quasigroup \mathcal{D} , possibly by taking limits which are uniform with respect to a set of parameters. Such an algebra may be called an "emergent algebra". In this approach, the compatibility condition between algebraic information \mathcal{A} and differential information \mathcal{D} , usually expressed as the differentiability of operations from \mathcal{A} with respect to \mathcal{D} , is replaced by the "emergence" of algebraic operations and relations from \mathcal{D} (see the comments and

references at the end of section 5 for a discussion of replacement of differentiability by emergence).

Outline of the paper. In section 2 we explain how this emergence phenomenon appears in fundamental studies concerning constructions of the intrinsically defined tangent bundle of a sub-riemannian manifold, like Bellaïche [1], Gromov [10], Margulis and Mostow [15] [16]. In previous papers [2] [3] [5] we proposed a general object which allow a differential calculus on a metric space, called "dilatation structure", and then used it to obtain old and new results concerning sub-riemannian geometry, in particular we used it in order to clarify the last section of Bellaïche paper, where he proposes to construct the nilpotent group operation on the metric tangent space, at a point of a sub-riemannian manifold, by using adapted coordinate systems and arguments of uniform convergence. We explain that the motivation of this paper is to extract the minimal algebraic (and analytical) information from dilatation structures, in order to prove the appearance of emergent algebraic structures without using a distance (by reasoning independent of metric geometry).

In section 3 we introduce idempotent right quasigroups and induced operations (a sort of bundle of isotopes of the right quasigroup operation). In section 4 we associate to any contractible group, or to any group with a contractive automorphism group, an idempotent right quasigroup. We show in proposition 4.5 that the group operation (of the contractible group) can be reconstructed as an emergent operation from uniform limits of isotopes of the associated idempotent right quasigroup.

This result motivates us to introduce in section 5 the notion of uniform irq, definition 5.1. The main property of a uniform irq is given in theorem 5.2, namely that from a uniform irq X we can construct a kind of tangent bundle which associates to any $x \in X$ a contractible group operation (like addition of vectors in the tangent space, only that it may be non-commutative). This tangent bundle gives us a minimal framework for differentiability, therefore we justify the use of uniform irqs as a generalization of differential structures.

In section 6 we study two examples of emergent algebraic structures. First example concerns contractible groups. We prove in theorem 6.1 that contractible groups are the same as distributive uniform irqs. We then show that such right quasigroups are in fact quasigroups. The second example concerns symmetric spaces in the sense of Loos [14]. We prove that already some properties of the inverse operation in a symmetric space are true for any uniform irq, then we propose a notion of uniform symmetric quasigroup, seen as a uniform quasigroup with a distributivity property for a "approximate inverse" operation inv_k , for any $k \in \mathbb{N}^*$ and prove that uniform symmetric quasigroups generate a family of symmetric spaces which contains the riemannian symmetric spaces.

2 Motivation

A particular class of locally compact groups which admit a contractive automorphism group is made by Carnot groups. These groups mysteriously appear as models of metric tangent spaces in sub-riemannian or Carnot-Carathéodory geometry (first proved in Mitchell [17]).

Non-holonomic spaces were discovered in 1926 by G. Vrănceanu [21], [22]. The Carnot-Carathéodory distance on a non-holonomic space is inspired by the work from 1909 of Carathéodory [6] on the mathematical formulation of thermodynamics. The modern study of non-holonomic spaces from the viewpoint of distance geometry, also known as sub-riemannian geometry, advanced steadily due to different lines of research: hypoelliptic operators Hörmander [12], harmonic analysis on homogeneous groups Folland, Stein [8], probability theory on groups Hazod [11], Siebert [20], studies in geometric analysis in metric spaces in relation with sub-riemannian geometry Bellaïche [1], Gromov [10], groups with polynomial growth Gromov [9], or Margulis type rigidity results Pansu [18].

Carnot groups are a kind of non-commutative vector spaces, in the sense that the group composition (of "vectors") is non-commutative and the multiplication by scalars is replaced by a contractive automorphism group. The fact that such a structure appears in relation with non-holonomic spaces (manifolds endowed with a completely non-integrable distribution) is very non-trivial and deserves an explanation.

Carnot groups seem to appear in this context as a consequence of a deformation theory of pseudo-differential operators. The proofs (maybe the most complete to be found in the big paper of Bellaïche [1], see also Margulis, Mostow [15] [16]) involve very careful development into series and manipulation of iterated brackets of vector fields, along with estimates of the order of remainders, therefore it is not clear if the Carnot group structure appears there by necessity or it is just the remnant manifestation of the mathematical tools used to explore the geometry of sub-riemannian spaces. Related to this question is Gromov proposal, made in his paper [10], to look at sub-riemannian spaces "from within", that is to deduce the basic results in sub-riemannian geometry in intrinsic manner. Gromov proposes to use only the Carnot-Carathéodory distance for this, but it seems that more than the distance is needed in order to get the Carnot group structure of the metric tangent space at a point in a (regular) sub-riemannian manifold.

A hint that this is true is a result of Siebert [20], which essentially characterizes Carnot groups as groups with a contractive group of automorphisms. In the papers [2] [3] [4] [5] we achieve this intrinsic description of a sub-riemannian space, as a particular example of a dilatation structure on a metric space.

