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A Priori Inequalities between Energy Release Rate and Energy Concentration for 3D Quasistatic Brittle Fracture Propagation

Marius Buliga
'Simion Stoilow' Institute of Mathematics of the Romanian Academy, Romania

Abstract
We study the properties of absolute minimal and equilibrium states of generalized Mumford–Shah functionals, with applications to models of quasistatic brittle fracture propagation. The main results, theorems 7.3, 8.4 and 9.1, concern *a priori* inequalities between energy release rate and energy concentration for 3D cracks with complex shapes, seen as outer measures living on the crack edge.

Keywords: 3D brittle fracture, energy methods, Mumford–Shah functional

1. Introduction

A new direction of research in brittle fracture mechanics begins with the article of Mumford and Shah[1] regarding the problem of image segmentation. This problem, which consists in finding the set of edges of a picture and constructing a smoothed version of that picture, it turns out to be intimately related to the problem of brittle crack evolution. In the aforementioned article, Mumford and Shah propose the following variational approach to the problem of image segmentation: let $g : \Omega \subset \mathbb{R}^2 \to [0,1]$ be the original picture, given as a distribution of grey levels (1 is white and 0 is black), let $u : \Omega \to R$ be the smoothed picture and let K be the set of edges. The set K represents the set where u has jumps, i.e. $u \in C^1(\Omega \setminus K, R)$. The pair formed by the smoothed picture u and the set of edges K then minimizes the functional:

$$I(u,K) \; = \; \int_\Omega \alpha \, |\nabla u|^2 \, \mathrm{d}x \; + \; \int_\Omega \beta \, |u-g|^2 \, \mathrm{d}x \; + \; \gamma \mathcal{H}^1(K).$$

The parameter α controls the smoothness of the new picture u, β controls the L^2 distance between the smoothed picture and the original one and γ controls the total length of the edges given by this variational method. The authors remark that for $\beta = 0$ the functional I might be useful for an energetic treatment of fracture mechanics.

An energetic approach to fracture mechanics is naturally suited to the explanation of brittle crack appearance under imposed boundary displacements. The idea is presented in what follows.

The state of a brittle body is described by a displacement–crack pair. $(\mathbf{u}, K)$ is such a pair if K is a crack – seen as a surface – which appears in the body and $\mathbf{u}$ is a displacement of the broken body under the imposed boundary displacement, i.e. $\mathbf{u}$ is continuous in the exterior of the surface K and $\mathbf{u}$ equals the imposed displacement $\mathbf{u}_0$ on the exterior boundary of the body.

Let us suppose that the total energy of the body is a Mumford–Shah functional of the form

$$E(\mathbf{u},K) \; = \; \int_\Omega w(\nabla \mathbf{u}) \, \mathrm{d}x \; + \; F(\mathbf{u}_0, K).$$

* **Corresponding author:**
Marius Buliga 'Simion Stoilow' Institute of Mathematics of the Romanian Academy, PO BOX 1-764,014700 Bucharest, Romania
Email: Marius.Buliga@imar.ro

The first term of the functional E represents the elastic energy of the body with the displacement $\mathbf{u}$. The second term represents the energy consumed to produce the crack K in the body, with the boundary displacement $\mathbf{u}_0$ as the parameter. Then the crack that appears is supposed to be the second term of the pair $(\mathbf{u}, K)$ which minimizes the total energy E.

After the rapid establishment of mathematical foundations, starting with De Giorgi and Ambrosio[2], Ambrosio[3,4], the development of such models continued with Francfort and Marigo[5,6], Mielke[7], Dal Maso et al.[8] and Buliga[9,10,11].

In this paper we introduce and study equilibrium and absolute minimal states of Mumford–Shah functionals, in relation to a general model of quasistatic brittle crack propagation.

On the space of the states of a brittle body, which are admissible with respect to an imposed Dirichlet condition, we introduce a partial order relation. Namely the state $(\mathbf{u}, K)$ is 'smaller than' $(\mathbf{v}, L)$ if $L \subset K$ and $E(\mathbf{u}, K) \leq E(\mathbf{v}, L)$. Equilibrium states for the Mumford–Shah energy E are then minimal elements of this partial order relation. Absolute minimal states are just minimizers of the energy E.

Both equilibrium states and absolute minimal states are good candidates for solutions of models for quasistatic brittle crack propagation. Usually such models, based on Mumford–Shah energies, take into consideration only the absolute minimal states. However, it seems that equilibrium states are better, because it is physically sound to define a state of equilibrium $(\mathbf{u}, K)$ of a brittle body as one with the property that its total energy $E(\mathbf{u}, K)$ cannot be lowered by increasing the crack further.

For this reason we study here the properties of equilibrium and absolutely minimal states of general Mumford–Shah energies. This study culminates with an inequality between the energy release rate and the elastic energy concentration, both defined as outer measures living on the edge of the crack. This result generalizes for tri-dimensional cracks with complex geometries what is known about brittle cracks with simple geometry in two dimensions. In the 2D case, for cracks with simple geometry, classical use of complex analysis lead us to an equality between the energy release rate and the elastic energy concentration at the tip of the crack. We prove that for absolute minimal states (corresponding to cracks with complex geometry) such an equality still holds, but for general equilibrium states we only have an inequality. Roughly stated, such a difference in the properties of the equilibrium and absolute minimal states comes from the mathematical fact that the class of first variations around an equilibrium state is only a semigroup.

This research might be relevant for 3D brittle fracture criteria applied for cracks with complex geometries. Indeed, it is very difficult even to formulate 3D fracture criteria, because in three dimensions a crack of arbitrary shape does not have a finite number of 'crack tips' (as in 2D classical theory), but an 'edge' which is a collection of piecewise smooth curves in 3D space.

2. Notation

Partial derivatives of a function f with respect to coordinate x_j are denoted by $f_{,j}$. We use the convention of summation over the repeating indices. The open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$ is denoted by $B(x, r)$.

We assume that the body under study has an open, bounded, with locally Lipschitz boundary, reference configuration $\Omega \subset \mathbb{R}^n$, with $n = 1, 2$ or 3. We use Hausdorff measures $\mathcal{H}^k$ in $\mathbb{R}^n$. For example, if $n = 3$ then $\mathcal{H}^n$ is the volume measure, $\mathcal{H}^{n-1}$ is the area measure and $\mathcal{H}^{n-2}$ is the length measure. If $n = 2$ then $\mathcal{H}^n$ is the area measure, $\mathcal{H}^{n-1}$ is the length measure and $\mathcal{H}^{n-2}$ is the counting measure.

Definition 2.1. *A smooth diffeomorphism with compact support in Ω is a function $\phi : \Omega \to \Omega$ with the following properties:*

(i) ϕ is bijective;
(ii) ϕ and ϕ^{-1} are C^∞ functions;
(iii) ϕ equals the identity map of Ω near the boundary $\partial\Omega$:

$$\text{supp } (id_\Omega - \phi) \subset\subset \Omega.$$

The set of all diffeomorphisms with compact support in Ω is denoted by $\mathcal{D}$ or $\mathcal{D}(\Omega)$.

The set $\mathcal{D}(\Omega)$ is obviously non-void because it contains at least the identity map id_Ω. Note also that it is a group with respect to function composition.

For any C^∞ vector field η on Ω there is a unique associated one-parameter flow, which is a function $\phi : I \times \Omega \to \Omega$, where $I \subset \mathbb{R}$ is an open interval around $0 \in \mathbb{R}$, with the properties:

(f1) $\forall t \in I$, the function $\phi(t,\cdot) = \phi_t(\cdot)$ satisfies (i) and (ii) from Definition 2.1;
(f2) $\forall t, t' \in I$, if $t - t' \in I$ then we have $\phi_{t'} \circ \phi_t^{-1} = \phi_{t-t'}$;
(f3) $\forall\, t \in I$, we have $\eta = \dot{\phi}_t \circ \phi_t^{-1}$, where $\dot{\phi}_t$ means the derivative of $t \mapsto \phi_t$.

The vector field $\eta = 0$ generates the constant flow $\phi_t = id_\Omega$. If η has compact support in Ω then the associated flow $t \mapsto \phi_t$ is a curve in $\mathcal{D}$.

