NUMERICAL SCHEMES FOR THE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We consider semidiscrete approximation schemes for the linear Schrödinger equation and analyze whether the classical dispersive properties of the continuous model hold for these approximations. For the conservative finite difference semi-discretization scheme of the linear Schrödinger equation, we show that, as the mesh-size tends to zero, the semidiscrete approximate solutions loose the dispersion property. We prove this property by constructing solutions concentrated at the points of the spectrum where the second order derivatives of the symbol of the discrete laplacian vanish. Therefore this phenomenon is due to the presence of numerical spurious high-frequencies.

To recover the dispersive properties of the solutions at the discrete level, we introduce three numerical remedies: Fourier filtering; numerical viscosity; two-grid preconditioner. For each of them we prove Strichartz-like estimates and the local space smoothing effect, uniformly on the mesh size. The methods we employ are based on classical estimates for oscillatory integrals. These estimates allow us to treat nonlinear problems with L^2-initial data, without additional regularity hypotheses. We prove the convergence of the proposed methods for nonlinearities that cannot be handled by energy arguments and which, even in the continuous case, require Strichartz estimates.

1. Introduction

Let us consider the linear (LSE) and the nonlinear (NSE) Schrödinger equations

$$\begin{cases} iu_t + \Delta u = 0, \ x \in \mathbb{R}^d, \ t \neq 0, \\ u(0,x) = \varphi(x), \ x \in \mathbb{R}^d. \end{cases} \qquad (1.1)$$

This equation is solved by $u(x,t) = S(t)\varphi$, where $S(t) = e^{it\Delta}$ is the free Schrödinger operator. The linear semigroup has two important properties. First, the conservation of the L^2-norm

$$\|u(t)\|_{L^2(\mathbb{R}^d)} = \|\varphi\|_{L^2(\mathbb{R}^d)} \qquad (1.2)$$

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and a dispersive estimate of the form:

$$|S(t)\varphi(x)| = |u(t,x)| \leq \frac{1}{(4\pi|t|)^{d/2}}\|\varphi\|_{L^1(\mathbb{R}^d)},\ x\in\mathbb{R}^d,\ t\neq 0. \tag{1.3}$$

The Space-Time Estimate

$$\|S(\cdot)\varphi\|_{L^{2+4/d}(\mathbb{R},\,L^{2+4/d}(\mathbb{R}^d))} \leq C\|\varphi\|_{L^2(\mathbb{R}^d)}, \tag{1.4}$$

due to Strichartz [37], is deeper. It guarantees that the solutions decay in some sense as t becomes large and that they gain some spatial integrability.

Inequality (1.4) was generalized by Ginibre and Velo [16]. They proved the *Mixed Space-Time Estimate*

$$\|S(\cdot)\varphi\|_{L^q(\mathbb{R},\,L^r(\mathbb{R}^d))} \leq C(q,r)\|\varphi\|_{L^2(\mathbb{R})} \tag{1.5}$$

for the so-called $d/2$-admissible pairs (q,r), excepting the limit case $(q,r) = (\infty,2)$ in dimension $d=2$. We recall that a σ-admisible pair satisfies (cf. [23]): $2\leq q,r\leq\infty$ and

$$\frac{1}{q} = \sigma\left(\frac{1}{2}-\frac{1}{r}\right). \tag{1.6}$$

The extension to the inhomogeneous linear Schrödinger equation is due to Yajima [43] and Cazenave and Weissler [7]. The estimates presented before play an important role in the proof of the well-posedness of the nonlinear Schrödnger equation (NSE). Typically the dispersive estimates are used when the energy methods fail to provide well-posedness results for nonlinear problems.

These estimates can be extended to a larger class of equations for which the laplacian is replaced by any self-adjoint operator so that the L^∞-norm of the fundamental solution behaves like $t^{-d/2}$ [23].

The nonlinear problem with nonlinearity $F(u) = |u|^{p-1}u$, $p < 4/d$ and initial data in $L^2(\mathbb{R}^d)$ has been first analyzed by Tsutsumi [40]. The author proved that, in this case, NSE is globally well posed in $L^\infty(\mathbb{R},\,L^2(\mathbb{R}^d))\cap L^q_{loc}(\mathbb{R},\,L^r(\mathbb{R}^d))$, where (q,r) is an $d/2$-admissible pair depending on the nonlinearity F. Also, Cazenave and Weissler [8] proved the local existence in the critical case $p=4/d$. The case of H^1-solutions has been analyzed by Baillon, Cazenave and Figueira [3], Lin and Strauss [26], Ginibre and Velo [14, 15], Cazenave [5], and, in a more general context, by Kato [21, 22].

The Schrödinger equation has another remarkable property guaranteeing the gain of one half space derivative in $L^2_{x,t}$ (cf. [32], [10], [11] and [24]):

$$\sup_{x_0,R}\frac{1}{R}\int_{B(x_0,R)}\int_{-\infty}^{\infty}|(-\Delta)^{1/4}e^{it\Delta}\varphi|^2dtdx \leq C\|\varphi\|^2_{L^2(\mathbb{R}^d)}. \tag{1.7}$$

It has played a crucial role in the study of the nonlinear Schrödinger equation with nonlinearities involving derivatives (see [25]). For other deep results on the Schrödinger equations we refer to [38], [6] and the bibliography therein.

