THE TOPOLOGICAL SUBSTRATUM OF THE DERIVATIVE (I)

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From the early days when the derivative was regarded as a velocity, a rate of change, until the modern viewpoint, that of a local approximation of a map by a map from a given class of operators, the concept of derivative has kept some topological features. These features are hidden in all kinds of geometrical interpretation of derivatives and one of our purpose is to show that these interpretations lie on a topological substratum. Thus it seems to be natural to ask if there is any way to find a class of differentiable maps using topological hypothesis. We named this questions the topological issue of the derivative. In this paper we show that the problem is to find a certain group of transformation which generates the derivative; this group describes the topological structure of the space in the neighbourhood of one of its points.

1. THE G-DERIVATIVE

In the following pages we use the notation $f : X \to Y$ for maps with two arguments in X with values in Y; the trace of f is the usual map $\varphi_f : X \to Y$ by $\varphi_f(x) = f(x, x)$. For $f : X \to Y$ and $g : Y \to Z$, $gf : X \to Z$ is the map $(x, y) \mapsto g(f(x, x), f(x, y))$.

Let $(G, .)$ be a group endowed with a compatibile partial-order relation which directs G, i. e.:

1. for any $m \in G$, if $h \leqslant n$ then $m \cdot h \leqslant m \cdot n$ and $h \cdot m \leqslant n \cdot m$
2. for any $m, h \in G$ there is a $g \in G$ with $g \geqslant h$ and $g \geqslant m$.

Also, let $(\ ) : G \times X \times X \to X$ be an action of G on a Hausdorff space X, i.e.

1. the map $y \mapsto (\ )(g, x, y) = g(x, y)$ is continous and the trace of $(x, y) \mapsto g(x, y)$ is 1_X;
2. for any $f, g \in G$ $f(x, g(x, y)) = (f \cdot g)(x, y)$ (or $(g \cdot f)(x, y)$ for right-action);
3. if $f(x, y) = g(x, y)$ for any x, y then $f = g$;
4. $1(x, y) = y$ for any x, y.

D I. 1. 1. Le $f : X \to X$ be a continuous map and let $x, y \in X$. We say that f is G-derivable in x along y if the limit of the net $(g^{-1}fg(x, y))_{g\in G}$ (or $(g\, fg^{-1}(x, y))_{g\in G}$ for righ-action) exists. In this case we note $\lim g^{-1}fg\,(x, y) = (D_G f)(x, y)$, the G-derivative of f in x along y.

A natural extension of D I. 1.1 is made like this: let $f : X \to Y$ be a continous map and $x, y \in X$. Suppose that we have two groups G_1 and G_2 with two actions on X and Y and a group-morphism $\lambda : G_1 \to G_2$ which preserves the order-relations and $\lambda(G_1)$ is a subnet of G_2. We say that f is λ-derivable in x along y iff the net $(\lambda(g^{-1})\ fg(x, y))_{g\in G}$ has a limit and this limit is $(D_\lambda f)(x, y)$, the λ-derivative of f in x along y.

P I. 1.1: *Let $f : X \to X$ be a G-derivable map in x along y and let $h \in G$. Then f is G-derivable in x along $h(x, y)$ and*

$$h(D_G f)(x, y) = (D_G f)h(x, y). \tag{I. 1.1}$$

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Proof. From D I. 1.1, f is G-derivable in x along x and $(D_G f)(x,x) = \varphi_f(x)$ so

$$h(D_G f)(x, y) = h(\varphi_f(x)), \ (D_G f)(x, y) = \lim hh'^{-1} fh'(x, y) =$$

$$= (\lim m^{-1} fm) h(x, y) = (D_G f) h(x, y).$$

We shall see that P I. 1.1 assures us that D_G is a kind of "homogeneous" operator.

P I. 1.2 : *Let $f, g : X \to X$ be two G-derivable maps in x, $\varphi_f(x)$. If gf is a G-derivable map in x then*

$$(D_G gf)(x, y) = (D_G g)(D_G f)(x, y) \text{ (the chain rule).} \tag{I. 1.2}$$

Proof.

$$\lim h^{-1} gfh(x, y) = \lim h^{-1} ghh^{-1} fh(x, y) =$$

$$= (\lim h^{-1} gh)(\lim h^{-1} fh)(x, y) = (D_G g)(D_G f)(x, y).$$

P I. 1.3. *G is a commutative group iff for any $h \in G$ $D_G h = h$.*

Proof. If G is commutative then

$$\lim m^{-1} hm(x, y) = \lim hm^{-1} m(x, y) = h(x, y),$$

so $(D_G h)x, y) = h(x, y)$. *Conversely, if for any $h \in G$, $D_G h$ exists then* PI1.1 *assures us about commutativity of G.*

We shall give some examples of D_G or D_λ derivatives.