A crack set K is a piecewise Lipschitz surface with a boundary. This means that there exist bi-Lipschitz functions $(f_\alpha)_{\alpha\in 1\ldots M}$, each of them defined over a relatively open subset D_α of $\mathbb{R}^{n-1}_+ = \{y \in \mathbb{R}^{n-1} : y_{n-1} \geq 0\}$, with ranges in $\mathbb{R}^n$, such that:

$$K = \bigcup_{\alpha=1}^{M} f_\alpha(D_\alpha),$$

$$\text{if } \alpha \neq \beta \text{ then } f_\alpha(D_\alpha \setminus \partial\mathbb{R}^{n-1}_+) \cap f_\beta(D_\beta \setminus \partial\mathbb{R}^{n-1}_+) = \emptyset.$$

The edge of the crack K is defined by

$$dK = \bigcup_{\alpha=1}^{M} f_\alpha(D_\alpha \cap \partial\mathbb{R}^{n-1}_+).$$

We denote further by $B_r(dK)$ the tubular neighborhood of radius r of dK, given by the formula

$$B_r(dK) = \bigcup_{x\in dK} B(x,r).$$

We denote by $[f] = f^+ - f^-$ the jump of the function f over the surface K with respect to the field of normals $\mathbf{n}$.

3. Mumford–Shah-Type Energies

Definition 3.1. *We describe the state of a brittle body by a pair* $(\mathbf{v}, S)$*. The crack is seen to be a piecewise Lipschitz surface* S *in the topological closure* $\overline{\Omega}$ *of the reference configuration* Ω *of the body and* $\mathbf{v}$ *represents the displacement of the body from the reference configuration. The displacement* $\mathbf{v}$ *has to be compatible with the crack, i.e.* $\mathbf{v}$ *has the regularity* C^1 *outside the surface* S*.*

The space of states of the brittle body with reference configuration Ω *is denoted by* $Stat(\Omega)$*.*

The main hypothesis in models of brittle crack propagation based on Mumford–Shah-type energies is as follows.

Brittle fracture hypothesis. *The total energy of the body subject to the boundary displacement* $\mathbf{u}_0$ *depends only on the state of the body* $(\mathbf{v}, S)$ *and it has the expression*

$$E(\mathbf{v}, S) = \int_\Omega w(\nabla \mathbf{v})\, \mathrm{d}x + F(S; \mathbf{u}_0). \tag{3.1}$$

The first term of this functional is the elastic energy associated with the displacement $\mathbf{v}$*; the second term represents the energy needed to produce the crack* S*, with the boundary displacement* $\mathbf{u}_0$ *as a parameter.*

We suppose that the elastic energy potential w is a smooth, non-negative function.

The most simple form of the function F is the Griffith-type energy:

$$F(S; \mathbf{u}_0) = \text{Const.} \cdot \text{Area}(S),$$

that is, the energy consumed to create the crack S is proportional, through a material constant, to the area of S.

One may consider expressions of the surface energy F, different from (3.1); for example,

$$F(\mathbf{v}, S) = \int_S \phi(\mathbf{v}^+, \mathbf{v}^-, \mathbf{n})\, \mathrm{d}s,$$

where $\mathbf{n}$ is a field of normals over S, $\mathbf{v}^+$ and $\mathbf{v}^-$ are the lateral limits of $\mathbf{v}$ on S with respect to the directions $\mathbf{n}$ and $-\mathbf{n}$, respectively, and ϕ has the property

$$\phi(\mathbf{v}^+, \mathbf{v}^-, \mathbf{n}) = \phi(\mathbf{v}^-, \mathbf{v}^+, -\mathbf{n}).$$

The function ϕ, depending on the displacement of the 'lips' of the crack, is a potential for surface forces acting on the crack. The expression (3.1) does not lead to such forces.

In general, we shall suppose that the function F has the properties:

(h1) is sub-additive: for any two crack sets A, B we have

$$F(A \cup B; \mathbf{u}_0) \ \leq \ F(A; \mathbf{u}_0) \ + \ F(B; \mathbf{u}_0);$$

(h2) for any $x \in \Omega$ and $r > 0$, let us denote by δ_r^x the dilatation of center x and coefficient r:

$$\delta_r^x(y) = x + r(y - x).$$

Then, there is a constant $C \geq 1$ such that, for any $A \subset \Omega$ with $F(A; \mathbf{u}_0) < +\infty$, we have

$$F(\delta_r^x(A) \cap \Omega; \mathbf{u}_0) \ \leq \ Cr^{n-1} F(A; \mathbf{u}_0).$$

The particular case $F(A; \mathbf{u}_0) = G\mathcal{H}^{n-1}(A)$ satisfies these two assumptions. In general these assumptions are satisfied for functions $F(\cdot; \mathbf{u}_0)$ which are measures absolutely continuous with respect to the area measure $\mathcal{H}^{n-1}$.

A weaker property than (h2) is the property (h3) below. We do not explain here why (h3) is weaker than (h2), but remark that (h3) is satisfied by the same class of examples given for (h2).

For any $A \subset \Omega$, let us denote by $B(A, r)$ the tubular neighborhood of A:

$$B(A, r) \ = \ \bigcup_{x \in A} B(x, r).$$

We shall suppose that F satisfies the following property:

(h3) for any $A \subset \Omega$ such that $F(A; \mathbf{u}_0) < +\infty$, we have

$$\limsup_{r \to 0} \frac{F(\partial B(A, r) \ \cap \ \Omega; \mathbf{u}_0)}{r} \ < \ +\infty.$$

4. The Space of Admissible States of a Brittle Body

Definition 4.1. *The class of admissible states of a brittle body with respect to the crack F and with respect to the imposed displacement* $\mathbf{u}_0$ *is defined as the collection of all states* $(\mathbf{v}, S)$ *such that:*

(a) $\mathbf{u} = \mathbf{u}_0$ on $\partial\Omega \setminus S$;
(b) $F \subset S_u$.

This class of admissible states is denoted by $Adm(F, \mathbf{u}_0)$.

An admissible displacement $\mathbf{u}$ is a function which has to be equal to the imposed displacement on the boundary of Ω (condition (a)). Any such function $\mathbf{u}$ is reasonably smooth in the set $\Omega \backslash S_u$ and the function $\mathbf{u}$ is allowed to have jumps along the set S. Physically the set represents the collection of all cracks in the body under the displacement $\mathbf{u}$. The condition (b) tells us that the collection of all cracks associated with an admissible displacement $\mathbf{u}$ contains F, at least.

For some states $(\mathbf{u}, S)$, the crack set S may have parts lying on the boundary of Ω, that is, $S \cap \partial\Omega$ is a surface with positive area. In such cases we think about $S \cap \partial\Omega$ as a region where the body has been detached from the machine which imposed upon the body the displacement $\mathbf{u}_0$.

In a weak sense the whole space of states of a brittle body may be identified with the space of special functions with bounded deformation $\mathbf{SBD}(\Omega)$; see[12]. Indeed, to every displacement field $\mathbf{u}$, which is a special function with bounded deformation, we associate the state of the brittle body described by $(\mathbf{u}, \overline{\mathbf{S}}_u)$, where generally for any set A we denote by $\overline{A}$ the topological closure of A. (Note that, technically, the crack set $\overline{\mathbf{S}}_u$ may not be a collection of surfaces with Lipschitz regularity.)

On the space of states of a brittle body we introduce a partial order relation. The definition is connected to Definition 4.1 and the brittle fracture hypothesis.

Definition 4.2. *Let* $(\mathbf{u}, S), (\mathbf{v}, L) \in Stat(\Omega)$ *be two states of a brittle body with reference configuration* Ω*. If*

(a) $S \subset L$,
(b) $\mathbf{u} = \mathbf{v}$ *on* $\partial\Omega \setminus L$,
(c) $E(\mathbf{v}, L) \leq E(\mathbf{u}, S)$,

then we write $(\mathbf{v}, L) \leq (\mathbf{u}, S)$*. This is a partial order relation.*

There are many pairs $(\mathbf{u}, S), (\mathbf{v}, L) \in Stat(\Omega)$ such that $(\mathbf{v}, L) \leq (\mathbf{u}, S)$ and $(\mathbf{u}, S) \leq (\mathbf{v}, L)$, but $\mathbf{u} \neq \mathbf{v}$. Nevertheless, such pairs have the same total energy E, the same crack set $S = L$ and $\mathbf{u} = \mathbf{v}$ on $\partial\Omega \setminus L$.

For a given boundary displacement $\mathbf{u}_0$ and for a given initial crack set K, on the set of admissible states $Adm(\mathbf{u}_0, K)$ we have the same partial order relation.

Definition 4.3. *An element $(\mathbf{u}, S) \in Adm(\mathbf{u}_0, K)$ is minimal with respect to the partial order relation $\leq$ if, for any $(\mathbf{v}, L) \in Adm(\mathbf{u}_0, K)$, the relation $(\mathbf{v}, L) \leq (\mathbf{u}, S)$ implies $(eu, S) \leq (\mathbf{v}, L)$.*

The set of equilibrium states with respect to a given crack K and imposed boundary displacement $\mathbf{u}_0$ is denoted by $Eq(\mathbf{u}_0, K)$ and it consists of all minimal elements of $Adm(\mathbf{u}_0, K)$ with respect to the partial order relation $\leq$.