In this paper we analyze whether semidiscrete schemes for LSE have dispersive properties similar to (1.3), (1.5) and (1.7), uniform with respect to the mesh sizes. The study of these dispersion properties for these approximation schemes is relevant for introducing convergent schemes in the nonlinear context. Since, as mentionated above, the well-posedness of the nonlinear Schrödinger equation requires a fine use of the dispersion properties, the convergence of the numerical schemes cannot be proved if these dispersion properties are not verified at the numerical level.

Estimates similar to (1.7) on discrete solutions will give sufficient conditions to guarantee their compactness and thus the convergence towards the solution of the nonlinear Schrödinger equation. Without such an estimate, despite the uniform boundedness of the discrete solutions in the space $L^\infty(\mathbb{R}, l^2(h\mathbb{Z}^d)) \cap L^q_{loc}(\mathbb{R}, l^r(h\mathbb{Z}^d))$, one cannot pass to the limit in the nonlinear term.

To better illustrate the problems we shall address, let us first consider the conservative semidiscrete numerical scheme

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = 0, \quad t > 0, \\ \qquad u^h(0) = \varphi^h. \end{cases} \tag{1.8}$$

Here u^h stands for the infinite unknown vector $\{u^h_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d}$, $u_{\mathbf{j}}(t)$ being the approximation of the solution at the node $x_{\mathbf{j}} = \mathbf{j}h$, and Δ_h is the classical second order finite difference approximation of Δ:

$$(\Delta_h u^h)_{\mathbf{j}} = h^{-2}\sum_{k=1}^{d}(u^h_{\mathbf{j}+e_k} + u^h_{\mathbf{j}-e_k} - 2u^h_{\mathbf{j}}).$$

In the one-dimensional case, the lack of uniform dispersive estimates for the solutions of (1.8) has been observed by the authors in [19, 20]. In that case the symbol of the Laplacian, ξ^2, is replaced by a discrete one $4/h^2\sin^2(\xi h/2)$ which vanishes its first and second derivative at the points $\pm\pi/h$ and $\pm\pi/2h$ of the spectrum. By concentrating wave packets at these pathological points it is possible to prove the lack of any uniform estimate of the type (1.3), (1.5) or (1.7). For the semidiscrete Schrödinger equation we also refer to [34]. In that paper the authors analyze the Schrödinger equation on the lattice $h\mathbb{Z}^d$ without concentrating on parameter h. They obtain Strichartz-like estimates in a class of exponents q and r larger than in the continuous one, none being independent of the parameter h.

In the case of fully discrete schemes, Nixon [28] considers an approximation of the one-dimensional KdV equation based on the backward Euler approximation of the linear semigroup and proves space time estimates for that approximation. For the Schrödinger equation in [18] necessary and sufficient conditions to guarantee the existence, at the discrete level, of dispersive properties for the Schrödinger equation are given.

The paper is organized as follows. In Section 2 we consider a numerical scheme for the one-dimensional cubic NSE and prove the existence of solutions that blow-up in any auxiliar space $L^q_{loc}(\mathbb{R},\, l^r(h\mathbb{Z}))$. This implies that there are no uniform dispersive estimates for the linear semigroup generated by scheme (1.8). If there exists any dispersive estimate similar to the ones in (1.5), the nonlinear problem will admit solutions which will remain bounded in some auxiliary space $L^q_{loc}(\mathbb{R},\, l^r(h\mathbb{Z}))$ which is not the case.

In Section 3 we analyze the conservative approximations (1.8). We prove that this scheme does not gain any uniform integrability or local smoothing of the solutions with respect to the initial data. Afterwards, once we have understood what are the pathologies of the numerical scheme (1.8), we propose a frequency filtering of initial data which will recover both integrability and local smoothing of the continuous model.

We then introduce two numerical schemes for which the estimates are uniform. The first one uses an artificial numerical viscosity term and the second one involves a two-grid algorithm to precondition the initial data. Both approximation schemes of the linear semigroup converge and have uniform dispersion properties. This allows us to build two convergent numerical schemes for the NSE in the class of $L^2(\mathbb{R}^d)$ initial data.

In Section 4 we introduce a numerical scheme containing a numerical viscosity term of the form $ia(h)\Delta_h u$. We prove that choosing a convenient $a(h)$ we are able to recover the properties mentioned above. We then consider an approximation of NSE based on the approximation of LSE introduced before and prove the convergence of its solutions towards the solutions of NSE.

Section 5 is dedicated to the analysis of the method based on the two-grid preconditioning of the initial data. We analyze the action of the linear semigroup $\exp(it\Delta_h)$ on the subspace of $l^2(h\mathbb{Z}^d)$ constituted by the slowly oscillating sequences generated by the two-grid method. Once we obtain Strichartz-like estimates in this subspace we apply them to approximate the NSE. The nonlinear term is approximated in a such way that permits the use of the Strichartz estimates in the class of slowly oscillating sequences.

2. On the cubic NSE

In this section we will consider an approximation of the one-dimensional NSE with nonlinearity $2|u|^2u$ which has explicit solutions. This will constitute a first example of a numerical scheme for NSE that has solutions which blow up in any $L^q_{loc}(\mathbb{R},\, l^r(h\mathbb{Z}))$-norm with $r > 2$.

To explain the necessity of analyzing the dispersive properties at numerical level let us consider the following discretization of the NSE that was proposed in [1] and is accordingly often referred to as the Ablowitz-Ladik

NSE of the form:

$$i\partial_t u_n^h + (\Delta_h u^h)_n = |u_n^h|^2 (u_{n+1}^h + u_{n-1}^h), \tag{2.9}$$

with initial condition $u^h(0) = \varphi^h$, φ^h being an approximation of the initial data in NSE.