Let G be the following group :

$$G_n = \{o_k^n : R^n \to R^n \ o_k^n(x, y) = x + k(y - x), \ k > 0\}. \tag{I. 1.3}$$

For any n we define an order-relation by

$$o_k^n \geqslant o_l^n \text{ iff } k \leqslant l. \tag{I. 1.4}$$

Now let $\lambda : G_n \to G_p$ be the following group-morphism

$$\lambda(o_k^n) = o_k^l. \tag{I. 1.5}$$

For $f : R^n \to R^p$ let $f(x, y) = f(y)$. Then

$$\lambda(o_k^{n-1}) f o_k^{n-1}(x, y) = f(x) + k\,(f(x + k(y - x)) - f(x)), \text{ so}$$

$$(Df)(x, y) = f(x) + Df(x)(y - x), \tag{I. 1.6}$$

where $Df(x)(\mathbf{v})$ is the Gâteax-derivative of f in x along $\mathbf{v}$. If we think at $Df(x)$ like a homogeneous map from the tangent-space in x to the tangent-space in $f(x)$ the right-member of the previous equality forms the Gâteaux-derivative of f in x. It is also clear now that P I. 1.1 shows in this case that the operator D is homogenous.

Another example is the following: let G_h be the group of transformations defined by

(I. 1.7) $o_{k,l} \in G_h$ iff $o_{k,l}((x_1, y_1), (x_2, y_2)) = (x_1 + k(x_2 - x_1), y_1 + l(y_2 - y_1))$,

where $kl \neq 0$. We say that

(I. 1.8) $$o_{k,l} \geqslant o_{m,n} \text{ iff } k \leqslant m \text{ and } l \leqslant n.$$

Let λ be the group-morphism:

(I. 1.9) $$\lambda : G_h \to G_1 \text{ by } \lambda(o_{k,l}) = o_{kl}.$$

For any functional $g : R^2 \to R$ we define g by

(I. 1.9) $g((x_1, y_1), (x_2, y_2)) = g(x_1, y_1) + g(x_2, y_2) - g(x_1, y_2) - g(x_2, y_1)$.

g is a continuous functional iff g is bidimensional continuous (see [3]). If the limit exists we have the following equality:

(I. 1.10) $\lim \lambda(o_{k,l}^{-1}) g o_{k,l}((x_1, y_1), (x_2, y_2)) = (D_h g)(x_1, y_1)(x_2 - x_1)(y_2 - y_1)$,

where $D_h g(x, y)$ is the hyperbolic derivative of g in (x, y) (see [3])

(I. 1.11) $$D_h g(x, y) = \lim_{\substack{h \to 0 \\ k \to 0}} \frac{g(x+h, y+k) + g(x, y) - g(x+h, y) - g(x, y+k)}{hk}.$$

Let X be a Hausdorff space and $M \subseteq X$ be an n-manifold in X; also let $\mathcal{A}$ be a maximal atlas of M.

For any map $h \in \mathcal{A}$ we define

(I. 1.12) $$\tilde{h} : \mathcal{C}(R^n, R^n) \to \mathcal{C}(X, R^n) \text{ by } \tilde{h}(\varphi) = \varphi h$$

and for any parametrization p of M we define

(I. 1.13) $$\tilde{p} : \mathcal{C}(X, R^n) \to \mathcal{C}(R^n, R^n) \text{ by } \tilde{p}(f) = fp$$

In this way we can see a manifold in X like a sort of manifold in $\mathcal{C}(X, R^n)$ which has maps on $\mathcal{C}(R^n, R^n)$ instead R^n and parametrizations defined on $\mathcal{C}(R^n, R^n)$ too. An atlas of M becomes a family of parametrizations on $\mathcal{C}(R^n, R^n)$ and conversely.

A curve $c : R \to M$ becomes $\tilde{c} : \mathcal{C}(X, R^n) \to \mathcal{C}(R, R^n)$ by $\tilde{c}(f)(t) = f(c(t))$. Let G_1 be the following group of transformations of $\mathcal{C}(X, R^n)$:

(I. 1.14) $\quad o_k : \mathcal{C}(X, R^n) \to \mathcal{C}(X, R^n).\ o_k(f, g) = f + k(g - f),$

and G_2 be the group

(I. 1.15) $\quad o_k : \mathcal{C}(R, R^n) \to \mathcal{C}(R, R^n),\ o_k(r, u)(t) = r(t) + u(kt) - r(kt).$

The order-relation and the morphism are defined like in (I. 1.4) and (I. 1.5). The limit $\lim \lambda(o_k)\ \tilde{c}\, o_k^{-1}(f, g)$, where c is a curve, exists only if $g(c(0)) = f(c(0))$ because

$$\lambda(o_k)\ \tilde{c}\, o_k^{-1}(f, g)(t) = f(c(t)) + k^{-1}(g(c(kt)) - f(c(kt))).$$

If the limit exists, for $f, g \in \mathfrak{X}(x) = \{f \in \mathfrak{X} f(x) = 0\}$, $x = c(0)$.