An element $(\mathbf{u}, S) \in Adm(\mathbf{u}_0, K)$ with the property that for any $(\mathbf{v}, L) \in Adm(\mathbf{u}_0, K)$ we have $E(\mathbf{u}, S) \leq E(\mathbf{v}, L)$ is called an absolute minimal state. The set of absolute minimal states is denoted by $Absmin(\mathbf{u}_0, K)$.

The physical interpretation of equilibrium states can be made as follows. An equilibrium state $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ is one such that any other state $(\mathbf{v}, L) \in Adm(\mathbf{u}_0, K)$, which is comparable to $(\mathbf{u}, S)$ with respect to the relation $\leq$, has the property $(\mathbf{u}, S) \leq (\mathbf{v}, L)$. In other words, equilibrium states are those with the following property: the total energy E cannot be made smaller by enlarging the crack set S or by modifying the displacement $\mathbf{u}$ compatible with the crack set S and imposed boundary displacement $\mathbf{u}_0$.

Absolute minimal states are just equilibrium states with minimal energy.

Remark 4.4. *There might exist several minimal elements of $Adm(\mathbf{u}_0, K)$, such that any two of them are not comparable with respect to the partial order relation $\leq$.*

For given expressions of the functions w and F, we formulate the following hypothesis.

Equilibrium hypothesis (EH). *For any piecewise C^1 imposed boundary displacement $\mathbf{u}_0$ and any crack K the set of equilibrium states $Eq(\mathbf{u}_0, K)$ is not empty.*

Without supplementary hypothesis on the total energy E, the EH does not imply that the set of absolute minimal states $Absmin(\mathbf{u}_0, K)$ is non-empty. Therefore, the following hypothesis is stronger than EH.

Strong equilibrium hypothesis (SEH). *For any piecewise C^1 imposed boundary displacement $\mathbf{u}_0$ and any crack K the set of equilibrium states $Absmin(\mathbf{u}_0, K)$ is not empty.*

5. Models of Quasistatic Evolution of Brittle Cracks

We shall describe here two models of quasistatic brittle crack propagation, Francfort and Marigo[5,6] and Mielke [[7], Section 7.6], the other proposed by Buliga[11,10]. At first sight the models seem to be identical, but subtle differences exist. Further, instead of referring to a particular different model, we shall write about a general model of brittle crack propagation based on energy functionals as if there is only one general model, with different variants, according to the choice among axioms listed further namely axioms (A1) – (A5) Def5.2 for the model of Buliga, respectively axioms (A1), (A2), (A3′), (A4) for the model of Francfort and Marigo. Whenever necessary, the exposition will contain variants of statements or assumptions which specialize the general model to one of the actual models in use.

As an input of the model we have an initial crack set $K \subset \overline{\Omega}$ and a curve of imposed displacements $t \in [0, T] \mapsto \mathbf{u}_0(t)$ on the boundary of Ω, the initial configuration of the body.

We like to think about the configuration Ω as being an open, bounded subset of $\mathbb{R}^n$, $n = 1, 2, 3$, with sufficiently regular boundary (that is, piecewise Lipschitz boundary).

The initial crack set K has the status of an initial condition. Thus, we suppose that $\partial (\mathbb{R}^n \setminus \Omega) = \partial\Omega$. For the same configuration Ω we may consider any crack set $K \subset \overline{\Omega}$ as an initial crack. The crack set K may be empty.

Remark 5.1. *Models suitable for the evolution of brittle cracks under applied forces would be of great interest. The present formulations of the models of brittle crack propagation allow only the introduction of conservative force fields, as is done in [7] or [6]. The reason for this is that models based on energy minimization cannot deal with arbitrary force fields. In the case of a conservative force field, it is enough to introduce the potential of the force field inside the expression of the total energy of the fractured body. Thus, in this particular case we do not have to change substantially the formulation of the model presented here, but only to slightly modify the expression of the energy functional.*

In order to simplify the model presented here, we suppose that no conservative force fields are imposed on Ω or parts of $\partial\Omega$. In the models described in [7] or [6] such forces may be imposed.

Definition 5.2. *A solution of the model is a curve of states of the brittle body* $t \in [0, T] \mapsto (\mathbf{u}(t), S_t)$ *such that:*

(A1) *(initial condition)* $K \subset S_0$;
(A2) *(boundary condition) for any* $t \in [0, T]$ *we have* $\mathbf{u}(t) = \mathbf{u}_0(t)$ *on* $\partial\Omega \setminus S_t$;
(A3) *(quasistatic evolution) for any* $t \in [0, T]$ *we have* $(\mathbf{u}(t), S_t) \in Eq(\mathbf{u}_0(t), S_t)$;
(A4) *(irreversible fracture process) for any* $t \leq t'$ *we have* $S_t \subset S_{t'}$;
(A5) *(selection principle) for any* $t \leq t'$ *and for any state* $(\mathbf{v}, S_t) \in Adm(\mathbf{u}_0(t'), S_t)$ *we have* $E(\mathbf{v}, S_t) \geq E(\mathbf{u}(t'), S_{t'})$.

From Definition 4.3 we see that (A2) is just a part of (A3). The axiom (A2) is present in the previous definition only for expository reasons.

The selection principle (A5) enforces the irreversible fracture process axiom (A4). Indeed, we may have severe non-uniqueness of solutions of the model. The axiom (A5) selects among all solutions satisfying (A1), ..., (A4) the ones that are energetically economical. The crack set S_t does not grow too fast, according to (A5). For an imposed displacement $\mathbf{u}_0(t')$, the body with the crack set $S_{t'}$ is softer than the same body with the crack set S_t, for any $t \leq t'$.

As presented in Definition 5.2, the model has been proposed in Buliga[11]. In the models described in [7] and [5, 6] we do not need the selection principle (A5) and the axiom (A3) takes the stronger form:

(A3') *(quasistatic evolution) for any* $t \in [0, T]$ *we have* $(\mathbf{u}(t), S_t) \in Absmin(\mathbf{u}_0(t), S_t)$ *A supplementary energy balance condition in imposed in [5, 6]. As explained by Mielke [[7], Section 7.6], the time incremental formulation of the model of Francfort and Marigo can nevertheless be reduced to the axioms (A1), (A2), (A3'), (A4).*

6. The Existence Problem

The existence of equilibrium or absolutely minimal states clearly depends on the ellipticity properties of the elastic energy potential w (as shown, for example, in [4, 12] or [5]). This is related to the existence of minimizers of the elastic energy functional, as shown by relation (7.1) further on. Some form of ellipticity of the function w is sufficient, but it is not clear if such conditions are also necessary. Much effort, especially of a mathematical nature, has been spent on this problem.

In this paper we are not concerned with the existence problem, however. Our purpose is to find general properties of solutions of brittle fracture propagation models based on Mumford–Shah functionals. These properties do not depend on particular forms of the elastic energy potential w, but on the hypothesis made in the general model. As any other model, the one studied in this paper is better fitted to some physical situations than others. If some property of solutions of this model are incompatible with a particular physical case, then we must deduce that the model is not fitted for this particular case (meaning that at least one of the hypotheses of the model is not suitable to this physical case). We are thus able to provide complementary information to that provided by the existence problem. See the conclusions section for more on this subject.

7. Absolute Minimal States Versus Equilibrium States

The differences between the models come from the differences between equilibrium states and absolute minimal states.

Absolute minimal states are equilibrium states, but not any equilibrium state is an absolute minimal state.

Let us denote by $(\mathbf{u}, S)$ an equilibrium state of the body with respect to the imposed displacement $\mathbf{u}_0$ and initial crack set K.

Consider first the class of all admissible pairs $(\mathbf{v}, S')$ with a fixed crack set S such that $S = S$. We have, as an application of Definition 4.3, that

$$\int_\Omega w(\nabla \mathbf{u})\, \mathrm{d}x \;\leq\; \int_\Omega w(\nabla \mathbf{v})\, \mathrm{d}x \quad \forall\, \mathbf{v},\ \mathbf{v} \;=\; \mathbf{u}_0 \text{ on } \partial\Omega \setminus \mathrm{s},\ \mathbf{v} \in C^1(\Omega \setminus \mathrm{s}). \tag{7.1}$$

Thus any equilibrium state minimizes the elastic energy functional (in the class of admissible pairs with the same associated crack set). A sufficient condition for the existence of such minimizers is the polyconvexity of the elastic energy potential w.