Let us assume the existence of a positive T such that for any $h > 0$, there exists $u^h \in L^\infty([0,T],\, l^2(h\mathbb{Z}^d))$ solution of (2.9).

The uniform boundedness of $\{u^h\}_{h>0}$ in $L^\infty([0,T],\, l^2(h\mathbb{Z}))$ does not allow us to prove its convergence towards the solution of the NSE. We recall that, as explained above, in order to prove the well-posedness of NSE we have to introduce an auxiliary space $L^q_{loc}(\mathbb{R},\, L^r(\mathbb{R}))$ with suitable q and r. One then needs to analyze whether the solutions of (2.9) belong to one of the auxiliary spaces $L^q_{loc}(\mathbb{R},\, l^r(h\mathbb{Z}))$, a property that will guarantee that any eventually limit point of $\{u^h\}_{h>0}$ belongs to $L^q([0,T],\, L^r(\mathbb{R}))$.

For any $h > 0$, equation (2.9) has explicit travelling soliton solutions. We remark that any solution u^h satisfies

$$u_n^h(t) = \frac{1}{h}\, u_n^1\left(\frac{t}{h^2}\right),\ n \in \mathbb{Z},\ t \geq 0,$$

where u^1 is the solution on the mesh size $h = 1$. In this case there are explicit solutions of (2.9) (cf. [2], p. 84) of the form:

$$u_n^1(t) = A \exp(i(an - bt)) \operatorname{sech}(cn - dt)$$

for suitable constants A, a, b, c, d (for the explicit values we refer to [2]).

The solutions of (2.9) obtained by scaling on t of this one are not uniformly bounded as $h \to 0$ in any auxiliary space $L^q([0,T],\, l^r(h\mathbb{Z}))$ with $r > 2$. More precisely, a scaling argument shows that

$$\frac{\|u^h\|_{L^q([0,T],\, l^r(h\mathbb{Z}))}}{\|u^h(0)\|_{l^2(h\mathbb{Z})}} = h^{\frac{1}{r} + \frac{2}{q} - \frac{1}{2}} \frac{\|u^1\|_{L^q([0,T/h^2],\, l^r(\mathbb{Z}))}}{\|u^1(0)\|_{l^2(\mathbb{Z})}}.$$

Observe that for any $t > 0$, the $l^r(\mathbb{Z})$-norm behaves as a constant:

$$\|u^1(t)\|_{l^r(\mathbb{Z})} \simeq \Big(\int_{\mathbb{R}} \operatorname{sech}^r(cx - dt)dx \Big)^{1/r} = \Big(\int_{\mathbb{R}} \operatorname{sech}^r(cx)dx \Big)^{1/r}.$$

Thus, for all $T > 0$ and $h > 0$ the solution u^1 satisfies

$$\|u^1\|_{L^q([0,T/h^2],\, l^r(\mathbb{Z}))} \simeq (Th^{-2})^{1/q}.$$

Consequently for any $r > 2$ the solution u^h on the lattice $h\mathbb{Z}$ satisfies:

$$\frac{\|u^h\|_{L^q([0,T],\, l^r(h\mathbb{Z}))}}{\|u^h(0)\|_{l^2(h\mathbb{Z})}} \simeq h^{\frac{1}{r} - \frac{1}{2}} \to \infty,\ h \to 0.$$

In view of the above example, in the case of a general numerical scheme for NSE one cannot expect that its solutions will have a limit point in

$L^q_{loc}(\mathbb{R},\, L^r(\mathbb{R}))$. We point out that this is compatible with the convergence of the numerical scheme (2.9) for smooth initial data [1, 2]

This motivates us to follow, at the semidiscrete level, the main steps of the theory of the well posedness of NSE and analyze whether we can derive similar dispersive properties for the linear part of the numerical scheme. However, this does not imply the existence of $\varphi \in L^2(\mathbb{R})$ and $\varphi^h \in l^2(h\mathbb{Z})$ such that $\varphi^h \to \varphi$ in $L^2(\mathbb{R})$ and $\|\varphi^h\|_{L^q([0,T],\, l^r(h\mathbb{Z}))} \to \infty$. The existence of such an example remains an open problem.

The blow-up of the solutions of the nonlinear problem (2.9) implies that there are no uniform dispersive estimates for the linear equation $iu^h_t + \Delta_h u^h = 0$. Indeed, if were exists any dispersive estimates similar to (1.5), the nonlinear problem would admit solutions which would remain bounded in some auxiliary space $L^q_{loc}(\mathbb{R},\, l^r(h\mathbb{Z}))$ which is not the case.

3. A conservative scheme

In this section we analyze the conservative scheme (1.8). This scheme satisfies the classical properties of consistency and stability which imply L^2-convergence. As we have seen in the previous section, there are no uniform dispersive properties for its solutions. We will treat to understand these pathologies by constructing explicit solutions for scheme (1.8) for which at the discrete level, nor (1.5) or 1.7 hold uniformly with respect to parameter h.

In our analysis, we make use of the semidiscrete Fourier transform (SDFT) (we refer to [39] for the mains properties of the SDTF). For any $v \in l^2(h\mathbb{Z}^d)$ we define its SDFT at the scale h by:

$$\widehat{v}(\xi) = (\mathcal{F}_h v)(\xi) = h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} e^{-i\xi\cdot\mathbf{j}h} v_{\mathbf{j}},\ \xi \in [-\pi/h, \pi/h]^d. \tag{3.10}$$

To avoid the presence of constants, we will use the notation $A \lesssim B$ to report the inequality $A \leq \textit{constant} \times B$, where the constant is independent of h. The statement $A \simeq B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.