(I. 1.16) $\quad (\mathrm{D}_\lambda c)(f, g)(t) = f(x) + X_c(g) - X_c(f),$

where X is the tangent vector in x at c.

If we want to make the same construction on M, not on the family of functionals on M, we meet two problems: first — how to define a particular group when we do not know anything about X — and the second — if we consider manifolds in normed vector spaces, we can easily see that we request a group of transformations of X, not of M. Indeed, we need the group of transformations with the following form:

(I. 1.17) $\quad o_k(x, y) = x + k(y - x),$

but if $x, y \in M$ nothing assures us that $x + k(y - x) \in M$. That are the reasons of working with functionals.

We can say that the morphism λ describes the usual differential operator used to define the notion of a tangent vector at M.

In the following example we will define another morphism for any 2-dimensional manifold M. This example will show the difference between the manifold and the differential structure which arises on the manifold.

Let $\mathcal{H} = \{(x, y) \in R^2 | x^2 + y^2 < 1\}$ a submanifold of the canonical manifold R^2. We define o $\mathcal{H}$ the following metric:

(I. 1.18) $\quad \mathbf{G}(x, y) = \mathbf{G}_{ij}\mathbf{i}_i \otimes \mathbf{i}_j,$

where $(\mathbf{i}_1, \mathbf{i}_2)$ is the canonical base in R^2 and

(I. 1.19)
$$\mathbf{G} = \begin{bmatrix} \dfrac{1 - y^2}{(1 - x^2 - y^2)^2} & \dfrac{xy}{(1 - x^2 - y^2)^2} \\[2ex] \dfrac{xy}{(1 - x^2 - y^2)^2} & \dfrac{1 - x^2}{(1 - x^2 - y^2)} \end{bmatrix}$$

This Riemannian manifold is the most familiar model for the hyperbolic plane: the Poincaré ([4]). For our purposes we will choose another manifold, diffeomorphic with $\mathcal{H}$,

(I. 1.20) $$H = \{(x_1, x_2, x_3) \in R^3 | x_3 > 0, x_3^2 - x_1^2 - x_2^2 = 1\}.$$

Let $\mathbf{L}_+$ be the Lorentz proper group in $\mathcal{R}^3$ and $\overline{\mathbf{L}}_+$ the subgroup of $\mathbf{L}_+$ formed by transformations described in canonical base of $\mathcal{R}^3$ by the matrix:

(I. 1.21) $$\begin{bmatrix} v_1^2 v^{-2}(\beta - 1) & v_1 v_2 v^{-2}(\beta - 1) & -v_1\beta \\ v_1 v_2 v^{-2}(\beta - 1) & v_2^2 v^{-2}(\beta - 1) & -v_2\beta \\ -v_1\beta & -v_2\beta & \beta \end{bmatrix}$$

where $v^2 = v_1^2 + v_2^2 < 1$, $v_1, v_2 \in R$ are two real parameters, and $\beta = (1 - v^2)^{-\frac{1}{2}}$.

For any $x = (x_1, x_2, x_3) \in H$ there is a unique $\mathbf{L}_x \in \overline{\mathbf{L}}_+$ with

(I. 1.22) $$\mathbf{L}_x((0,0,1)) = (x_1, x_2, x_3),$$

determined by the parameters $v_1 = x_1 x_3^{-1}$, $v_2 = x_2 x_3^{-1}$. So, for any two points $x_1, x_2 \in H$, there is a unique $\mathbf{L}_s \in \overline{\mathbf{L}}_+$ with

(I. 1.23) $$\mathbf{L}_s(x_1) = x_2.$$

For any $x \in H$ we define $(\mathbf{e}_1(x), \mathbf{e}_2(x))$ a base of $T_x(H)$ by:

(I. 1.24)
$$\mathbf{e}_1(x) = \left(\frac{x_1^2(x_3 - 1)}{x_1^2 + x_2^2} + 1\right)\mathbf{i}_1 + \frac{x_1 x_2(x_3 - 1)}{x_1^2 + x_2^2}\mathbf{i}_2 - x_1\mathbf{i}_3$$
$$\mathbf{e}_2(x) = \frac{x_1 x_2(x_3 - 1)}{x_1^2 + x_2^2}\mathbf{i}_1 + \left(\frac{x_2^2(x_3 - 1)}{x_1^2 + x_2^2} + 1\right)\mathbf{i}_2 - x\mathbf{i}_3$$

The Frechét derivative of $\mathbf{L}_s$ in x_1, $(D\mathbf{L}_s)(x_1) : T_{x_1}(H) \to T_{x_2}(H)$, actions like this:

(I. 1.25) $$(D\mathbf{L}_s)(x_1)(\mathbf{e}_i)(x_1)) = \mathbf{e}_i(x_2).$$

Endowed with the metric $\mathbf{G} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2$, H becomes a Riemannian manifold diffeomorphic with $\mathcal{H}$ by:

(I. 1.26) $$(x_1, x_2, x_3) \mapsto (x_1 x_3^{-1}, x_2 x_3^{-1}).$$

We will look at $\mathcal{H}$ to see geometrical features and at H, where we have the group $\overline{\mathbf{L}}_+$, for moving from a point to another. $\overline{\mathbf{L}}_+$ is a hyperbolic analogus of the euclidean group of translations.