The elastic energy potential function $w : M^{n\times n}(\mathbb{R}) \to \mathbb{R}$ associates to any strain $\mathbf{F} \in M^{n\times n}(\mathbb{R})$ (here $n = 2$ or 3) the real value $w(\mathbf{F}) \in \mathbb{R}$. If this function is smooth enough then we can define the (Cauchy) stress tensor as coming from the elastic energy potential:

$$\sigma(\mathbf{u}) \;=\; \frac{\partial w(\mathbf{F})}{\partial \mathbf{F}}(\nabla \mathbf{u}).$$

The variational inequality (7.1) implies that in the sense of distributions we have

$$\operatorname{div} \sigma(\mathbf{u}) \;=\; 0$$

and that on the crack set S we have

$$\sigma(\mathbf{u})^{+}\mathbf{n} \;=\; \sigma(\mathbf{u})^{-}\mathbf{n} \;=0,$$

where the signs $+$ and $-$ denote the lateral limits of $\sigma(\mathbf{u})$ with respect to the field of normals $\mathbf{n}$.

7.1. Configurational Relations for Absolute Minimal States

We can also make smooth variations of the pair $(\mathbf{u}, S)$. Here appears the first difference between the absolute minimal and equilibrium states. We suppose further that $S \setminus K \neq \emptyset$; in fact, we suppose that $S \setminus K$ is a surface with positive area.

If $(\mathbf{v}, L) \in Adm(\mathbf{u}_0, K)$ is an admissible state and $\phi \in \mathcal{D}$ is a diffeomorphism of Ω with compact support, such that $K \subset \phi(K)$, then $(\mathbf{v} \circ \phi^{-1}, \phi(S))$ is admissible too.

If $(\mathbf{u}, S)$ is an absolute minimal state then, as an application of Definition 4.3, we have

$$E(\mathbf{u}, S) \;\leq\; E(\mathbf{u} \circ \phi^{-1}, \phi(S)) \quad \forall \phi \in \mathcal{D}\,,\, K \subset \phi(K). \tag{7.2}$$

We may use (7.2) in order to derive a first variation equality.

We shall restrict further to the group $\mathcal{D}(K)$ of diffeomorphisms $\phi \in \mathcal{D}$ such that $\operatorname{supp}\,(\phi - id) \cap K = \emptyset$. Vector fields η which generate one-parameter flows in $\mathcal{D}(K)$ are those with the property $\operatorname{supp} \eta \cap K = \emptyset$. Further we shall work only with such vector fields.

We shall admit that for any smooth vector field η there exist the derivatives at $t = 0$ of the functions

$$t \;\mapsto\; \int_{\Omega} w(\nabla(\mathbf{u} \circ \phi_t^{-1}))\, \mathrm{d}x, \quad t \;\mapsto\; F(\phi_t(K); \mathbf{u}_0),$$

where ϕ_t is the one-parameter flow generated by the vector field η. The relation (7.2) implies then that

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} F(\phi_t(S); \mathbf{u}_0) \;=\; - \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \int_{\Omega} w(\nabla(\mathbf{u} \circ \phi_t^{-1}))\, \mathrm{d}x. \tag{7.3}$$

Let us compute the right-hand side of (7.3). We have

$$- \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \int_{\Omega} w(\nabla(\mathbf{u} \circ \phi_t^{-1}))\, \mathrm{d}x \;=\; \int_{\Omega} \left\{ -w(\nabla \mathbf{u}) \operatorname{div} \eta \;+\; \sigma(\mathbf{u})_{ij} (\nabla \mathbf{u})_{ik} (\nabla \eta)_{kj} \right\}\, \mathrm{d}x.$$

For any vector field η, let us define, for any $x \in S$, $\lambda(x) = \eta(x) \cdot \mathbf{n}(x)$, $\eta^{T}(x) = \eta(x) - \lambda)(x)\mathbf{n}(x)$, where $\mathbf{n}$ is a fixed field of normals over S.

With this notation, and recalling that the divergence of the stress field equals 0, we have

$$- \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \int_{\Omega} w(\nabla(\mathbf{u} \circ \phi_t^{-1}))\, \mathrm{d}x \;=\; \int_{S} [w(\nabla \mathbf{u})] \lambda \,\mathrm{d}\, \mathcal{H}^{n-1}$$

$$+ \lim_{r \to 0} \int_{\partial B_r(\mathrm{d}S)} \left\{ [w(\nabla \mathbf{u})] \lambda \;-\; [\sigma(\mathbf{u})_{ij} (\nabla \mathbf{u})_{ik}] \eta_k \mathbf{n}_j \right\}\, \mathrm{d}\, \mathcal{H}^{n-1}. \tag{7.4}$$

Definition 7.1. *We introduce three kinds of variations in terms of a vector field η which generates a one-parameter flow $\phi_t \in \mathcal{D}(K)$:*

(a) *(crack neutral variations) for $\eta \;=\; 0$ on S, we have $\phi_t(S) \;=\; S$ for any t;*
(b) *(crack normal variations) for $\eta \;=\lambda \mathbf{n}$ on $S \setminus K$, with $\lambda : S \to \mathbb{R}$ a scalar, smooth function, such that $\lambda(x) = 0$ for any $x \in K \cup \mathrm{d}S$;*
(c) *(crack tangential variations) for $\eta \cdot \mathbf{n} = 0$ on S.*

For case (a) of crack neutral variations, the relation (7.4) gives no new information when compared with (7.1).

In case (b) of crack normal variations, the relation (7.4) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} F(\phi_t(K); \mathbf{u}_0) \;=\; \int_{S} [w(\nabla \mathbf{u})] \lambda \,\mathrm{d}\, \mathcal{H}^{n-1}.$$

In the particular case $F(S; \mathbf{u}_0) = \mathcal{H}^{n-1}(S)$ we obtain

$$\int_S \{[w(\nabla \mathbf{u})] + H\} \lambda \, d\mathcal{H}^{n-1} = 0,$$

where $H = -\mathrm{div}_s \mathbf{n} = -\operatorname{div} \mathbf{n} + \mathbf{n}_{i,j}\mathbf{n}_i\mathbf{n}_j$ is the mean curvature of the surface S. Therefore, we have

$$[w(\nabla \mathbf{u})(x)] + H(x) = 0 \tag{7.5}$$

for any $x \in S \setminus K$.

In case (c) of crack tangential variations, the relation (7.4) implies that

$$\frac{d}{dt}_{|t=0} F(\phi_t(S); \mathbf{u}_0) =$$

$$= \lim_{r \to 0} \int_{\partial B_r(dS)} \left\{ [w(\nabla \mathbf{u})]\lambda - [\sigma(\mathbf{u})_{ij}(\nabla \mathbf{u})_{ik}]\eta_k \mathbf{n}_j \right\} d\mathcal{H}^{n-1}. \tag{7.6}$$

This last relation admits a well-known interpretation, briefly explained in the next subsection.

7.2. Absolute Minimal States for $n = 2$

Let us consider the case $n = 2$ and the function

$$F(S; \mathbf{u}_0) = G\,\mathcal{H}^1(S),$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure, i.e. the length measure. Let us suppose, for simplicity, that the initial crack set K is empty and the crack set S of the absolute minimal state $(\mathbf{u}, S)$ has only one edge, i.e. $dS = \{x_0\}$. Let us choose a vector field η with compact support in Ω such that η is tangent to S. The equality (7.6) then becomes

$$G\,\eta(x_0) \cdot \tau(x_0) = \lim_{r \to 0} \int_{\partial B_r(x_0)} \left\{ [w(\nabla \mathbf{u})]\eta \cdot \mathbf{n} - [\sigma(\mathbf{u})_{ij}(\nabla \mathbf{u})_{ik}]\eta_k \mathbf{n}_j \right\} d\mathcal{H}^{n-1},$$

where $\tau(x)$ is the unitary tangent in $x \in K$ at K. If we suppose, moreover, that the crack S is straight near x_0, and the material coordinates are chosen such that near x_0 we have $\eta(x) = \tau(x) = (1, 0)$, then the equality (7.6) takes the form

$$G = \lim_{r \to 0} \int_{\partial B_r(x_0)} \left\{ [w(\nabla \mathbf{u})]\mathbf{n}_1 - [\sigma(\mathbf{u})_{ij}(\nabla \mathbf{u})_{i1}]\mathbf{n}_j \right\} d\mathcal{H}^{n-1}. \tag{7.7}$$

We recognize in the right-hand term of (7.7) the integral J of Rice; therefore, at the edge of the crack the integral J has to be equal to the constant G, interpreted as the constant of Griffith.