Taking SDFT in (1.8) we obtain that $u^h(t) = S^h(t)\varphi$ solution of (1.8) satisfies

$$i\widehat{u}^h_t(t,\xi) + p_h(\xi)\widehat{u}^h(t,\xi) = 0,\ t \in \mathbb{R},\ \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^d, \tag{3.11}$$

where the function $p_h : [-\pi/h, \pi/h]^d \to \mathbb{R}$ is defined by

$$p_h(\xi) = \frac{4}{h^2}\sum_{k=1}^{d} \sin^2\left(\frac{\xi_k h}{2}\right). \tag{3.12}$$

Solving ODE (3.11) we obtain that u^h is given in Fourier variable by

$$\widehat{u}^h(t,\xi) = e^{-itp_h(\xi)}\widehat{\varphi}^h(\xi),\ \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^d. \tag{3.13}$$

Observe that the new symbol is different from the continuous one: $|\xi|^2$. In the one-dimensional case, the symbol $p_h(\xi)$ changes convexity at the points $\xi = \pm\pi/2h$ and has critical points also at $\xi = \pm\pi/h$, two properties that the continuous symbol does not have. Using that

$$\inf_{\xi\in[-\pi/h,\pi/h]} |p_h''(\xi)| + |p_h'''(\xi)| > 0,$$

in [20] (see also [34] for $h = 1$) it has been proved that

$$\|u^h(t)\|_{l^\infty(h\mathbb{Z})} \lesssim \|\varphi^h\|_{l^1(h\mathbb{Z})}\Big(\frac{1}{t^{1/2}} + \frac{1}{(th)^{1/3}}\Big). \tag{3.14}$$

Note that the estimate (3.14) blows-up as $h \to 0$, therefore it does not yield uniform Strichartz estimates.

In dimension d, similar results can be obtained in terms of the number of nonvanishing principal curvatures of the symbol and its gradient. Observe that, at the points $\xi = (\pm\pi/2h, \dots, \pm\pi/2h)$ all the eigenvalues of the hessian matrix $H_{p_h} = (\partial_{ij} p_h)_{ij}$ vanish. We also mention that if k-components of the vector ξ equal $\pm\pi/2h$ then the rank of H_{p_h} at these points is $d-k$ instead of d in the continuous case. This will imply that in this case the solutions of equation (1.8) will behave as $t^{-(d-k)/2}(th)^{-hk/3}$ instead of $t^{-d/2}$.

On the other hand, at the points $\xi = (\pm\pi/h, \dots, \pm\pi/h)$, the gradient of the symbol $p_h(\xi)$ vanishes. As we will see, these pathologies affect the dispersive properties of the semidiscrete scheme (1.8).

The first pathology, i.e. the fact that $H_p((\pm\pi/2h, \dots, \pm\pi/2h)) = 0$, shows that there are no uniform estimates similar to (1.3) at the discrete level. Consequently the solutions of the semidiscrete scheme (1.8) have no uniform (with respect to h) $L^q(l^r(h\mathbb{Z}^d))$ integrability properties. This condition is necessary to prove the uniform boundedness of the semidiscrete solutions.

Because of the second pathology, i.e. the existence of critical points of the semidiscrete symbol, solutions of (1.8) do not fulfill the regularizing property (1.7) uniformly on $h > 0$ needed to guarantee the compactness of the semidiscrete solutions. This constitute an obstacle when passing to the limit as $h \to 0$ in the nonlinear semidiscrete models.

3.1. **Lack of uniform dispersive estimates.** As we have seen in Section 2 there are not uniform dispersive estimates for the solutions of (1.8). This means that there is no uniform decay of solutions as in (1.3) nor a space time estimate similar to (1.5). In this section we construct explicit examples of solutions of equation (1.8) for which all the classical estimates of the continuous case (1.5) blow-up.

Theorem 3.1. *Let $T > 0$, $r_0 \geq 1$ and $r > r_0$. Then*

$$\sup_{h>0,\, \varphi\in l^{r_0}(h\mathbb{Z}^d)} \frac{\|S^h(T)\varphi\|_{l^r(h\mathbb{Z}^d)}}{\|\varphi\|_{l^{r_0}(h\mathbb{Z}^d)}} = \infty \tag{3.15}$$

and

$$\sup_{h>0,\,\varphi\in l^{r_0}(h\mathbb{Z}^d)} \frac{\|S^h(\cdot)\varphi\|_{L^1((0,T),l^r(h\mathbb{Z}^d)}}{\|\varphi\|_{l^{r_0}(h\mathbb{Z}^d)}}=\infty. \tag{3.16}$$

Let I^h be an interpolator, piecewise constant or linear. For any fixed $T>0$, the uniform boundedness principle guarantees the existence of a function $\varphi\in L^2(\mathbb{R}^d)$ and a sequence φ^h such that $I^h\varphi^h\to\varphi$ in $L^2(\mathbb{R}^d)$ and the corresponding solutions u^h of (1.8) satisfy

$$\|I^h u^h\|_{L^1((0,T),\,L^r(\mathbb{R}^d))}\to\infty.$$

This guarantees the existence of an initial datum φ and approximations φ^h such that the solutions of (1.8) have no limit point in any auxiliary space $L^q_{loc}(\mathbb{R},\,L^r(\mathbb{R}^d))$.