In this paper we try to explain that this is a more general phenomenon, namely a manifestation of interesting emergent operations obtained from iterated deformations (or isotopes) of idempotent right quasigroups. The results presented here don't belong to distance geometry, but they are inspired by the previous mentioned studies of dilatation structures on metric spaces [2] [3] [4].

In the language used in this paper a dilatation structure is basically an idempotent right quasigroup endowed with a compatible distance. We show here that

some of the results obtained previously in metric spaces are in fact true in the realm of uniform idempotent right quasigroups (definition 5.1). Therefore our question, concerning the apparition of Carnot groups as models of non-commutative tangent spaces in sub-riemannian geometry, gets the following interesting answer: uniform idempotent right quasigroups offer a minimal concept in order to have a tangent bundle of a space, therefore a decent differential calculus. This minimal concept allow to construct emergent, more complex, algebraic structures, like the one of contractible group or of a symmetric space in the sense of Loos [14].

3 Irqs and induced operations

Definition 3.1 *A right quasigroup is a set X with a binary operation $*$ such that for each $a, b \in X$ there exists a unique $x \in X$ such that $a * x = b$. We write the solution of this equation $x = a \setminus b$.*

A quasigroup is a set X with a binary operation $$ such that for each $a, b \in X$ there exist unique elements $x, y \in X$ such that $a * x = b$ and $y * a = b$. We write the solution of the last equation $y = b / a$.*

An idempotent right quasigroup (irq) is a quasigroup with the property that the operation $$ is idempotent: for any $x \in X$ we have $x * x = x$.*

Remark 3.2 *Maybe the most well known example of a right quasigroup comes from knot theory. Indeed, in the unpublished correspondence of J.C. Conway and G.C. Wraith from 1959, they used the name "wrack" for a self-distributive right quasigroup generated by a link diagram. Later, Fenn and Rourke [7] proposed the name "rack" instead. The correspondence between the notation used here and the notation used by Fenn and Rourke is:*

$$x * y = y^x$$

Joyce [13] studied and used a particular case of a rack, named "quandle". Quandles are self-distributive idempotent right quasigroups. In the language of Reidemeister moves, the axioms of a (rack ; quandle ; irq) correspond respectively to the (2,3 ; 1,2,3 ; 1,2) Reidemeister moves. See further remark 6.2 for the relation between quandles and contractible groups.

An equivalent definition for a idempotent right quasigroup (irq) is the following: $(X, *, \setminus)$ is a irq if X is a set X endowed with two operations $*$ and \setminus which satisfy the following axioms: for any $x, y \in X$

$$(P1) \quad x * (x \setminus y) = x \setminus (x * y) = y$$

$$(P2) \quad x * x = x \setminus x = x$$

We use these operations to define the sum, difference and inverse operations of the irq.

Definition 3.3 *Let $(X, *, \setminus)$ be a irq. For any $x, u, v \in X$ we define the following operations:*

- (a) the difference operation is $(xuv) = (x * u) \setminus (x * v)$. By fixing the first variable x we obtain the difference operation based at x : $v -^x u = dif^x(u, v) = (xuv)$.
- (b) the sum operation is $)xuv(= x \setminus ((x * u) * v)$. By fixing the first variable x we obtain the sum operation based at x : $u +^x v = sum^x(u, v) =)xuv($.
- (c) the inverse operation is $inv(x, u) = (x * u) \setminus x$. By fixing the first variable x we obtain the inverse operator based at x : $-^x u = inv^x u = inv(x, u)$.

For any $k \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ we define the following operations:

- $x *_1 u = x * u, x \setminus_1 u = x \setminus u,$
- for any $k > 0$ let $x *_k u = x * (x *_k u)$ and $x \setminus_{k+1} u = x \setminus (x \setminus_k u),$
- for any $k < 0$ let $x *_k u = x \setminus_{-k} u$ and $x \setminus_k u = x *_k u.$

Remark 3.4 Let $(X, *)$, (Y, \circ) be (right) quasigroups. An isotopy from X to Y is a triple of invertible functions (α, β, γ) defined from X to Y with the property that for any $x, y, z \in X$ we have $\alpha(x) \circ \beta(y) = \gamma(z)$.

It is then obvious that the operation $+^x$ is isotopic with $*$, with $\alpha(u) = x * u$, $\beta(v) = v$ and $\gamma(w) = x * w$.

For any $k \in \mathbb{Z}^*$ the triple $(X, *_k, \setminus_k)$ is a irq. We denote the difference, sum and inverse operations of $(X, *_k, \setminus_k)$ by the same symbols as the ones used for $(X, *, \setminus)$, with a subscript "k".

These operations are interesting because they have properties related with group operations.

Proposition 3.5 In any irq $(X, *, \setminus)$ we have the relations:

- (a) $(u +^x v) -^x u = v$
- (b) $u +^x (v -^x u) = v$
- (c) $v -^x u = (-^x u) +^{x*u} v$
- (d) $-^{x*u} (-^x u) = u$
- (e) $u +^x (v +^{x*u} w) = (u +^x v) +^x w$
- (f) $-^x u = x -^x u$
- (g) $x +^x u = u$

Proof. By remark 3.4 the operation $+^x$ is isotopic with $*$, therefore (a), (b) are just P1 for the operation $+^x$.