Let $x \in H$. We define on H the transformation $o_k(x, y) = z$, with tne following properties :

1. $|k|\ d_h(x, y) = d(x, z)$,

2. if $k > 0$ then z is on the h-ray determined by x and y and if $k < 0$ then z is on the opposite h-ray.

On $\mathcal{H}$ the metric determines the h-distance

$$d_h(x, y) = \frac{1}{2} \ln(x, y ; x^*, y^*), \tag{I. 1.27}$$

where x^*, y^* are the intersection-points of the h-line determined by x, y, which is an orthogonal circle on the circle $\mathcal{C}(0, 1)$, with the circle $\mathcal{C}(0, 1)$ and

$$(x, y ; x^*, y^*) = \frac{d_h(x, x^*) d_h(y, x^*)}{d_h(x, u^*) d_h(y, y^*)} \tag{I. 1.28}$$

So

$$d_h((0, 0), (x, y)) = \frac{1}{2} \ln \frac{1 + \|(x, y)\|}{1 - \|(x, u)\|}, \tag{I. 1.29}$$

whence

$$o_k((0, 0), (x, y)) = \tag{I. 1.30}$$

$$= \frac{(1 + \|(x, y)\|)^k - (1 - \|(x, y)\|)^k}{(1 + \|(x, y)\|)^k + (1 - \|(x, u)\|)^k} \left(\frac{x}{\|(x, \dot{y})\|}, \frac{y}{\|(x, \dot{y})\|} \right)$$

If we look on H by diffeomorphism (I. 1. 26) we can reach the form of o_k because

$$o_k(x, y) = \mathbf{L}_s(o_k((0,0, 1), \mathbf{L}_s^{-1}(y))) \tag{I. 1.31}$$

where $\mathbf{L}_s \in \mathbf{L}_+$, $\mathbf{L}_s((0, 0, 1)) = x$.

Let G_h be the group of these transformations for all $k \in R$, $k > 0$, Let M be a 2-dimensional manifold in X. $\mathcal{H}$ has a maximal atlas on R^2 containing a single map so we can work with maps on $\mathcal{H}$ instead R^2. In this case we have the same correspondence for maps and parametrizations on $\mathcal{H}$ like in the previous case. Let $\mathcal{F}_h$ be the family of functionals $f : M \to \mathcal{H}$. We define $\tilde{G}_h$ by :

$$\tilde{o}_k : \mathcal{F}_h \to \mathcal{F}_h \in G_h \text{ iff } \tilde{o}_k(f, g)(x) = o_k(f(x), g(x)), \tag{I. 1.32}$$

and

$$\lambda : \tilde{G}_h \to G_2 \text{ by } \lambda(\tilde{o}_k) = o_k. \tag{I. 1.33}$$

The order-reltion on $\tilde{G}_h$ is like in (I. 1.4).

We note the G-derivative determined by λ with D_h · From (I. 1.16) it follows that $X_c(g) = (D_\lambda c)\ (0, g)$, We define the tangent h-vector at c in $x = c(0)$ by :

(I. 1.34) $$X_c^h(g) = (D_h c)\ (0, g), \text{ where } 0 = (0, 0, 1).$$

So the same 2-dimensional manifold M has two different morphisms which give to M two different differential structures. The study of D_h is connected with the older problem of an intrinsic analysis on a hyperbolic plane, but here our purpose is to light the topological problem of derivatives, which is :

2. FINDING THE GROUP G BY TOPOLOGICAL HYPOTHESIS

Let (X, τ) be a Hausdorff space, with τ the set of open neighbourhoods from X, and let $X(\tau)$ be the set of all filters over τ · $X(\tau)$ is naturally endowed with the topology τ^e; X is embedded in $X(\tau)$ by the map $x \in X \to \mathcal{V}(x) \in X(\tau)$, where $\mathcal{V}(x)$ is the family of open neighbourhoods of x. We shall identify X with $\{\mathcal{V}(x) | x \in X\} \subseteq X(\tau)$.