Proof of Theorem 3.1. First, observe that it is sufficient to deal with the one-dimensional case. For any sequence $\{\psi_j\}_{j\in\mathbb{Z}}$ set $\varphi_{\mathbf{j}}=\psi_{j_1}\dots\psi_{j_d}$, $\mathbf{j}=(j_1,j_2,\dots,j_d)$. Then, for any t the following holds:

$$(S^h(t)\varphi)_{\mathbf{j}}=(S^{1,h}(t)\psi)_{j_1}(S^{1,h}(t)\psi)_{j_2}\dots(S^{1,h}(t)\psi)_{j_d},$$

where $S^{1,h}(t)$ is the linear semigroup generated by the equation (1.8) in the one-dimensional case. Thus it is obvious that (3.15) and (3.16) hold in dimension $d\geq 2$, once we prove them in the one-dimensional case $d=1$.

In the following we will consider the one-dimensional case $d=1$. Using the properties of the SDTF it is easy to see that $S^h(t)\varphi=S^1(t/h^2)\varphi$. A scaling argument in (3.15) and (3.16) shows that

$$\frac{\|S^h(T)\varphi\|_{l^q(h\mathbb{Z})}}{\|\varphi\|_{l^{q_0}(h\mathbb{Z})}}=h^{\frac{1}{q}-\frac{1}{q_0}}\frac{\|S^1(T/h^2)\varphi\|_{l^q(\mathbb{Z})}}{\|\varphi\|_{l^{q_0}(\mathbb{Z})}} \tag{3.17}$$

and

$$\frac{\|S^h(\cdot)\varphi\|_{L^1((0,T),\,l^q(h\mathbb{Z}))}}{\|\varphi\|_{l^{q_0}(h\mathbb{Z})}}=h^{2+\frac{1}{q}-\frac{1}{q_0}}\frac{\|S^1(\cdot)\varphi\|_{L^1((0,T/h^2),\,l^q(\mathbb{Z}))}}{\|\varphi\|_{l^{q_0}(\mathbb{Z})}}. \tag{3.18}$$

Let us introduce the operator $S_1(t)$ defined by

$$(S_1(t)\varphi)(x)=\int_{-\pi}^{\pi}e^{-itp_1(\xi)}e^{ix\xi}\widehat{\varphi}(\xi)d\xi. \tag{3.19}$$

We point out that for any sequence $\{\varphi_j\}_{j\in\mathbb{Z}}$, $S_1(t)\varphi$ as in (3.19), which is defined for all $x\in\mathbb{R}$, is in fact the band-limited interpolator of the semi-discrete function $S^1(t)\varphi$. The results of Magyar et al. [27] (see also Plancherel and Polya [29]) on band-limited functions show that the following inequality holds for any $q>q_0\geq 1$ and for all continuous, 2π-periodic functions $\widehat{\varphi}$:

$$\frac{\|S^1(t)\varphi\|_{l^q(\mathbb{Z})}}{\|\varphi\|_{l^{q_0}(\mathbb{Z})}}\geq c(q,q_0)\frac{\|S_1(t)\varphi\|_{L^q(\mathbb{R})}}{\|\varphi\|_{L^{q_0}(\mathbb{R})}}. \tag{3.20}$$

In view of this property it is sufficient to deal with the operator $S_1(t)$.

Denoting $\tau = T/h^2$, by (3.17) and (3.18) the proof of (3.15) and (3.16) is reduced to the proof of the following ones on the new operator $S_1(t)$:

$$\lim_{\tau\to\infty} \tau^{\frac{1}{2}\left(\frac{1}{q_0}-\frac{1}{q}\right)} \sup_{\operatorname{supp}\widehat{\varphi}\subset[-\pi,\pi]} \frac{\|S_1(\tau)\varphi\|_{l^q(\mathbb{R})}}{\|\varphi\|_{l^{q_0}(\mathbb{R})}} = \infty \tag{3.21}$$

and

$$\lim_{\tau\to\infty} \tau^{-1+\frac{1}{2}\left(\frac{1}{q_0}-\frac{1}{q}\right)} \sup_{\operatorname{supp}\widehat{\varphi}\subset[-\pi,\pi]} \frac{\|S_1(\cdot)\varphi\|_{L^1((0,\tau),\,L^q(\mathbb{R}))}}{\|\varphi\|_{L^{q_0}(\mathbb{R})}} = \infty. \tag{3.22}$$

The following lemma is the key point in the proof of the last two estimates.

Lemma 3.1. *There exists a positive constant c such that for all τ sufficiently large, there exists a function φ_τ that satisfies $\|\varphi_\tau\|_{L^p(\mathbb{R})} \simeq \tau^{1/3p}$ for all $p \geq 1$ and*

$$|(S_1(t)\varphi_\tau)(x)| \geq \frac{1}{2} \tag{3.23}$$

for all $|t| \leq c\tau$ and $|x - tp_1'(\pi/2)| \leq c\tau^{1/3}$.

Remark 3.1. *comparatie cu cazul continuu*

The proof of Lemma 3.1 will be given later.