(a) We apply two times P1, as explained further:

$$\begin{aligned} (u +^x v) -^x u &= (x * u) \setminus (x * (x \setminus ((x * u) * v))) = \\ &= (x * u) \setminus ((x * u) * v) = v \end{aligned}$$

(b) The proof is similar, only the order of application of P1 is reversed, first for $(x * u) * ((x * u) \setminus \dots)$, then for $x * (x \setminus \dots)$:

$$u +^x (v -^x u) = x \setminus ((x * u) * ((x * u) \setminus (x * v))) = x \setminus (x * v) = v$$

(c) Let us denote by A the expression $A = (x * u) * (-^x u)$. This expression enters in the right hand side of the equality from (c): $(-^x u) +^{x*u} v = (x * u) \setminus (A * v)$. We compute the expression A , using P1: $A = (x * u) * (-^x u) = (x * u) * ((x * u) \setminus x) = x$. We use then $A = x$ in the right hand side of the equality (c) and we obtain:

$$(-^x u) +^{x*u} v = (x * u) \setminus (x * v) = v -^x u$$

(d) We use again the expression A from the previous computation and the fact that $A = x$, then we use P1: $-^{x*u} (-^x u) = A \setminus (x * u) = x \setminus (x * u) = u$.

(e) We compute the left hand side (LHS) and the right hand side (RHS) separately.

$$LHS = x \setminus \{(x * u) * [(x * u) \setminus (((x * u) * v) * w)]\} = x \setminus (((x * u) * v) * w)$$

$$RHS = x \setminus \{[x * (x \setminus ((x * u) * v))] * w\} = x \setminus (((x * u) * v) * w)$$

Therefore $LHS = RHS$.

(f) Here we use P2: $x -^x u = (x * u) \setminus (x * x) = (x * u) \setminus x = -^x u$.

(g) We use P2, then P1: $x +^x u = x \setminus ((x * x) * u) = x \setminus (x * u) = u$. The proof is done. \square

Let us comment the relations from proposition 3.5. If we replace all the superscripts " $x * u$ " of the signs "+" and "-" by the superscript " x " in the relations (a) to (g) then we see some interesting patterns.

For example (e) becomes $u +^x (v +^x w) = (u +^x v) +^x w$, which expresses the associativity of the operation $+^x$. Therefore (e) is a generalized associativity relation.

Relation (d) takes the form $-^x (-^x u) = u$, that is the inverse operator is an involution.

Relation (c) becomes $v -^x u = (-^x u) +^x v$, which can be seen as a definition of the expression of the left hand side. Otherwise said $v -^x u$ is the left translate of v (with respect to $+^x$) by the inverse of u .

Relations (a) and (b) tell us that the inverse of the left translate by u is the left translate by $-^x u$ (here we don't need to change the superscript " x ").

Relation (g) is transformed into $x +^x u = u$, that is x is a left neutral element of $+^x$.

Without changing the superscripts, let us take $v = x$ in (c) and use (f):

$$-^x u = (-^x u) +^{x*u} v$$

If we modify the superscripts then we get that x is also a right neutral element. All in all we obtain that $+^x$ is a group operation, with neutral element x and inverse $-^x$.

Another group of relations will be useful further.

Proposition 3.6 *In any irq $(X, *, \setminus)$ the following relations are true:*

$$(h) \quad u -^x u = (xuu) = x * u$$

$$(i) \quad u -^x x = (xxu) = u$$

$$(j) \quad ((xuu) (xuv) (xuw)) = (xvw) .$$

(k) *Finally, with the notations from definition 3.3, for any $p, q \in \mathbb{Z}^*$ and any $x, u, v \in X$ we have the distributivity property:*

$$(x *_q v) -^x_p (x *_q u) = (x *_p q u) *_q (v -^x_{pq} u)$$

Proof. (h) and (i) come from straightforward computations using P2 and the definition of (xuw) .

(j) Here we use (h) in the left hand side of the equality:

$$\begin{aligned} ((xuu) (xuv) (xuw)) &= ((x * u) (xuv) (xuw)) = \\ &= ((x * u) ((x * u) \setminus (x * v)) (xuw)) = \\ &= ((x * u) * ((x * u) \setminus (x * v))) \setminus ((x * u) * (xuw)) = \\ &= (x * v) \setminus ((x * u) * ((x * u) \setminus (x * w))) = (x * v) \setminus (x * w) = (xvw) \end{aligned}$$

For proving (k) we compute:

$$\begin{aligned} (x(x *_q u)(x *_q v))_p &= (x *_p (x *_q u)) \setminus_p (x *_p (x *_q v)) = \\ &= (x *_p q u) \setminus_p (x *_p q v) = (x *_p q u) *_q ((x *_p q u) \setminus_{pq} (x *_p q v)) = \\ &= (x *_p q u) *_q (xuv)_{pq} \quad \square \end{aligned}$$

4 Contractible groups

We can construct irqs from groups, as explained in the following definition.