D I. 2.1. Let $\delta : X \to \mathfrak{P}(X(\tau))$ be a map. If

1. for any $x \in X$, $\delta(x) \neq \emptyset$,
2. for any $\mathcal{V} \in \delta(x)$, $\mathcal{V}(x) \subseteq \mathcal{V}$,
3. for any $\mathcal{V}_1, \mathcal{V}_2 \in \delta(x)$, $\mathcal{V}_1 \setminus \mathcal{V}_2 \neq \emptyset$

then δ is a boundary map.

From D I. 2.1 it follows that for any $\mathcal{V}_1 \in \delta(x_1)$, $\mathcal{V}_2 \in \delta(x_2)$, if $x_1 \neq \neq x_2$ then $\mathcal{V}_1 \setminus \mathcal{V}_2 \neq \emptyset$, so for any $x \in X$ the subspace

(I. 2.1) $$X_x = (X \setminus \{x\}) \cup \delta(x)$$

of $X(\tau)$ is a Hausdorff space. The space X_x is, from a topological viewpoint, finer than X because the filter $\mathcal{V}(x)$ is replaced with a family of finer filters.

We can imagine that a boundary map is a correspondence $x \in X \to X_x$, where X_x is homeomorphic with $(X \setminus \{x\}) \cup \delta(x)$.

Let $f : X \to X$ be a continuous map. There is a unique continuous extension of f,

(I. 2.2) $$\tilde{f} : X(\tau) \to X(\tau),\ \tilde{f}/X = f.$$

D I. 2.2. If $\tilde{f}(X_x) \subseteq X_{f(x)}$ then f is δ-continuous in x. For any δ-continuous map f we define

(I. 2.3) $$\delta f : \bigcup_{x \in X} \delta(x) \to \bigcup_{x \in X} \delta(x),\ \delta f(\mathcal{V}) = \tilde{f}(\mathcal{V}),$$

the boundary of f. If f maps $(x, y) \mapsto f(x, y)$ then f is δ-continuous in x if the extension of the map $y \mapsto f(x, y)$ is δ-continous in x and the boundary of f in x is

(I. 2.4) $$\delta f(x) : \delta(x) \to \delta(\varphi_f(x)),\ \delta f(x)\ (\mathcal{V}) = \tilde{f}(x, \cdot)\ (\mathcal{V}).$$

The following example of a boundary map suggests the source of this notion.

For any $x \in R^n$ and for any open ray

$$s = \{x + t\mathbf{v} \mid t > 0, \mathbf{v} \neq 0\} \in S(x) \tag{I. 2.5}$$

let $\mathcal{V}(x)$ be the family of all open sets W for whic exists $V = \tilde{V}$, $U = \tilde{U}$ with

$$V \in \mathcal{V}(x), \quad s \subseteq U, \quad V \cap U \subseteq W \tag{I. 2.6}$$

$\mathcal{V}(s)$ is a finer filter than $\mathcal{V}(x)$ and for and different open rays $s_1, s_2 \in \mathcal{S}(x)$, $\mathcal{V}(s_1) \setminus \mathcal{V}(s_2) \neq \emptyset$. We define

$$\delta : R^n \to \mathfrak{L}(R^n(\tau)), \quad \delta(x) = \{\mathcal{V}(s) \mid s \in \mathcal{S}(x)\}. \tag{I. 2.7}$$

Let $f : R^n \to R^n$ be a continuous map. f is δ-continous in x iff for any net $(x_n)_n$ with :

1. $x_n \to x$;
2. there is a $\mathbf{v} \in \mathrm{R}^n$, $\|v\| = 1$, with $\left(\dfrac{x_n - x}{\|x_n - x)}\right) \to \mathbf{v}$, the net

$(f(x_n))_n$ has the same features 1 and 2. For any Frechét derivable map f, if the Frechét-derivative in $x \in R^n$ is invertible then it is obvious that f is δ-continuous in x.

Now we will try to characterize the topological structure of X in a neighbourhood of $x \in X$, seen in the space X_x. Let $i(x)$ be a base of open neigbourhoods for $\mathcal{V}(x) \cdot i(x)$ is directed by the relation $\supseteq$:

for any V, $U \in i(x)$ thereis a $W \in i(x)$ with $V \supseteq W$ and $U \supseteq W$.

We note

$$\mathrm{End}\ \delta(x) = \{h : X \to X \mid h(x) = x \text{ and } \tilde{h}/X_x : X_x \to X_x \text{ is a homeomorphism}\}, \tag{I. 2.8}$$

the group of endomorphisms of $\delta(x)$. There is a one-to-one correspondence between the homeomorphisms of X_x which does not move $\delta(x)$ and the endomorphisms of $\delta(x)$.