The equality (7.7) tells us that at the edge of a crack set belonging to an absolute minimal state the Griffith criterion is fulfilled with equality.

7.3. Configurational Inequalities

For equilibrium states that are not absolute minimal states we obtain just an inequality, instead of the equality from relation (7.6). Also, for such equilibrium states there is no relation like (7.5) between the mean curvature of the crack set and the jump of the elastic energy potential. We explain this further.

The reason for this lies in the fact that if $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ is an equilibrium state with $S \setminus K$ having positive area, and $\phi \in \mathcal{D}(K)$ is a diffeomorphism preserving the initial crack set K, then we do not generally have the relation (7.2).

Indeed, in order to be able to compare $(\mathbf{u}, S)$ with $(\mathbf{u} \circ \phi^{-1}, \phi(S))$, we have to impose $S \subset \phi(S)$. Only for these diffeomorphisms $\phi \in \mathcal{D}(K)$ is the relation (7.2) true. The class of these diffeomorphisms is not a group, like $\mathcal{D}(K)$, but only a semigroup. Technically, this is the reason for having only an inequality replacing (7.6), and for the disappearance of relation (7.5).

There is a necessary condition on the edge dS of the crack set S in order to have a trivial vector field η which generates a one-parameter flow $\phi_t \in \mathcal{D}(K)$ with $S \subset \phi_t(S)$ for any $t \in [0, T]$ (with $T > 0$ sufficiently small). This condition is $dS \setminus K \neq \emptyset$.

Thus, for $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ with $S \setminus K$ with positive area, and $\mathrm{d}S \setminus K \neq \emptyset$, we have

$$E(\mathbf{u}, S) \ \leq \ E(\mathbf{u} \circ \phi_t^{-1}, \phi_t(S)) \quad \forall t \in [0, T], \tag{7.8}$$

for any one-parameter flow $\phi_t \in \mathcal{D}(K)$ with $S \subset \phi_t(S)$ for any $t \in [0, T]$.

In relation (7.8), crack normal variations (case (b) of Definition 7.1) are prohibited. However, these types of variations lead us to the relation (7.5). We deduce that for an equilibrium state $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$, such that $S \setminus K$ has positive area, and $\mathrm{d}S \setminus K \neq \emptyset$, the relation (7.5) does not necessarily hold.

The crack tangential variations (case (c) of Definition 7.1) are allowed in relation (7.8) only for $t \geq 0$. That is why we get only a first variation inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} F(\phi_t(S); \mathbf{u}_0)$$

$$\geq \ \lim_{r \to 0} \int_{\partial B_r(dK)} \left\{ [w(\nabla \mathbf{u})]\lambda \ - \ [\sigma(\mathbf{u})_{ij} (\nabla \mathbf{u})_{ik}] \eta_k \mathbf{n}_j \right\} \ \mathrm{d}\mathcal{H}^{n-1}, \tag{7.9}$$

for any vector field η which generates one-parameter flow $\phi_t \in \mathcal{D}(K)$ with $S \subset \phi_t(S)$ for any $t \in [0, T]$.

The physical interpretation of relation (7.9) is as follows: the crack set S of an equilibrium state satisfies the Griffith criterion of fracture, but, distinct from the case of an absolute minimal state, there is an inequality instead of the previous equality. We are aware of at least one example where this inequality is strict. This case concerns a crack set in three dimensions formed by a pair of intersecting, transversal planar cracks. Such a crack set has an edge (in the form of a cross), but also a ‘tip’ (at the intersection of the edges of the planar cracks). The physical implications of the inequality (7.9) are that such a 3D crack may propagate in different ways, either along a crack tangential variation, or along a more topologically complex shape, by loosing its ‘tip’.

We may interpret the Griffith criterion of fracture, in the form given by relation (7.9), as a first-order stability condition for the crack S associated to the state of a brittle body. Surprisingly then, absolute minimal states are first-order neutral (stable and unstable), even if globally stable (as global minima of the total energy). There might exist equilibrium states for which we have a strict inequality in relation (7.9). Such states are surely not absolute minimal, but they seem to be first-order stable, if our interpretation of (7.9) is physically sound.

7.4. Concentration of Energy from Comparison with Admissible States

We can obtain energy concentration estimates from comparison of the energy of the equilibrium state $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ with other particular admissible pairs.

Let $x_0 \in \Omega$ be a fixed point and $r > 0$ such that $B(x_0, r) \subset \Omega$. We construct the following admissible pair $(\mathbf{v}_r, S_r)$:

$$\mathbf{v}_r(x) \ = \ \begin{cases} \mathbf{u}(x) & \text{if } x \in \Omega \setminus B(x_0, r), \\ 0 & \text{if } x \in \Omega \cap B(x_0, r), \end{cases}$$

$$S_r \ = \ S \cup \partial B(x_0, r).$$

We then have the inequality $E(\mathbf{u}, S) \ \leq \ E(\mathbf{v}_r, S_r)$, for any $r > 0$ sufficiently small. We use the properties (h1) and (h2) of F to deduce that, for any $x_0 \in \Omega$ and $r > 0$, we have

$$\int_{B(x_0, r)} w(\nabla \mathbf{u}) \, \mathrm{d}x \ \leq \ C\Omega_n(x; \mathbf{u}_0) \, r^{n-1}, \tag{7.10}$$

where $\Omega_n(x_0; \mathbf{u}_0)$ is a number defined by

$$\Omega_n(x_0; \mathbf{u}_0) = F(\partial B(x_0, 1); \mathbf{u}_0).$$

In the case of a Griffith-type surface energy $F(S; \mathbf{u}_0) = G\mathcal{H}^{n-1}(S)$ we have

$$\Omega_n(x_0; \mathbf{u}_0) = G\omega_n,$$

with ω_n the area of the boundary of the unit ball in n dimensions, that is, $\omega_1 = 2$, $\omega_2 = 2\pi$, $\omega_3 = 4\pi^2$.

This inequality leads us to the following energy concentration property for $\mathbf{u}$:

$$\limsup_{r \to 0} \frac{\int_{B(x_0, r)} w(\nabla \mathbf{u}) \, \mathrm{d}x}{r^{n-1}} \ \leq \ C\Omega_n(x_0; \mathbf{u}_0). \tag{7.11}$$

The term from the left-hand side of the relation (7.11) is the concentration factor of the elastic energy around the point x_0.

The relation (7.11) shows that the distribution of elastic energy of the body in the state $(\mathbf{u}, S)$ is what we expect it to be, from the physical viewpoint. Indeed, let us go back to the case $n = 2$. It is well known that in the case of linear elasticity in two dimensions, if $(\mathbf{v}, S)$ is a displacement–crack pair such that div $\sigma(\mathbf{v}) = 0$ outside S and $\sigma(\mathbf{v})^+\mathbf{n} = \sigma(\mathbf{v})^-\mathbf{n} = 0$ on S, then $\mathbf{v}$ behaves like $\sqrt{r}$ near the edge of the crack; hence the elastic energy behaves like r^{-1}. We then recover the relation (7.11) for $n = 2$.

The relation (7.11) does imply that the elastic energy concentration has an upper bound, but it does not imply that the energy concentration is positive at the tip of the crack. In the case $n = 2$, for example, and for the general form of the elastic energy density, the relation (7.11) tells us that if there is a concentration of energy (that is, if the density of elastic energy goes to infinity around the point x in the reference configuration) then the elastic energy density behaves like r^{-1}. But it might happen that the elastic energy density is nowhere infinite. In this case we simply have

$$\limsup_{r\to 0} \frac{\int_{B(x_0,r)} w(\nabla \mathbf{u})\, \mathrm{d}x}{r^{n-1}} = 0,$$

which is not in contradiction with (7.11).

From the hypothesis (h3) on the surface energy F we get a slightly different estimate. We need first a definition.

Definition 7.2. *For the equilibrium state $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ and for any open set $A \subset \Omega$ we define*

$$CE(\mathbf{u}, S)(A) = \limsup_{r\to 0} \frac{\int_{B((\mathrm{d}S\cap A,r)\cap\Omega} w(\nabla \mathbf{u})\, \mathrm{d}x}{r} \ ,$$

$$CF(S; \mathbf{u}_0)(A) = \limsup_{r\to 0} \frac{F(\partial B(\mathrm{d}S\cap A, r); \mathbf{u}_0)}{r}.$$

The functions $CE(\mathbf{u}, S)(\cdot)$, $CF(S; \mathbf{u}_0)(\cdot)$ are sub-additive functions which by well-known techniques induce outer measures over the σ-algebra of Borelian sets in Ω.