Definition 4.1 *Let G be a group with operation $(x, y) \mapsto xy$, inverse $x \mapsto x^{-1}$ and neutral element e , and $\delta : G \rightarrow G$ a bijective function such that $\delta(e) = e$. We construct the irq $G(\delta) = (G, *, \setminus)$ defined by:*

$$x * u = x \delta(x^{-1}u) \quad , \quad x \setminus u = x \delta^{-1}(x^{-1}u)$$

For example, if $G = \mathbb{R}^n$ with addition and $\varepsilon > 0$, then let $\delta_\varepsilon(x) = \varepsilon x$. The irq operations are then

$$x *_k u = x + \varepsilon^k (-x + u) \quad , \quad x \setminus_k u = x + \varepsilon^{-k} (-x + u)$$

For any x and u the point $x * u$ is the result of the homothety of coefficient ε , based at x and applied to u . If for example $\varepsilon = \frac{1}{2}$ then $x * u$ is the middle point between x and u , or the arithmetic average of x and u .

The induced operations are:

$$\begin{aligned} v -\frac{x}{k} u &= (xuv)_k = x + \varepsilon^k (-x + u) - u + v \\ u +\frac{x}{k} v &=)xuv(k = u + \varepsilon^k (-u + x) - x + v \end{aligned}$$

If we neglect the term in ε^k then the difference and the sum operations are the translates by x of the difference and sum operation in \mathbb{R}^n . This is a general phenomenon which is related with contractible groups, as explained by proposition 4.5, further on.

In the case of the general irq $G(\delta)$, if δ is a group morphism then we get for the induced operations the same expressions as previously:

$$\begin{aligned} v -\frac{x}{k} u &= (xuv)_k = x \delta^k (x^{-1} u) u^{-1} v \\ u +\frac{x}{k} v &=)xuv(k = u \delta^k (u^{-1} x) x^{-1} v \end{aligned}$$

One particular case is the one of contractible groups.

Definition 4.2 *A contractible group is a pair (G, α) , where G is a topological group with neutral element denoted by e , and $\alpha \in \text{Aut}(G)$ is an automorphism of G such that:*

- α is continuous, with continuous inverse,
- for any $x \in G$ we have the limit $\lim_{n \rightarrow \infty} \alpha^n(x) = e$.

For a contractible group (G, α) , the automorphism α is compactly contractive (Lemma 1.4 (iv) [20]), that is: for each compact set $K \subset G$ and open set $U \subset G$, with $e \in U$, there is $N(K, U) \in \mathbb{N}$ such that for any $x \in K$ and $n \in \mathbb{N}$, $n \geq N(K, U)$, we have $\alpha^n(x) \in U$.

If G is locally compact then α compactly contractive is equivalent with: each identity neighbourhood of G contains an α -invariant neighbourhood.

A even more particular case is the one of locally compact groups admitting a contractive automorphism group. We begin with the definition of a contracting automorphism group [20], definition 5.1.

Definition 4.3 *Let G be a locally compact group. An automorphism group on G is a family $T = (\tau_t)_{t>0}$ in $\text{Aut}(G)$, such that $\tau_t \tau_s = \tau_{ts}$ for all $t, s > 0$.*

The contraction group of T is defined by

$$C(T) = \left\{ x \in G : \lim_{t \rightarrow 0} \tau_t(x) = e \right\} \quad .$$

The automorphism group T is contractive if $C(T) = G$.

Next is proposition 5.4 [20], which gives a description of locally compact groups which admit a contractive automorphism group.

Proposition 4.4 *For a locally compact group G the following assertions are equivalent:*

- (i) G admits a contractive automorphism group;
- (ii) G is a simply connected Lie group whose Lie algebra admits a positive graduation.

The proof of the next proposition is an easy application of the previously explained facts.

Proposition 4.5 *Let (G, α) be a contractible group and $G(\alpha)$ be the associated irq. Then*

$$\lim_{k \rightarrow \infty} v \overset{-x}{\underset{k}{\dashv}} u = x u^{-1} v \quad , \quad \lim_{k \rightarrow \infty} u \overset{+x}{\underset{k}{\dashv}} v = u x^{-1} v$$

uniformly with respect to x, u, v in a compact set.

If G is a group which admits a contractive automorphism group $T = (\tau_t)_{t>0}$ then we define for any $t > 0$ the irq $G(\tau(t))$ which has the associated operations denoted respectively by $v \overset{-x}{\underset{t}{\dashv}} u$ and $u \overset{+x}{\underset{t}{\dashv}} v$. Then we have:

$$\lim_{t \rightarrow 0} v \overset{-x}{\underset{t}{\dashv}} u = x u^{-1} v \quad , \quad \lim_{t \rightarrow 0} u \overset{+x}{\underset{t}{\dashv}} v = u x^{-1} v$$

uniformly with respect to x, u, v in a compact set.

A particular class of locally compact groups which admit a contractive automorphism group is made by Carnot groups. They are related to sub-riemannian or Carnot-Carathéodory geometry, which is the study of non-holonomic manifolds endowed with a Carnot-Carathéodory distance. Non-holonomic spaces were discovered in 1926 by G. Vrănceanu [21], [22]. The Carnot-Carathéodory distance on a non-holonomic space is inspired by Carathéodory [6] work from 1909 on the mathematical formulation of thermodynamics. Such spaces appear in applications to thermodynamics, to the mechanics of non-holonomic systems, in the study of hypo-elliptic operators cf. Hörmander [12], in harmonic analysis on homogeneous cones cf. Folland, Stein [8], and as boundaries of CR-manifolds. We briefly describe the structure of Carnot groups.