The map δ (I. 2.4) is a group-morphism from End $\delta(x)$ to

$$\mathrm{End}(x) = \{\delta h = \tilde{h}/\delta(x) \mid h \in \mathrm{End}(x)\} \tag{I. 2.9}$$

so

$$\ker \delta = \{h \in \mathrm{End}\,\delta(x) \mid \delta h = 1_{\delta(x)}\} \tag{I. 2.10}$$

is a normal subgroup of End $\delta(x)$.

The following set is another subgroup of End $\delta(x)$:

$$\mathrm{End}_i\,\delta(x) = \{h \in \mathrm{End}\ \delta(x) \mid h(i(x)) \subseteq i(x) \text{ and for any } V, W \in i(x)\ h(V) = h(W) \text{ iff } V = W\}, \tag{I. 2.11}$$

where $h(V) = \{h(y) | y \in V\}$. With the notation End (i) for the group of endomorphisme of the net $i(x)$, the map

(I. 2.12) $$i : \mathrm{End}_i\delta(x) \to \mathrm{End}(i),\ i(h)(V) = h(V)$$

is a group-morphism. We note

(I. 2.13) $$\Theta_i\delta(x) = \ker\ \delta \cap \mathrm{End}_i(x)$$
$$\Theta\delta(x) = \ker\ \delta \cap \ker\ i.$$

$\Theta\delta(x)$ is a normal subgroup of $\Theta_i\delta(x)$.
For any δ the group

(I. 2.14) $$\Theta_i\delta(x)/\Theta\delta(x) \rightleftarrows i(\Theta_i\delta x))$$

is called the fundamental group of $i(x)$ relative to δ. If $\delta(x) = \{\Upsilon(x)\}$ the correponding group is named the fundamental group of $i(x)$.
Let $X = R^n$ and for $x = 0$ let

(I. 2.15) $$i(0) = \{D(0,\ r) | r > 0\},\ D(0,\ r) = \{y\ |\ \|y\| < r\}.$$

The fundamental group of $i(0)$ is isomorphic with the group (I. 2.18).
Indeed, $f \in \Theta i(0)$, (for the simplest boundary map we neglect the notation with δ) if $f(x) = f(y)$ iff $\|x\| = \|y\|$, so for any $x \neq 0$

(I. 2.16) $$f(x) = k(\|x\|)\ \mathbf{Q}(x)\frac{x}{\|x\|},\ k : R_+ \to R_+ \text{ is continuously increasing } \mathbf{Q}(x)\mathbf{Q}^T(x) = 1,\ \mathbf{Q} \text{ is continuou}$$

$f \in \Theta(0)$ iff $\|x\| = \|f(x)\|$ so

(I. 2.17) $$f(x) = \mathbf{Q}(x)x,\ \mathbf{Q} \text{ is continous and } \mathbf{Q}(x)\mathbf{Q}^T(x) = 1.$$

It is obvious now that the factor group is isomorphic with the group of transformations

(I. 2.18) $$f(x) = k(\|x\|)x',\ k \text{ is continuously increasing, } x' = \frac{x}{\|x\|}.$$

For any group of continuous maps $x \mapsto \mathbf{Q}(x)$ and for any group-morphism $k(\ ) \mapsto \mathbf{Q}_k(\ )$ there is a group isomorphic with the fundamental group formed by

(I. 2.19) $$f(x) = k(\|x\|)\ \mathbf{Q}_k(x)x'.$$

From the previous example we can see that the group G that we are trying to reach at is isomorphic with a fundamental group of a $i(0)$. Supposing that there is a subgroup

(I. 2.20) $$\theta_i\delta(x) \subseteq \Theta_i\delta(x),\ \theta_i\delta(x) \rightleftarrows \Theta_i\delta(x)/\Theta\delta(x)$$

that subgroup may not be unique. The selection will be made by the boundary maps.

Let $I \subseteq \mathfrak{L}(X)$ be a net with the order-relation $\subseteq$. Then there is at most one point $x \in X$ with

1. for any $V \in \mathcal{V}(x)$ there is an $A \in I$, $A \subseteq V$.
2. for any $y \in X$ with 1. $\mathcal{V}(y) \subseteq \mathcal{V}(x)$, only if $\emptyset \notin I$. We denote

$$x = \text{Lim } I. \tag{I. 2.21}$$

Let δ be a boundary map for which there is a $V \in i(x)$ with for any $W \in i(x)$ there is a map $h \in \Theta_i\delta(x)$, $h(V) \subseteq W$.
For any group $\theta_i\delta(x)$ with (I. 2.20) we construct

$$\mathbf{Y}_U = \{h \in \theta_i\delta(x) / h(V) \subseteq U\}, \tag{I. 2.22}$$

for any $U \in i(x)$. For any $\mathcal{V} \in X(\tau)$

$$\mathbf{Y}_U(\mathcal{V}) = \{\tilde{h}(\mathcal{V}) / h \in \mathbf{Y}_U\}. \tag{I. 2.23}$$

Let

$$\tilde{G}(\mathcal{V}) = \text{Lim } \mathbf{Y}_U(\mathcal{V}),\ G = \theta_i\delta(x). \tag{I. 2.24}$$

be the limit-function of G relative to y.