We now prove (3.21). The proof of (3.22) is similar. In view of Lemma 3.1, for sufficiently large τ the following holds:

$$\sup_{\operatorname{supp}\widehat{\varphi}\subset[-\pi,\pi]} \frac{\|S_1(\tau)\varphi\|_{L^q(\mathbb{R})}}{\|\varphi\|_{L^{q_0}(\mathbb{R})}} \gtrsim \tau^{\frac{1}{3q}-\frac{1}{3q_0}}.$$

Thus

$$\lim_{\tau\to\infty} \tau^{\frac{1}{2}\left(\frac{1}{q_0}-\frac{1}{q}\right)} \sup_{\operatorname{supp}\widehat{\varphi}\subset[-\pi,\pi]} \frac{\|S_1(\tau)\varphi\|_{L^q(\mathbb{R})}}{\|\varphi\|_{L^{q_0}(\mathbb{R})}} \gtrsim \lim_{\tau\to\infty} \tau^{\frac{1}{6}(\frac{1}{q_0}-\frac{1}{q})} = \infty,$$

which finishes the proof of (3.21). □

A finer analysis can be done. Let us consider the class of functions $\widehat{\varphi}$ with their support in

$$\Omega^k = \left\{\xi = (\xi_1, \ldots, \xi_d) \in [-\pi,\pi]^d : \xi_{k+1}, \ldots, \xi_d \neq \frac{\pi}{2}\right\},\ 1 \leq k \leq d.$$

Then the following

$$\sup_{\widehat{\varphi}\in\Omega^k} \frac{\|S_1(\tau)\varphi\|_{L^q(\mathbb{R})}}{\|\varphi\|_{L^{q_0}(\mathbb{R})}} \gtrsim \tau^{\frac{k}{3}(\frac{1}{q}-\frac{1}{q_0})}\tau^{\frac{d-k}{2}(\frac{1}{q}-\frac{1}{q_0})}$$

holds for large enough τ. As a consequence we also obtain (3.15) and (3.16). This shows that on the hyperplane $(x_{k+1}, \ldots, x_d)$ we have the right decay $\tau^{-(d-k)/2}$ and the bad one $\tau^{-k/3}$ on the hyperplane $(x_1, \ldots, x_k)$.

Proof of Lemma 3.1. The techniques used below are similar to those used in [13] to get lower bounds on oscillatory integrals.

We define the relevant initial data through its Fourier transform. Let us first fix a positive function $\widehat{\varphi}$ supported on $(-1,1)$ such that $\int_{-\pi}^{\pi}\widehat{\varphi}=1$. For all positive τ, we set:

$$\widehat{\varphi}_\tau(\xi)=\tau^{1/3}\widehat{\varphi}(\tau^{1/3}(\xi-\pi/2)).$$

We define φ_τ as the inverse Fourier transform of $\widehat{\varphi}_\tau$. Observe that $\widehat{\varphi}_\tau$ is supported in the interval $(\pi/2-\tau^{-1/3},\pi/2+\tau^{-1/3})$ and $\int_{-\pi}^{\pi}\widehat{\varphi}_\tau=1$. Also using that $\varphi_\tau(x)=\varphi_1(\tau^{1/3}x)$ we get $\|\varphi_\tau\|_{L^p(\mathbb{R})}\simeq\tau^{-1/3p}$ for any $p\geq 1$.

The mean value theorem applied to the integral occurring in the right hand side of (3.19) shows that

$$|S_1(t)\varphi_\tau(x)|\geq\left(1-2\tau^{-1/3}\sup_{\xi\in\operatorname{supp}\widehat{\varphi}_\tau}|x-tp_1'(\xi)|\right)\int_{-\pi}^{\pi}\widehat{\varphi}_\tau(\xi)d\xi. \tag{3.24}$$

Using that the second derivative of p_1 vanishes at $\xi=\pi/2$ we obtain the existence of a positive constant c_1 such that

$$|x-tp_1'(\xi)|\leq|x-tp_1'(\pi/2)|+tc_1|\xi-\pi/2|^2,\ \xi\sim\pi/2.$$

In particular for all $\xi\in[\pi/2-\tau^{-1/3},\pi/2+\tau^{-1/3}]$ the following holds

$$|x-tp_1'(\xi)|\leq|x-tp_1'(\pi/2)|+tc_1\tau^{-2/3}.$$

Thus there exists a (small enough) positive constant c such that for all x and t satisfying $|x-tp_1'(\pi/2)|\leq c\tau^{1/3}$ and $t\leq c\tau$:

$$2\tau^{-1/3}\sup_{\xi\in\operatorname{supp}\widehat{\varphi}_\tau}|x-tp_1'(\xi)|\leq\frac{1}{2}.$$

This yields (3.23) and finishes the proof. □

3.2. **Lack of uniform local smoothing effect.** In order to analyze the local smoothing effect at the discrete level we introduce the discrete fractional derivatives on the lattice $h\mathbb{Z}^d$. We define for any $s\geq 0$, the fractional derivative $(-\Delta_h)^{s/2}u$ at the scale h as:

$$((-\Delta_h)^{s/2}u)_{\mathbf{j}}=\int_{[-\pi/h,\pi/h]^d}p_h^{s/2}(\xi)e^{i\mathbf{j}\cdot\xi h}\mathcal{F}_h(u)(\xi)d\xi,\ \mathbf{j}\in\mathbb{Z}^d \tag{3.25}$$

where $\mathcal{F}_h(u)$ is the SDFT of the sequence $\{u_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d}$ at the scale h .

Concerning the local smoothing effect we have the following result:

Theorem 3.2. *Let be $T>0$ and $s>0$. Then*

$$\sup_{h>0,\varphi\in l^2(h\mathbb{Z}^d)}\frac{h^d\sum\limits_{|\mathbf{j}|h\leq 1}|((-\Delta_h)^{s/2}S^h(T)\varphi)_{\mathbf{j}}|^2}{\|\varphi\|^2_{l^2(h\mathbb{Z}^d)}}=\infty \tag{3.26}$$

and

$$\sup_{h>0,\varphi\in l^2(h\mathbb{Z}^d)} \frac{h^d \sum_{|\mathbf{j}|h\leq 1} \int_0^T |((-\Delta_h)^{s/2} S^h(t)\varphi)_{\mathbf{j}}|^2 dt}{\|\varphi\|^2_{l^2(h\mathbb{Z}^d)}} = \infty. \tag{3.27}$$

In contrast with the proof of Theorem 3.1 we cannot reduce it to the one-dimensional case. This is due to the extra factor $p_h^{s/2}(\xi)$ which does not allow us to use separation of variables . The proof consists in reducing (3.26) and (3.27) to the case $h=1$ and the following lemma.