Definition 4.6 *A Carnot (or stratified homogeneous) group is a pair (N, V_1) consisting of a real connected simply connected group N with a distinguished subspace V_1 of the Lie algebra $Lie(N)$, such that the following direct sum decomposition occurs:*

$$n = \sum_{i=1}^m V_i \quad , \quad V_{i+1} = [V_1, V_i]$$

The number m is the step of the group. The number $Q = \sum_{i=1}^m i \dim V_i$ is called the homogeneous dimension of the group.

Because the group is nilpotent and simply connected, the exponential mapping is a diffeomorphism. We shall identify the group with the algebra, if is not locally otherwise stated.

The structure that we obtain is a set N endowed with a Lie bracket and a group multiplication operation, related by the Baker-Campbell-Hausdorff formula. Remark that the group operation is polynomial.

Any Carnot group admits a contractive automorphism group. For any $\varepsilon > 0$, the associated morphism is:

$$x = \sum_{i=1}^m x_i \mapsto \delta_\varepsilon x = \sum_{i=1}^m \varepsilon^i x_i$$

This is a group morphism and a Lie algebra morphism.

5 Uniform irq

Motivated by proposition 4.5, we introduce the notion of uniform idempotent right quasigroup.

Definition 5.1 *A uniform irq $(X, *, \backslash)$ is a separable uniform space X endowed with continuous irq operations $*$, \backslash such that:*

- *the operation $*$ is compactly contractive: for each compact set $K \subset X$ and open set $U \subset X$, with $x \in U$, there is $N(K, U) \in \mathbb{N}$ such that for any $u \in K$ and $n \in \mathbb{N}$, $n \geq N(K, U)$, we have $x *_n u \in U$;*
- *the following limits exist*

$$\lim_{k \rightarrow \infty} v -^x_k u = v -^x_\infty u \quad , \quad \lim_{k \rightarrow \infty} u +^x_k v = u +^x_\infty v$$

and are uniform with respect to x, u, v in a compact set.

The main property of a uniform irq is the following.

Theorem 5.2 *Let $(X, *, \backslash)$ be a uniform irq. Then for any $x \in X$ the operation $(u, v) \mapsto u +^x_\infty v$ gives X the structure of a contractible group with the contraction $\alpha(u) = x * u$.*

Proof. Let us use the relations from proposition 3.5, for the irq $(X, *_k, \backslash_k)$. The uniformity assumptions from definition 5.1 allow us to pass to the limit with $k \rightarrow \infty$ in these relations. We obtain the following list:

- (a) $(u +^x_\infty v) -^x_\infty u = v$
- (b) $u +^x_\infty (v -^x_\infty u) = v$
- (c) the inverse $-^x_\infty u$ (see (f) below) exists and $v -^x_\infty u = (-^x_\infty u) +^x_\infty v$

- (d) $-_{\infty}^x (-_{\infty}^x u) = u$
- (e) $u +_{\infty}^x (v +_{\infty}^x w) = (u +_{\infty}^x v) +_{\infty}^x w$
- (f) $-_{\infty}^x u = x -_{\infty}^x u$ exists as a particular case of the limit $x -_{\infty}^x u$ with $v = x$,
- (g) $x +_{\infty}^x u = u$.

As a consequence $(X, +_{\infty}^x)$ is a group with neutral element x and inverse $u \mapsto -_{\infty}^x u$. It is left to prove that $\alpha(u) = x * u$ is a group automorphism. But this is a consequence of passage to the limit in relation (k) proposition 3.6, for $q = 1$ and $p \rightarrow \infty$. \square

We have seen that to any uniform irq we can associate a bundle of contractible groups $x \in X \mapsto (X, +_{\infty}^x, x*)$. This bundle resembles a lot to a tangent bundle, namely: to any $x \in X$ is associated a contractible group with x as neutral element, which can be seen as the tangent space at x . In fact this is a correct picture in the case of a manifold, in the following sense: if we look to a small portion of the manifold then we know that there is a chart of this small portion, which puts it in bijection with an open set in \mathbb{R}^n . We have seen that we can associate to \mathbb{R}^n a uniform irq by using as the operation $*$ a homothety with fixed ratio $\varepsilon < 1$. This uniform irq is transported on the manifold by the chart. If we ignore the facts that we are working not with the whole manifold, but with a small part of it, and not with \mathbb{R}^n , but with an open set, then indeed we may identify, for any point x in the manifold, a neighbourhood of the point with a neighbourhood of the tangent space at the point, such that the operation of addition of vectors in the tangent space at x is just the operation $+_{\infty}^x$ and scalar multiplication by (any integer power of) ε is just $x*$.

The same is true in the more complex situation of a sub-riemannian manifold, as shown in [4], in the sense that (locally) we may associate to each point x a "dilatation" of ratio ε , which in turn gives us a structure of uniform irq. In the end we get a bundle of Carnot group operations which can be seen as a tangent bundle of the sub-riemannian manifold. (In this case we actually have more structure given by the Carnot-Carathéodory distance, which induces also a "group norm" on each Carnot group.)

We shall comment now the fact that a uniform irq can be seen as a differential structure. For this it is enough to have a definition of differentiable functions between two uniform irqs.

Definition 5.3 *Let $(X, *, \backslash)$ and $(Y, \circ, \backslash\backslash)$ be two uniform irqs. A function $f : X \rightarrow Y$ is differentiable if there is a function $Tf : X \times X \rightarrow Y$ such that*

$$\lim_{k \rightarrow \infty} f(x) \backslash\backslash_k f(x *_k u) = Tf(x, u)$$

uniformly with respect to x, u in compact sets.