P I. 2.1 : 1. *For any $h \in G$ and for any $\mathcal{V}$* $\tilde{h}\tilde{G}(\mathcal{V}) = \tilde{G}(\mathcal{V})$;

2. $h(\mathcal{V}) = \mathcal{V}$ *for any* $h \in \tilde{G}$ *iff* $\tilde{G}(\mathcal{V}) = \mathcal{V}$;

3. $\tilde{G}\tilde{G}(\mathcal{V}) = \tilde{G}(\mathcal{V})$ *for any* $\mathcal{V}$.

Proof : 1. For any $h \in G$ $i(h)$ is a one-to-one map so

$$\{hg / g \in \mathbf{Y}_U\} = \mathbf{Y}_{i(h)(U)}.$$

Also, because h is a continuous map,

$$\tilde{h}(\text{Lim } \mathbf{Y}_U(\mathcal{V})) = \text{Lim } \tilde{h}\mathbf{Y}_U(\mathcal{V}) = \text{Lim } \mathbf{Y}_U(\mathcal{V}).$$

2. Suppose that $h(\mathcal{V}) = \mathcal{V}$ for any $h \in G$. Then $\mathbf{Y}_U(\mathcal{V}) = \{(\mathcal{V})\}$ for any U so $\tilde{G}(\mathcal{V}) = \mathcal{V}$. If $\tilde{G}(\mathcal{V}) = \mathcal{V}$ then, from 1., it follows that $\tilde{h}(\mathcal{V}) = \tilde{h}\tilde{G}(\mathcal{V}) = \tilde{G}(\mathcal{V}) = \mathcal{V}$ for any h.
3. Results after 1 and 2.

Let $\mathfrak{Z}(x)$ be the family of the subgroups of $\Theta i(x)$ which satisfies (I. 2.20). For any $G, H \in \mathfrak{Z}(x)$ we say that G is finer than H if the limit-function of G is finer than the limit-function of H, i. e.

$$\text{for any } \mathcal{V}\quad \tilde{G}(\mathcal{V}) \supseteq \tilde{H}(\mathcal{V}). \tag{I. 2.25}$$

For a fixed group $G \in \mathfrak{Z}(x)$ we have the order-relation defined by

$$h \geqslant g \text{ iff } h(V) \subseteq g(V). \tag{I. 2.26}$$

With our hypothesis the relation $\leqslant$ directs G and G conserves $\leqslant$. Therefore we can define the G-derivative in X. We make the notation $D_G f(x, y) = D_G f(y)$ and for the moment we are working only with G-derivative in x.

P I. 2.2 : 1. *For any* $\mathcal{V}$ *there is a* $\mathcal{W}$ *with*

(I. 2.27) $$(\widetilde{D_G f})\,(\tilde{G}(\mathcal{V})) = \tilde{G}(\mathcal{W}) \text{ if } (D_G f) \text{ is continuous}$$

2. *For any* $\mathcal{V} = \mathcal{V}(y)$

(I. 2.28) $$(\widetilde{D_G f})(\tilde{G}(\mathcal{V})) \subseteq \tilde{f}(\tilde{G}\mathcal{V}).$$

Proof. 1. We saw, from P I. 1.1, that for any $h \in G$

$$h(D_G f) = (D_G f)h.$$

The previous equality holds for extensions, so

$$\tilde{h}(\widetilde{D_G f}) = (\widetilde{D_G f})\tilde{h}.$$

Now, let $\mathcal{V}$ be a filter. Then

$$\tilde{h}(\widetilde{D_G f})(\tilde{G}(\mathcal{V})) = (\widetilde{D_G f})\,\tilde{h}(\tilde{G}(\mathcal{V}) = (\widetilde{D_G f})(\tilde{G}(\mathcal{V}))$$

because of P I. 2.1, 1. Therefore, the filter $(\widetilde{D_G f})\,(\tilde{G}(\mathcal{V}))$ is conserved by any $h \in G$; from P I. 2.1 it follows that the filter $\mathcal{W} = (\widetilde{D_G f})(\tilde{G}(\mathcal{V}))$ satisfies (I. 2.27).