The function $CE(\mathbf{u}, S)(\cdot)$ is called the elastic energy concentration measure associated with the equilibrium state $(\mathbf{u}, S)$. Likewise, the function $CF(S; \mathbf{u}_0)(\cdot)$ is called the surface energy concentration measure associated with $(\mathbf{u}, S)$.

Theorem 7.3. *Let $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ be an equilibrium state. Then for any open set $A \subset \Omega$ we have*

$$CE(\mathbf{u}, S)(A) \le CF(S; \mathbf{u}_0)(A).$$

Proof. We consider, for any closed subset A of Ω, the admissible state $(\mathbf{u}_{r,A}, S_{r,A})$ given by

$$\mathbf{u}_{r,A}(x) = \begin{cases} \mathbf{u}(x) & \text{if } x \in \Omega \setminus B(\mathrm{d}S\cap A, r), \\ 0 & \text{if } x \in \Omega \cap B(\mathrm{d}S\cap A, r), \end{cases}$$

$$S_{r,A} = S \cup \partial B(\mathrm{d}S\cap A, r).$$

The state $(\mathbf{u}, S)$ is an equilibrium state and $(\mathbf{u}_{r,A}, S_{r,A})$ is a comparable state; therefore we obtain

$$\int_{B(\mathrm{d}S\cap A,r)\cap\Omega} w(\nabla \mathbf{u})\, \mathrm{d}x \le F(\partial B(\mathrm{d}S\cap A, r); \mathbf{u}_0).$$

We get eventually

$$\limsup_{r\to 0} \frac{\int_{B(\mathrm{d}S\cap A,r)\cap\Omega} w(\nabla \mathbf{u})\, \mathrm{d}x}{r} \le \limsup_{r\to 0} \frac{F(\partial B(\mathrm{d}S\cap A, r); \mathbf{u}_0)}{r}. \qquad \square$$

Theorem 7.3 shows that an equilibrium state satisfies a kind of Irwin-type criterion. Indeed, an Irwin criterion is formulated in terms of stress intensity factors. Closer inspection reveals that really it is formulated in terms of the elastic energy concentration factor and that, for special geometries of the crack set and for linear elastic materials, we are able to compute the energy concentration factor as a combination of stress intensity factors.

8. Energy Release Rate and Energy Concentration

From relations (7.3) and (7.6), we deduce that a good generalization of the J integral of Rice (which is classically a number) might be a functional

$$\eta\ ,\ \mathrm{supp}\ \eta \subset\subset \Omega\ \mapsto\ -\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \int_{\Omega} w(\nabla(\mathbf{u}.\phi_t^{-1}))\ \mathrm{d}x,$$

where ϕ_t is the flow generated by η.

Definition 8.1. *For any equilibrium state $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ and for any vector field η which generates a one-parameter flow $\phi_t \in \mathcal{D}(K)$, such that (there is a $T > 0$ with) $S \subset \phi_t(S)$ for all $t \in [0, T]$, we define the energy release rate along the vector field η by*

$$ER(\mathbf{u}, S)(\eta)\ =\ -\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \int_{\Omega} w(\nabla(\mathbf{u} \circ \phi_t^{-1}))\ \mathrm{d}x. \tag{8.1}$$

Denote by $\mathcal{V}(K, S)$ the family of all vector fields η generating a one-parameter flow $\phi_t \in \mathcal{D}(K)$, such that there is a $T > 0$ with $S \subset \phi_t(S)$ for all $t \in [0, T]$. Formally this set plays the role of the tangent space at the identity for the (infinite-dimensional) semigroup of all $\phi \in \mathcal{D}(K)$ such that $S \subset \phi(S)$.

Note that $ER(\mathbf{u}, S)(\eta)$ is a linear expression in the variable η. Indeed, we have

$$ER(\mathbf{u}, S)(\eta)\ =\ \int_{\Omega} \left\{\sigma(\nabla\mathbf{u})_{ij}\mathbf{u}_{i,k}\eta_{k,j}\ -\ w(\nabla\mathbf{u})\ \mathrm{div}\ \eta\right\}\ \mathrm{d}x.$$

Nevertheless, the set $\mathcal{V}(K, S)$ is not a vector space (mainly because the class of all $\phi \in \mathcal{D}(K)$ such that $S \subset \phi(S)$ is only a semigroup and not a group). Therefore, the energy release rate is not a linear functional in a classical sense.

Definition 8.2. *With the notation from Definition 8.1, the total variation of the energy release rate in a open set $D \subset \Omega$ is defined by*

$$|ER|(\mathbf{u}, S)(D)\ =\ \sup\ ER(\mathbf{u}, S)(\eta), \tag{8.2}$$

over all vector fields $\eta \in \mathcal{V}(K, S)$, with support in D, $\mathrm{supp}\ \eta \subset D$, such that, for all $x \in \Omega$, we have $\|\eta(x)\| \leq 1$.

The function $|ER|(\mathbf{u}, S)(\cdot)$ is positive and sub-additive and therefore induces an outer measure over the σ-algebra of Borelian sets in Ω.

We call this function the energy release rate associated with $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$.

The number $|ER(\mathbf{u}, S)|(D)$ measures the maximal elastic energy release rate that can be obtained by propagating the crack set S inside the set D, with sub-unitary speed, *by preserving its shape topologically*.

In the case $n = 2$, as explained in Section 7.2, let x_0 be the crack tip of the crack set S and J the Rice integral. Then for an open set $D \subset \Omega$ we have:

- $|ER(\mathbf{u}, S)|(D) = J$ if the crack tip belongs to D, that is, $x_0 \in D$;
- $|ER(\mathbf{u}, S)|(D) = 0$ if the crack tip does not belong to D.

For short, if we denote by δx_0 the Dirac measure centered at the crack tip x_0, we can write

$$|ER(\mathbf{u}, S)| = J\ \delta x_0.$$

It is therefore the appropriate generalization of the Rice integral in three dimensions.

Suppose that for any crack set L and boundary displacement $\mathbf{u}_0$ the surface energy has the expression

$$F(S; \mathbf{u}_0)\ =\ G\mathcal{H}^{n-1}(S).$$

Then $CF(S, \mathbf{u}_0)(\Omega)$ is just G times the perimeter (length if $n = 3$) of the edge of the crack S which is not contained in K (technically, it is the Hausdorff measure $\mathcal{H}^{n-2}$ of $\mathrm{d}S \setminus K$).

There is a mathematical formula which expresses the perimeter of the edge of an arbitrary crack set L as an 'area release rate'. Indeed, it is well known that the variation of the area of the crack set $\phi_t(L)$, along a one-parameter flow generated by the vector field $\eta \in \mathcal{V}(K, L)$, has the expression

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathcal{H}^{n-1}(\phi_t(S))\ =\ \int_{S} \mathrm{div}_{\tan}\eta\ \mathrm{d}\mathcal{H}^{n-1}(x),$$

where the operator $\mathrm{div}_{\tan}$ is the tangential divergence with respect to the surface S. If we denote by $\mathbf{n}$ the field of normals to the crack set S, then the expression of $\mathrm{div}_{\tan}$ operator is

$$\mathrm{div}_{\tan}\eta \;=\; \eta_{i,i} \;-\; \eta_{i,j}\mathbf{n}_i\mathbf{n}_j.$$

Further, the perimeter of $\mathrm{d}S \setminus K$, the edge of the crack set S outside K, admits the following description, similar in principle to the expression of the elastic energy release rate given in Definition 8.2:

$$\mathcal{H}^{n-2}(\mathrm{d}S \setminus K) \;=\; \sup\left\{ \int_S \mathrm{div}_{\tan}\eta \;\mathrm{d}\mathcal{H}^{n-1}(x) : \eta \in \mathcal{V}(K,S),\ \forall x \in X \ \ \|\eta(x)\| \le 1 \right\}.$$

By putting together this expression of the perimeter with relation (7.6) we obtain the following proposition.

Proposition 8.3. *If for any crack set L we have $F(L;\mathbf{u}_0) \;=\; G\mathcal{H}^{n-1}(L)$, then for any absolute minimal state $(\mathbf{u},S) \in$ Absmin$(\mathbf{u}_0,K)$ such that $S \setminus K \neq \emptyset$, we have*

$$|ER(\mathbf{u},S)|(\Omega) \;=\; CF(\mathbf{u},S)(\Omega).$$

At this point let us note that for a general equilibrium state in three dimensions $(\mathbf{u},S) \in Eq(\mathbf{u}_0,K)$ there is no obvious connection between the energy release rate $|ER(\mathbf{u},S)|$, as in Definition 8.2, and the elastic energy concentration $CE(\mathbf{u},S)$, as in Definition 7.2.