This definition corresponds to uniform differentiability in the metric case of dilatation structures (definition 16 and the comments after it in [2]) and by abstract nonsense the application Tf has nice properties, like $Tf(x, \cdot) : (X, +^x) \rightarrow (Y, +^{f(x)})$ is a morphism of contractible groups.

It is an interesting question if the emergent operation $+^x$ is differentiable. The answer is yes, provided one chooses the right uniform irq structure on $X \times X$, as explained in the metric case in section 8.2 [2]. The construction of a uniform irq on $X \times X$, starting from a uniform irq on X , such that the operation $+^x$ is differentiable, is clearly not canonical. But the operation $+^x$ is constructed in a "emergent" manner and this property seems to us stronger than differentiability (although we don't have a proof for this imprecise statement).

6 Emergent algebraic structures

In this section we describe two examples of emergent algebraic structures. We prove in theorem 6.1 that contractible groups are in bijection with distributive uniform irqs. We then show that such right quasigroups are in fact quasigroups. The second example concerns symmetric spaces in the sense of Loos [14]. We prove that already some properties of the inverse operation in a symmetric space are true for any uniform irq, then we propose a notion of uniform symmetric quasigroup, seen as a uniform quasigroup with a distributivity property for a "approximate inverse" operation inv_k , for any $k \in \mathbb{N}^*$ and prove that uniform symmetric quasigroups generate a family of symmetric spaces which contains the riemannian symmetric spaces.

6.1 Distributive uniform irqs

To any group G and contraction α we can associate a uniform irq, as explained by proposition 4.5. The operation $*$ is constructed from α and the group operation.

We could turn the things upside down and ask if we can recover the contraction and the group operation from the respective uniform irq. We shall construct the initial group operation from the more basic operations of the uniform irq, by a passage to the limit. In this sense the group operation is "emergent" as the limit of longer and longer strings of compositions of operations $*$ and \backslash .

Theorem 6.1 *Let (G, α) be a contractible group and $G(\alpha)$ be the associated uniform irq. Then the irq is distributive, namely it satisfies the relation*

$$x * (y * z) = (x * y) * (x * z) \quad (6.1.1)$$

*Conversely, let $(G, *, \backslash)$ be a distributive uniform irq. Then there is a group operation on G (denoted multiplicatively), with neutral element e , such that:*

- (i) $xy = x +_\infty^e y$ for any $x, y \in G$,
- (ii) for any $x, y, z \in G$ we have $(xyz)_\infty = xy^{-1}z$,
- (iii) for any $x, y \in G$ we have $x * y = x(e * (x^{-1}y))$.

In conclusion there is a bijection between distributive uniform irqs and contractible groups.

Proof. Recall that for a contractible group G, α we define $x * y = x\alpha(x^{-1}y)$. Therefore $\alpha(y) = e * y$. The distributivity relation (6.1.1) is indeed true, as shown by the following string of equalities:

$$\begin{aligned} x * (y * z) &= x \alpha(x^{-1}y \alpha(y^{-1}z)) = x \alpha(x^{-1}y) \alpha^2(y^{-1}z) = \\ &= x \alpha(x^{-1}y) \alpha(\alpha(y^{-1}x x^{-1}z)) = x \alpha(x^{-1}y) \alpha(\alpha(y^{-1}x)x^{-1} x\alpha(x^{-1}z)) = \\ &= (x * y) \alpha\left((x * y)^{-1} (x * z)\right) = (x * y) * (x * z) \end{aligned}$$

Conversely, let $e \in X$ and define $xy = x +_{\infty}^e y$. This is a contractible group operation, with contraction $\alpha(y) e * y$. We are left to show (ii) and (iii).

(ii). If $(G, *, \setminus)$ is a distributive uniform irq then we also have the following distributivity relations:

$$\begin{aligned} x * (y \setminus z) &= (x * y) \setminus (x * z) \\ x \setminus (y * z) &= (x \setminus y) * (x \setminus z) \end{aligned}$$

Indeed, the first relation is deduced from (6.1.1) if we replace z by $y \setminus z$. The second distributivity relation comes from (6.1.1) if we replace y by $x \setminus y$ and z by $x \setminus z$. In fact we also obtain that for any $k > 0$ $(G, *_k, \setminus_k)$ is distributive and moreover, for any $k, l \in \mathbb{Z}$ we have the general distributivity relation

$$x *_k (y *_l z) = (x *_k y) *_l (x *_k z)$$

With the help of these relations we obtain the following: $(xyz)_k = x *_k (y \setminus_k z)$ and $)xyz)_k = y *_k (x \setminus_k z)$. Therefore we get $)xyz)_k = (yxz)_k$. After we pass to the limit with $k \rightarrow \infty$ we obtain:

$$)xyz)_{\infty} = (yxz)_{\infty} \tag{6.1.2}$$

The following string of equalities is true:

$$(xe(eyz)_k)_k = x *_k (e \setminus_k (e *_k (y \setminus_k z))) = x *_k (y \setminus_k z) = (xyz)_k$$

After passing to the limit with k we obtain: $(xe(eyz)_{\infty})_{\infty} = (xyz)_{\infty}$, therefore we have $(xyz)_{\infty} = (xe(eyz)_{\infty})_{\infty} =)ex(eyz)_{\infty}(\infty = xy^{-1}z$.