Lemma 3.2. *Let be $s>0$. There is a positive constant c such that for all τ sufficiently large there exists a function φ_τ with $\|\varphi_\tau\|_{l^2(\mathbb{Z}^d)} = \tau^{d/2}$ and*

$$|((-\Delta_1)^{s/2} S^1(t)\varphi_\tau)_{\mathbf{j}}| \geq 1/2 \tag{3.28}$$

for all $|t|\leq c\tau^2$, $|\mathbf{j}|\leq c\tau$.

We postpone the proof of Lemma 3.2 and proceed with the proof of Theorem 3.2.

Proof of Theorem 3.2. We consider the case of (3.26), the other being similar. As in the previous section we reduce the proof to the case $h=1$. By the definition of $(-\Delta_h)^{s/2}$ for any $\mathbf{j}\in\mathbb{Z}^d$ we have that

$$\Big((-\Delta_h)^{s/2} S^h(t)\varphi\Big)_{\mathbf{j}} = h^{-s}\Big((-\Delta_1)^{s/2} S^1\left(t/h^2\right)\varphi\Big)_{\mathbf{j}}, \quad \mathbf{j}\in\mathbb{Z}^d.$$

Thus

$$\frac{h^d \sum_{|\mathbf{j}|h\leq 1} |((-\Delta_h)^{s/2} S^h(T)\varphi)_{\mathbf{j}}|^2}{\|\varphi\|^2_{l^2(h\mathbb{Z}^d)}} = \frac{h^{-2s} \sum_{|\mathbf{j}|\leq 1/h} |((-\Delta_1)^{s/2} S^1(T/h^2)\varphi)_{\mathbf{j}}|^2}{\|\varphi\|^2_{l^2(\mathbb{Z}^d)}}.$$

With c and φ_τ given by Lemma 3.2 and τ such that $c\tau^2 = T/h^2$, i.e. $\tau = (T/c)^{1/2}h^{-1}$, we have $\|\varphi_\tau\|^2_{l^2(\mathbb{Z})} = \tau^d$ and

$$\lim_{\tau\to\infty} \frac{h^{-2s} \sum_{|\mathbf{j}|\leq 1/h} |((-\Delta_1)^{s/2} S^1(T/h^2)\varphi_\tau)_{\mathbf{j}}|^2}{\|\varphi_\tau\|^2_{l^2(\mathbb{Z}^d)}} \gtrsim \lim_{\tau\to\infty} \frac{\tau^{2s}\tau^d}{\tau^d} = \infty.$$

This finishes the proof. □

Proof of Lemma 3.2. As in the proof of Lemma 3.1 we choose a function $\widehat{\varphi}$ supported in the unit ball with $\int_{\mathbb{R}^d}\widehat{\varphi} = 1$. Set for all $\tau\geq 1$

$$\widehat{\varphi}_\tau(\xi) = \tau^d \widehat{\varphi}\left(\tau(\xi - \pi_d)\right),$$

where $\pi_d = (\pi,\dots,\pi)$. We define φ_τ as the inverse Fourier transform of $\widehat{\varphi}_\tau$. Thus $\widehat{\varphi}_\tau$ is supported in $\{\xi : |\xi-\pi_d|\leq\tau^{-1}\}$, $\int_{[-\pi,\pi]^d}\widehat{\varphi}_\tau = 1$ and $\|\varphi_\tau\|_{l^2(\mathbb{Z}^d)} \simeq \tau^{d/2}$. Applying mean value theorem to the oscillatory integral occurring in the definition of $(-\Delta_1)^{s/2}S^1(t)\varphi$ and using that $p_1(\xi)$ behaves as a positive constant in the support of $\widehat{\varphi}_\tau$ we obtain that for some positive constant c_0:

$$\begin{aligned}|((-\Delta_1)^{s/2}S^1(t)\varphi_\tau)_{\mathbf{j}}| &\geq \left(1-2\tau^{-1}\sup_{\xi\in\operatorname{supp}\widehat{\varphi}_\tau}|\mathbf{j}-t\nabla p_1(\xi)|\right)\int_{[-\pi,\pi]^d}p_1^{s/2}(\xi)\widehat{\varphi}_\tau(\xi)d\xi\\ &\geq c_0\left(1-2\tau^{-1}\sup_{\xi\in\operatorname{supp}\widehat{\varphi}_\tau}|\mathbf{j}-t\nabla p_1(\xi)|\right)\int_{[-\pi,\pi]^d}\widehat{\varphi}_\tau(\xi)d\xi.\end{aligned}$$

Using that ∇p_1 vanishes at $\xi=\pi_d$ we obtain the existence of a positive constant c_1 such that

$$|\mathbf{j}-t\nabla p_1(\xi)|\leq|\mathbf{j}|+tc_1|\xi-\pi_d|,\ \xi\sim\pi^d.$$