The following theorem gives a relation between these two quantities.

Theorem 8.4. *Let $(\mathbf{u},S) \in Eq(\mathbf{u}_0,K)$ be an equilibrium state of the brittle body with reference configuration Ω, and let $D \subset \Omega$ be an arbitrary open set. Then we have the following inequality:*

$$|ER(\mathbf{u},S)|(D) \;\le\; CE(\mathbf{u},S)(D). \tag{8.3}$$

Remark 8.5. *For an arbitrary crack set L, we cannot* a priori *deduce from the EH the existence of a displacement $\mathbf{u}'$ with $(\mathbf{u}',L) \in Adm(\mathbf{u}_0,K)$ and such that for any other state $(\mathbf{v},L) \in Adm(\mathbf{u}_0,K)$ we have*

$$\int_\Omega w(\nabla\mathbf{u}')\,\mathrm{d}x \;\le\; \int_\Omega w(\nabla\mathbf{v})\,\mathrm{d}x.$$

From the mechanical point of view such an assumption is natural. There are mathematical results which support this hypothesis, but to the best of our knowledge, not with the regularity needed in this paper. Fortunately, we shall not need to make such an assumption in order to prove Theorem 8.4.

Proof of Theorem 8.4. (First part) Let us consider an arbitrary vector field $\eta \in \mathcal{V}(K,S)$, with compact support in D, such that for any $x \in \Omega$ we have $\|\eta(x)\| \le 1$.

In order to prove the theorem it is enough to show that

$$ER(\mathbf{u},S)(\eta) \;\le\; CE(\mathbf{u},S)(D). \tag{8.4}$$

Indeed, suppose (8.4) is true for any vector field $\eta \in \mathcal{V}(K,S)$, with compact support in D, such that for any $x \in \Omega$ we have $\|\eta(x)\| \;\le\; 1$. Then, by taking the supremum with respect to all such vector fields η and using Definition 8.2, we get the desired relation (8.3).

The inequality (8.4) is a consequence of Proposition 8.6 and relation (8.9), which are of independent interest. We shall resume the proof of Theorem 8.4, by giving the proof of the inequality (8.4), after we prove the afore-mentioned results. □

Let ϕ_t be the one-parameter flow generated by the vector field η. We can always find a curvilinear coordinate system $(\alpha_1,\ldots,\alpha_{n-1},\gamma)$ in the open set D such that

- on the part of the edge $\mathrm{d}S \cap \operatorname{supp}\eta$ of the crack set S we have $\gamma = 0$,

- the surface $\gamma = t$ (constant) is the boundary of an open set B_t such that

$$\phi_t(S) \setminus S \subset B_t \subset \operatorname{supp} \eta \subset D,$$

- there exists $T > 0$ such that for all $t \in [0, T]$ we have

$$B_t \subset B(\mathrm{d}S \cap D, t) \cap D, \tag{8.5}$$

where $B(\mathrm{d}S \cap D, t)$ is the tubular neighborhood of $\mathrm{d}S \cap D$ of radius t.

Consider also the one-parameter flow ψ_t, $t \geq 0$, which is equal to the identity outside the open set D and, in the curvilinear coordinates just introduced, it has the expression

$$\psi_t(x(\alpha_i, \gamma)) = x(\alpha_i, t + \gamma).$$

Notice that $\psi_t(\Omega) = \Omega \setminus B_t$. We shall use this notation for proving that the elastic energy concentration is a kind of energy release rate, after the following result.

Proposition 8.6. *With the notation defined previously before, we have*

$$\lim_{t \to 0} \frac{1}{t} \left(\int_{\Omega \setminus B_t} w(\nabla \mathbf{u}) \, \mathrm{d}x - \int_{\Omega \setminus B_t} w(\nabla(\mathbf{u} \circ \psi_t^{-1})) \, \mathrm{d}x \right) = 0. \tag{8.6}$$

Proof. Recalling that $\psi_t(\Omega) = \Omega \setminus B_t$, we use the change of variables $x = \psi_t(y)$ to prove that (8.6) is equivalent with

$$\lim_{t \to 0} \frac{1}{t} \left(\int_\Omega \left(w(\nabla \mathbf{u}(y)(\nabla \psi_t)^{-1}(y)) - w((\nabla \mathbf{u})(\psi_t(y))\right) \det \nabla \psi_t(y) \, \mathrm{d}y \right) = 0.$$

The previous relation is just

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \int_\Omega \left(w(\nabla \mathbf{u}(y)(\nabla \psi_t)^{-1}(y)) - w((\nabla \mathbf{u})(\psi_t(y))\right) \det \nabla \psi_t(y) \, \mathrm{d}y = 0. \tag{8.7}$$

We shall prove this from $(\mathbf{u}, S) \in Eq(\mathbf{u}_0, K)$ and from an approximation argument. We shall use the notation from Section 7.1.

Denote by ω the vector field which generates the one-parameter flow ψ_t. Let us compute, using integration by parts,

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \int_\Omega \left(w(\nabla \mathbf{u}(y)(\nabla \psi_t)^{-1}(y)) - w((\nabla \mathbf{u})(\psi_t(y)))\right) \det \nabla \psi_t(y) \, \mathrm{d}y$$

$$= \int_\Omega \left(\sigma_{ij} \mathbf{u}_{i,jk} \omega_k + \sigma_{ij} \mathbf{u}_{i,k} \omega_{k,j} \right) \, \mathrm{d}y. \tag{8.8}$$

For any $\gamma > 0$ sufficiently small, choose a smooth scalar function $f^\gamma : \Omega \to [0, 1]$ such that

(a) $f^\gamma(x) = 0$ for all $x \in B_\gamma, f^\gamma(x) = 1$ for all $x \in \Omega \setminus B_{2\gamma}$;
(b) as γ goes to 0 we have

$$\lim_{\gamma \to 0} \int_\Omega f^\gamma \left(\sigma_{ij} \mathbf{u}_{i,jk} \omega_k + \sigma_{ij} \mathbf{u}_{i,k} \omega_{k,j} \right) \, \mathrm{d}y = \int_\Omega \left(\sigma_{ij} \mathbf{u}_{i,jk} \omega_k + \sigma_{ij} \mathbf{u}_{i,k} \omega_{k,j} \right) \, \mathrm{d}y,$$

$$\lim_{\gamma \to 0} \int_\Omega f_j^\gamma \sigma_{ij} \mathbf{u}_{i,k} \omega_k \, \mathrm{d}y = 0.$$

For all sufficiently small $\gamma > 0$ it is true that

$$\int_\Omega \left(\sigma_{ij} \mathbf{u}_{i,jk} \omega_k^\gamma + \sigma_{ij} \mathbf{u}_{i,k} \omega_{k,j}^\gamma \right) \, \mathrm{d}y$$

$$= \int_\Omega \left(f^\gamma \left(\sigma_{ij} \mathbf{u}_{i,jk} \omega_k + \sigma_{ij} \mathbf{u}_{i,k} \omega_{k,j} \right) + f_j^\gamma \sigma_{ij} \mathbf{u}_{i,k} \omega_k \right) \, \mathrm{d}y.$$

Thus, from (a) and (b) above we get the equality

$$\lim_{\gamma\to 0}\int_\Omega \left(\sigma_{ij}\mathbf{u}_{i,jk}\omega_k^\gamma + \sigma_{ij}\mathbf{u}_{i,k}\omega_{k,j}^\gamma\right)\,\mathrm{d}y \;=\; \int_\Omega \left(\sigma_{ij}\mathbf{u}_{i,jk}\omega_k + \sigma_{ij}\mathbf{u}_{i,k}\omega_{k,j}\right)\,\mathrm{d}y.$$

Recall that $(\mathbf{u}, S)$ is an equilibrium state and therefore the stress field $\sigma = \sigma(\nabla\mathbf{u})$ has divergence equal to 0. Integration by parts shows that, for any sufficiently small $\gamma > 0$, we have

$$\int_\Omega \left(\sigma_{ij}\mathbf{u}_{i,jk}\omega_k^\gamma + \sigma_{ij}\mathbf{u}_{i,k}\omega_{k,j}^\gamma\right)\,\mathrm{d}y \;=\; \int_\Omega -\sigma_{ij,j}\left(\mathbf{u}_{i,k}\omega_k^\gamma\right)\,\mathrm{d}y \;= 0.$$

We have therefore obtained the relation

$$\int_\Omega \left(\sigma_{ij}\mathbf{u}_{i,jk}\omega_k + \sigma_{ij}\mathbf{u}_{i,k}\omega_{k,j}\right)\,\mathrm{d}y \;= 0.$$

This is equivalent to relation (8.7), and by computation (8.8). □

A straightforward consequence of (8.6) is that the elastic energy concentration is related to a kind of configurational energy release rate. Namely, we see that

$$\limsup_{t\to 0}\frac{1}{t}\int_{B_t} w(\nabla\mathbf{u})\,\mathrm{d}x \;=$$

$$= \limsup_{t\to 0}\frac{1}{t}\left(\int_\Omega w(\nabla\mathbf{u})\,\mathrm{d}x - \int_{\Omega\setminus B_t} w(\nabla(\mathbf{u}\circ\psi_t^{-1}))\,\mathrm{d}x\right). \tag{8.9}$$

We now turn back to the proof of Theorem 8.3. Recall that what is left to prove is relation (8.4).