(iii). We start from the following computation:

$$\begin{aligned})ex(e*(exy)_k)_k &= x *_k (e \setminus_k (e * (exy)_k)) = x *_k (e \setminus_k (e * (e *_k (x \setminus_k y)))) = \\ &= x *_k (e * (x \setminus_k y)) = (x *_k e) * y \end{aligned}$$

We pass to the limit with k and we obtain:

$$x * y =)ex(e*(exy)_{\infty})_{\infty} = x(e*(x^{-1}y))$$

The proof is done. \square

Remark 6.2 *In the language of racks and quandles, the previous theorem states that there is a bijective correspondence between contractible groups and a family of quandles. Indeed, as indicated in remark 3.2, a "distributive irq" is another name for a quandle. In this respect, "uniform quandle" could be an alternative name for a uniform distributive irq. These uniform distributive irqs are particular examples of topological quandles, studied in [19].*

A loop is a quasigroup $(X, *, \backslash, /)$ with an identity element e such that $x * e = e * x = x$ for any $x \in X$. Quasigroups are isotopic with loops, by a well known construction described further: for any $x \in X$ we define the operation on X

$$u \circ^x v = (u/x) * (x \backslash v) \quad (6.1.3)$$

Then (X, \circ^x) is a loop with identity element x .

Turning back to remark 3.4, we see that if X is a quasigroup the operations $+^x$ and \circ^x are isotopic. In relation to this, for Carnot groups or more general for normed conical groups (see definition 7 [3]) we can state the following corollary of proposition 8.4 [3].

Corollary 6.3 *Let G be a contractible group with contraction δ and uniformity coming from a left invariant norm compatible with the contraction. Let us define the operation $x * y = x \delta(x^{-1}y)$. Then $(G, *_k)$ is a quasigroup, and moreover, if we denote by \circ_k^x the isotope of $*_k$ as defined by (6.1.3), we have*

$$\lim_{k \rightarrow \infty} u \circ_k^x v = u +^x v$$

uniformly with respect to x, u, v in compact set.

Proof. The fact that $(G, *_k)$ is a quasigroup is a consequence of proposition 8.4 [3]. Indeed the solution of the equation $x *_k a = b$ is

$$x = b /_k a = \prod_{p=0}^{\infty} \delta^p (\delta(a^{-1}) b)$$

We can compute then $u \circ_k^x v$. We use the notation $\delta_k^x y = x *_k y$ and we write:

$$u \circ_k^x v = (u /_k x) *_k (x \backslash_k v) = \delta_k^{u/_k x} \delta_{k-1}^x v = (u /_k x) \delta_k \left((u /_k x)^{-1} x \right) x^{-1} v$$

As $k \rightarrow \infty$ we have $u /_k x \rightarrow u$ uniformly with respect to x, u, v in compact set (see for this the "affine" interpretation of $u /_k x$ as a ratio of a collinear triple in proposition 8.7 and the estimates provided by proposition 8.8 [3]). We can therefore pass to the limit and obtain the conclusion of the corollary. \square

6.2 Uniform symmetric quasigroups

Further we try to construct symmetric spaces in the sense of Loos [14] from uniform irqs.

Definition 6.4 (X, inv) is an algebraic symmetric space if $inv : X \times X \rightarrow X$ is an operation which satisfies the following axioms:

(L1) inv is idempotent: for any $x \in X$ we have $inv(x, x) = x$,

(L2) distributivity: for any $x, y, z \in X$ we have

$$inv(x, inv(y, z)) = inv(inv(x, y), inv(x, z))$$

(L3) for any $x, y \in X$ we have $inv(x, inv(x, y)) = y$,

(L4) for every $x \in X$ there is a neighbourhood $U(x)$ such that $inv(x, y) = y$ and $y \in U(x)$ then $x = y$.

If X is a manifold, inv is smooth (of class C^∞) and (L4) is true locally then (X, inv) is a symmetric space as defined by Loos [14] chapter II, definition 1.

We have seen previously that manifolds are particular cases of uniform irqs. Our problem is to propose a generalization of Loos symmetric spaces as uniform irqs with supplementary algebraic properties.

For a uniform irq a good candidate for the operation inv of an algebraic symmetric space is

$$inv_\infty(x, y) = \lim_{k \rightarrow \infty} inv_k(x, y) \quad , \quad inv_k(x, y) = (x *_k y) \setminus_k x$$

This limit exists because, as a consequence of proposition 3.5 (f), we have $inv_\infty(x, y) = x -^x_\infty y$. Indeed, from corollary 6.3 we deduce that axioms (L1), (L4) are true for inv_∞ . Moreover relation (d) proposition 3.5 looks like a weak version of axiom (L3). This is confirmed by the following proposition.

Proposition 6.5 Let $(X, *, \setminus)$ be a irq and $T(y, x) = (inv(x, y), x * y)$. Then $T \circ T = id$.