2. We know that

(I. 2.29) $$(D_G f)\,(f)\,(y) = \operatorname{Lim}\,\{h^{-1}fh(y) \mid h(V) \subseteq U\}$$

so for any $m \in G$

$$(D_G f)m(y) = m(D_G f)\,(y) = \operatorname{Lim}\{mh^{-1}fh(y) \mid h(V) \subseteq U\}.$$

Because we supposed that $(D_G f)$ is continuous

$$(\widetilde{D_G f})(\tilde{G}(\mathcal{V})(u))) = \operatorname{Lim}\{\operatorname{Lim}\{mh^{-1}fh(y) \mid h(V) \subseteq U\} \mid m(V) \subseteq U\}$$

For the Lim operator the theorem of iterated limits assures us that

$$(\widetilde{D_G f})(\tilde{G}(\mathcal{V}(y))) \subseteq \operatorname{Lim}\{fh(y) \mid h(V) \subseteq U\}$$

and because f is continuous the right member of the previous equality is equal with $\tilde{f}(\tilde{G}(\mathcal{V}(y)))$.

For any subgroup G of G which is directed by (1. 2.26) the previous two propositions holds if we use the G'-derivative and G'. In the second part of this paper we shall prove that any G'-derivative is a G''-derivative, where G''-is a commutative subgroup of G with the following qualitity: any member of G which commutes with any member of G'' is a member of G''. For the group (I. 2.18) G_n plays the part of G'' and the G''-derivative is the Gâteaux derivative. P I. 2.2 shows us what we called the topological substratum of geometrical interpretations of derivative. The filters defined by (I. 2.7) are conserved by G so P I. 2.2, 1. assures us that the family $\mathfrak{s}(x)$ (I. 2.5) is locally conserved by the Gâteaux derivative of any map f with $f(x) = x$. Also 2 shows us that on the same family of filters $(D_G f)$ and f act in the same way; this is, in a few words, the geometrical interpretation of the Gâteaux derivative.

Let $G \in \mathfrak{I}(x)$ and let δ be a boundary map with $\delta(x)$ conserved by G. Then G is a subgroup of $\Theta_i\delta(x)$.

It is obvious that

(I. 2.30) $$\Theta_i\delta(x)/\Theta\delta(x) \subseteq \Theta i(x)/\Theta(x) \rightleftarrows G$$

so

(I. 2.31) $$G \rightleftarrows \Theta_i\delta(x)/\Theta\delta(x)$$

G is isomorphic with the fundamental group of $i(x)$ relative to δ. So δ makes a selection in the family $\mathfrak{s}(x)$. In our particular case the boundary map (I. 2.7) makes a strong selection. Indeed, under some meaningful hypothesis, as we shall prove in the second part of this paper, the group G'' is unique and the G''-derivative is in our case the Gâteaux derivative for any G(I. 2.31).

We shall find the group G_h (I. 1.7). Let, on R^2, the following base of $\mathcal{V}(0)$:

(I. 2.32) $i(0) = \{D(a, b, r) | a, b, r>0\}$ where $D(a, b, r) = \{x | (ax_1)^2 + (bx_2)^2 < r^2\}$

The fundamental group of $i(0)$ is isomorphic with the group of maps

(I. 2.33) $(x_1, x_2) \mapsto (k(x_1)) \|x\|^{-1} x_1, l(x_2) \|x\|^{-1} x_2)$ where k, l are continuously increasing and $k(0_+) = l(0_+) = 0$.

Let

(I. 2.34) $$\delta(0) = \{\mathcal{V}_{ij} | i \cdot j = 1, 2\}$$

and

$$\mathcal{V}_{11} = \{V | \text{there are } k, l > 0 \text{ with } (x_1 + k, x_1) \times (x_2, x_2 + l) \subseteq V\}$$

$$\mathcal{V}_{12} = \{V | \text{there are } k>0,\ l<0 \text{ with } (x_1, x_1 + k) \times (x_2, x_2 + l) \subseteq V\}$$

(I. 2.35)

$$\mathcal{V}_{22} = \{V | \text{there are } k, l < 0 \text{ with } (x_1, x_2 + k) \times (x_2, x_2 + l) \subseteq V\}$$

$$\mathcal{V}_{21} = \{V | \text{there are } k<0,\ l > 0 \text{ with } (x_1, x_1 + k) \times (x_2, x_2 + l) \subseteq V\}$$

The group $\Theta(0)$ is a finite group isomorphic with Z_4 and, as in the previous case, for any group $G \in \mathfrak{J}\ \delta(0)$ the G-derivative is the hyperbolic derivative.

If we want to construct a G-derivative in any point we have to make the same construction everywhere. The problem which arises is the connection between the groups $G(x)$ and $G(v)$ and between $\delta(x)$ and $\delta(y)$ (which means a characterisation of the boundary map). The simplest way is to suppose that there is a map $(x, y) \mapsto f((x, y)$ which is a local homeomorphism (i. e. for any x the map $y \mapsto f(x, y)$ is a local homeomorphism) and for any $x\ f(x, y) = x_0$; also f is G-derivable and $D_G f = f$. In our particular cases such a map exists; it is the affine correspondence $(x, y) \mapsto y - x$.

In the end, the extended notion of λ-derivative arises no problems and P I. 2.2, 2. is still true.

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