Proof of Equation (8.4). By construction, for all sufficiently small $t > 0$ we have

$$\frac{1}{t}\int_{B(\mathrm{d}S,t)\cap D} w(\nabla\mathbf{u})\,\mathrm{d}x \;\geq\; \frac{1}{t}\int_{B_t} w(\nabla\mathbf{u})\,\mathrm{d}x$$

because $B_t \subset B(\mathrm{d}S, t)\cap D$. We write the right-hand side of this inequality as a sum of three terms:

$$\begin{aligned}\frac{1}{t}&\int_{B_t} w(\nabla\mathbf{u})\,\mathrm{d}x\\ &= \frac{1}{t}\left(\int_\Omega w(\nabla\mathbf{u})\,\mathrm{d}x - \int_\Omega w(\nabla(\mathbf{u}\circ\phi_t^{-1}))\,\mathrm{d}x\right)\\ &+ \frac{1}{t}\left(\int_\Omega w(\nabla(\mathbf{u}\circ\phi_t^{-1}))\,\mathrm{d}x - \int_{\Omega\setminus B_t} w(\nabla(\mathbf{u}\circ\psi_t^{-1}))\,\mathrm{d}x\right)\\ &+ \frac{1}{t}\left(\int_{\Omega\setminus B_t} w(\nabla(\mathbf{u}\circ\psi_t^{-1}))\,\mathrm{d}x - \int_{\Omega\setminus B_t} w(\nabla(\mathbf{u}))\,\mathrm{d}x\right).\end{aligned}$$

As t goes to 0, the first term converges to $EC(\mathbf{u}, S)(\eta)$ and the third term converges to 0 by Proposition 8.6. We want to show that

$$\lim_{t\to 0}\frac{1}{t}\left(\int_\Omega w(\nabla(\mathbf{u}\circ\phi_t^{-1}))\,\mathrm{d}x - \int_{\Omega\setminus B_t} w(\nabla(\mathbf{u}\circ\psi_t^{-1}))\,\mathrm{d}x\right) \;= 0. \tag{8.10}$$

The proof of this limit is identical to the proof of Proposition 8.6. Indeed, in that proof we worked with the one-parameter flow ψ_t generated by the vector field ω. This one-parameter flow is a semigroup (with respect to composition of functions), but after inspection of the proof it can be seen that we only used the following: for any $x \in \Omega\setminus S$

$$\lim_{t\to 0}\psi_t(x) \;=\; x \quad\text{and}\quad \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\psi_t(x) \;=\; \omega(x).$$

Therefore, we can modify the proof of Proposition 8.6 by considering, instead of ψ_t, the diffeomorphisms λ_t defined by

$$\lambda_t \;=\; \psi_t\circ\phi_t^{-1}.$$

The rest of the proof goes exactly as before, thus leading us to relation (8.10).

Eventually, we have

$$CE(\mathbf{u}, S)(D) = \limsup_{t\to 0} \frac{1}{t} \int_{B(\mathrm{d}S,t)\cap D} w(\nabla \mathbf{u})\, \mathrm{d}x$$

$$\geq \limsup_{t\to 0} \frac{1}{t} \int_{B(\mathrm{d}S,t)\cap D} w(\nabla \mathbf{u})\, \mathrm{d}x = ES(\mathbf{u}, S)(\eta)$$

$$+ \lim_{t\to 0} \frac{1}{t} \left(\int_{\Omega} w(\nabla(\mathbf{u} \circ \phi_t^{-1}))\, \mathrm{d}x - \int_{\Omega\setminus B_t} w(\nabla(\mathbf{u} \circ \psi_t^{-1}))\, \mathrm{d}x \right)$$

$$+ \lim_{t\to 0} \frac{1}{t} \left(\int_{\Omega\setminus B_t} w(\nabla(\mathbf{u} \circ \psi_t^{-1}))\, \mathrm{d}x - \int_{\Omega\setminus B_t} w(\nabla(\mathbf{u}))\, \mathrm{d}x \right) = ES(\mathbf{u}, S)(\eta)$$

and (8.4) is therefore proven. □

9. A Constraint on Some Minimal Solutions

Let us consider now a solution of the model of brittle crack propagation described in Section 5. More precisely, for given boundary conditions $\mathbf{u}_0(t)$ and an initial crack set K, we shall call a solution $(\mathbf{u}(t), S_t) \in Eq(\mathbf{u}_0(t), S_t)$ of the model described by axioms (A1),..., (A5), an ‘equilibrium solution’. Likewise, a solution $(\mathbf{u}(t), S_t) \in Absmin(\mathbf{u}_0(t), S_t)$ of the model described by axioms (A1),(A2),(A3’),(A4), will be called a ‘minimal solution’.

We shall deal with a minimal solution $(\mathbf{u}(t), S_t) \in Absmin(\mathbf{u}_0(t), S_t)$ for which the crack set S_t propagates smoothly, *without topological changes*. Namely, we shall suppose that there exists a vector field η with compact support in Ω, such that for all $t \in [0, T]$ we have $S_t = \phi_t(K)$, where ϕ_t is the one-parameter flow generated by η.

Because the problem is quasistatic, time enters only as a parameter; therefore, we may suppose moreover that for all $x \in \Omega$ we have $\eta(x) \leq 1$.

At each moment $t \in [0, T]$ we shall have $\eta \circ \phi_t \in \mathcal{V}(K, S_t)$.

Theorem 9.1. *Suppose that for any crack set L and boundary displacement* $\mathbf{u}_0$ *the surface energy has the expression*

$$F(S; \mathbf{u}_0) = G\mathcal{H}^{n-1}(S).$$

Let $(\mathbf{u}(t), S_t) \in Absmin(\mathbf{u}_0(t), S_t)$ *be a minimal solution, with* $S_0 = K$*, such that exists a vector field* η *with* $\|\eta(x)\| \leq 1$ *for all* $x \in \Omega$ *and for all* $t \in [0, T]$ *we have* $S_t = \phi_t(K)$*, where* ϕ_t *is the one-parameter flow generated by* η*.*

Then, for any $t \in [0, T]$ *and any open set* $D \subset \Omega$*, we have the equalities*

$$|ER(\mathbf{u}(t), \phi_t(S))|(D) = EC(\mathbf{u}(t), \phi_t(S))(D)$$

$$= CF(\phi_t(S); \mathbf{u}_0(t))(D) = G\mathcal{H}^{n-2}(\mathrm{d}S \setminus K). \tag{9.1}$$

Proof. Theorems 8.4 and 7.3 tell us that for any open set $D \subset \Omega$ and for any $t \in [0, T]$ we have

$$|ER(\mathbf{u}(t), \phi_t(S))|(D) \leq EC(\mathbf{u}(t), \phi_t(S))(D) \leq CF(\phi_t(S); \mathbf{u}_0(t))(D).$$

Proposition 8.3 tells that

$$CF(\phi_t(S); \mathbf{u}_0(t))(\Omega) = |ER(\mathbf{u}(t), \phi_t(S))|(\Omega).$$

We deduce that for any open set $D \subset \Omega$ and for any $t \in [0, T]$ the string of equalities (9.1) is true. □

This result is natural in 2D linear elasticity. Nevertheless, in the case of 3D elasticity, the constraints on the elastic energy concentration provided by Theorem 9.1 might be too hard to satisfy.

Indeed, from (9.1) we deduce that, in particular, the elastic energy concentration has to be absolutely continuous with respect to the perimeter measure of the edge of the crack.

10. Conclusions

We have proposed a general model of brittle crack propagation based on Mumford–Shah functionals. We have defined equilibrium and absolute minimal solutions of the model.

By a combination of analytical and configurational analysis, we have defined measures of the energy release rate and energy concentrations for equilibrium and absolute minimal solutions and we have shown that there is a difference between such solutions, as shown mainly by Theorems 7.3, 8.4 and 9.1.

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