Proof. We define $\delta : X \times X \rightarrow X \times X$ and $i : X \times X \rightarrow X \times X$ by:

$$\delta(y, x) = (x * y, x) \quad , \quad i(y, x) = (x, y)$$

Because X is a irq the transformation δ is invertible and $\delta^{-1}(y, x) = (x \setminus y, x)$. By direct computation we have $T = \delta^{-1} \circ i \circ \delta$. The conclusion follows by the obvious fact that $i \circ i = id$. \square

There is only one axiom left to investigate, namely (L2). We would like to define an operation inv_k , for any $k \in \mathbb{N}^*$, which satisfies (L2) and it is constructed from the operations of a uniform irq.

We have two hints about constructing such an operation. The first hint comes from the observation that contractible groups are symmetric spaces and corollary 6.3 states that the uniform irq of a contractible group is in fact a quasigroup. The second hint comes from the fact that a Lie group is a symmetric space with operation $inv(u, v) = uv^{-1}u$. Consider then the uniform irq associated to the Lie group G and a transformation $\delta : G \rightarrow G$, which is not a group morphism, but it is

continuous, with continuous inverse and compactly contractive. Then $G(\delta)$ is an uniform irq and after an elementary computation we find the interesting relations:

$$\text{inv}_k(u, v) = \text{inv}(u, v *_k u) \quad , \quad \text{inv}_\infty(u, v) = \text{inv}(u, v)$$

These two considerations motivate us to introduce the following definition of a uniform symmetric quasigroup.

Definition 6.6 *A uniform symmetric quasigroup (usq) is a uniform irq $(X, *)$ with the following properties:*

(i) $(X, *_k, \backslash_k)$ is a quasigroup for any $k \in \mathbb{N}^*$. For any $a, b \in X$ we denote by $x = a /_k b$ the solution of the equation $x *_k b = a$.

(ii) the function $\underline{\text{inv}}_k : X \times X \rightarrow X$ defined by

$$\underline{\text{inv}}_k(u, v) = \text{inv}_k(u, v /_k u) = (u *_k (v /_k u)) \backslash_k u$$

satisfies (L2) (is distributive).

(iii) the function inv_∞ satisfies (L4).

In order to cover all symmetric spaces (like the compact ones for example) we should ask that condition (i) is true only locally. (The locality condition in (iii) is of a different nature, see the following remark). In this paper we make only local considerations. The problem of locality in (i) can be solved, but this leads to more involved reasoning and notations, which only obscure the ideas presented here. For a way to treat the locality problem (but within the metric space theory) see [2] [3].

Remark 6.7 *The condition (iii) is not so severe. For example if the tangent space at x , namely $(X, +^x)$, is a Carnot group (which is the case for the symmetric spaces in the sense of Loos) then (iii) is trivially true.*

*Thus a more restrictive condition that (iii) is: for every $x \in X$ the group $(X, +^x)$ is locally compact (with respect to the topology inherited from the uniformity on X) and admits a contractive automorphism group $C(x)$ such that the application $\delta^x(y) = x * y$ belongs to $C(x)$.*

Proposition 6.8 *If $(X, *, \backslash)$ is a uniform symmetric quasigroup then (X, inv_∞) is an algebraic symmetric space. Conversely, let X be any riemannian symmetric space, $\varepsilon \in (0, 1)$, let the operation $*$ be defined by $x * \exp^x(X) = \exp^x(\varepsilon X)$, where \exp is the geodesic exponential. Then X is a uniform symmetric quasigroup.*

Proof. With $\text{inv} = \text{inv}_\infty$ the properties (L1), (L3) are true for any usq. (L4) is just the condition (iii). It is left to study the distributivity (L2). This is straightforward because $\underline{\text{inv}}_k(x, u *_k x) = \text{inv}_k(x, u)$, and $\underline{\text{inv}}_k(x, \cdot)$ converges uniformly on compact sets to $\text{inv}(x, \cdot)$, therefore we have the uniform limit

$$\lim_{k \rightarrow \infty} \underline{\text{inv}}_k(x, v) = \text{inv}(x, v)$$

But then we can pass to the limit in the distributivity relation for inv_k and eventually obtain the distributivity condition (L4) for inv_∞ .

For the second part of the proposition we first recall that we are making only local considerations, therefore we shall consider that any two points are joined by a unique geodesic. The fact that $(X, *)$ is a uniform irq reduces to an elementary computation using the geodesic exponential (basically that the derivative of geodesic exponential $exp^x(\varepsilon X)$ with respect to ε at $\varepsilon = 0$ equals X). We only have to check then if the operation $*$ is a quasigroup operation and if it satisfies (ii) 6.6. But this is obvious because we are using the geodesic exponential for the definition of this operation. Then x , $x * u$ and u are on the same geodesic and with respect to the coordinate system with origin in x along this geodesic, the coordinate of $x * u$ equals ε times the coordinate of u . This shows that x is determined by $x * u$ and u . Concerning the condition (ii), we claim that $inv_k(u, v) = inv(u, v)$ for any $k \in \mathbb{N}^*$. This is equivalent with $inv_k(u, v) = inv(v *_k u)$. Indeed let γ be the geodesic joining u with v , parameterized by length, with origin in u . If v has coordinate V , then $u *_k v$ is on γ , with coordinate $\varepsilon^k V$ and $inv_k(u, v)$ is then also on γ , with coordinate $(\varepsilon^k - 1)V$. But $inv(v *_k u)$ is on γ as well, with the same coordinate as $inv_k(u, v)$. \square

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