Then there exists a positive constant c such that for all $\mathbf{j}$ and t satisfying $|\mathbf{j}|\leq c\tau$ and $t\leq c\tau^2$ the following holds:

$$2\tau^{-1}\sup_{\xi\in\operatorname{supp}\widehat{\varphi}_\tau}|\mathbf{j}-t\nabla p_1(\xi)|\leq\frac{1}{2}.$$

Thus for all t and $\mathbf{j}$ as above (3.28) holds. This finishes the proof. □

3.3. **Filtering of the initial data.** As we have seen in the previous section the conservative scheme does not reflect the dispersive properties of the LSE. In this section we prove that a suitable filtering of the initial data in the Fourier space provides uniform dispersive properties and a local smoothing effect. The key point to recover the decay rates (1.3) at the discrete level is to choose initial data with their SDFT supported away from the pathological points

$$\mathcal{M}_1^h=\left\{\xi=(\xi_1,\dots,\xi_d)\in\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^d:\exists i\in\{1,\dots,d\}\text{ such that }\xi_i=\frac{\pi}{2h}\right\}$$

or

$$\mathcal{M}_2^h=\left\{\xi=(\xi_1,\dots,\xi_d)\in\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^d:\exists i\in\{1,\dots,d\}\text{ such that }\xi_i=\frac{\pi}{h}\right\}.$$

For any positive $\epsilon<\pi/2$ we define Ω_ϵ^h, the set of all the points inside the cube $[-\pi/h,\pi/h]^d$ whose distance is at least ϵ/h from the set where $p_h(\xi)$ vanishes at least one of its second order derivatives:

$$\Omega_{\epsilon,d}^h=\left\{\xi=(\xi_1,\dots,\xi_d)\in\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^d:\left|\xi_i\mp\frac{\pi}{2h}\right|\geq\frac{\epsilon}{h},\ i=1,\dots,d\right\}.$$

Let us define the class of functions $\mathcal{I}_\epsilon^h\subset l^2(h\mathbb{Z}^d)$, whose SDFT is supported on $\Omega_{\epsilon,d}^h$:

$$\mathcal{I}_{\epsilon,d}^h=\{\varphi\in l^2(h\mathbb{Z}^d):\ \operatorname{supp}\widehat{\varphi}\subset\Omega_{\epsilon,d}^h\}.\tag{3.29}$$

The following Theorem shows that for initial data that have been filtered in a convenient way, the semigroup $S^h(t)$ has the same long time behaviour as the continuous one and moreover this behaviour is independent of h.

Theorem 3.3. *Let be $0 < \epsilon < \pi/2$ and $p \geq 2$. There exists a positive constant $C(\epsilon, p, d)$ such that*

$$\|S^h(t)\varphi\|_{l^p(h\mathbb{Z}^d)} \leq \frac{C(\epsilon, p, d)}{|t|^{\frac{d}{2}\left(1-\frac{2}{p}\right)}} \|\varphi\|_{l^{p'}(h\mathbb{Z}^d)}, \ t \neq 0 \tag{3.30}$$

holds for all $\varphi \in l^{p'}(h\mathbb{Z}^d) \cap \mathcal{I}^h_{\epsilon,d}$, uniformly on $h > 0$.

Proof. A scaling argument reduces the proof to the case $h = 1$. For any $\varphi \in I_1^{\epsilon,d}$ the solution of (1.8) is given by $S^1(t)\varphi = K^{1,\epsilon} * \varphi$ where

$$K_d^{1,\epsilon}(t, \mathbf{j}) = \int_{\Omega^1_{\epsilon,d}} e^{itp_1(\xi)} e^{i\mathbf{j}\xi} d\xi, \ \mathbf{j} \in \mathbb{Z}^d.$$

As a consequence of Young's inequality it is sufficient to prove that

$$\|K_d^{1,\epsilon}(t)\|_{l^p(\mathbb{Z})} \leq c(\epsilon, d)|t|^{-d/2(1-1/p)} \tag{3.31}$$

for any $p \geq 2$ and for all $t \neq 0$. The case $p = 2$ easily follows by Plancherel's identity. We consider the case $p = \infty$, the others come by Hölder's inequality. Observe that for any $j = (j_1, \dots, j_d) \in \mathbb{Z}^d$ the following holds:

$$K_d^{1,\epsilon}(t, \mathbf{j}) = \prod_{l=1}^{d} K_1^{1,\epsilon}(t, j_l).$$

It is then sufficient to prove (3.31) in the one-dimensional case. Using that the second derivative of the function $\sin^2(\xi/2)$ is positive on $\Omega^1_{\epsilon,1}$ we obtain by the Van der Corput Lemma (Prop. 2, Ch. 8, p. 332, [36]) that $\|K_d^{1,\epsilon}(t)\|_{l^\infty} \leq c(\epsilon)|t|^{-1/2}$ which finishes the proof.

□

A similar result for the local smoothing effect can be stated. For a positive ϵ, let us define the set A^h_ϵ of all points situated at a distance of at least ϵ from the points $(\pm\pi/h)^d$:

$$A^h_\epsilon = \Big\{\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}]^d : \Big|\xi_i \mp \frac{\pi}{h}\Big| \geq \frac{\epsilon}{h}, \ i = 1, \dots, d\Big\}.$$

Observe that on A^h_ϵ the symbol $p_h(\xi)$ has no critical points different from the origin. A similar argument as in [24] shows that the linear semigroup $S^h(t)$ gains 1/2-space discrete derivative in $L^2_{t,x}$ with respect to the initial datum.

Theorem 3.4. *Let be $\epsilon > 0$. There exists a positive constant $C(\epsilon, d)$ such that for any $R > 0$*

$$h^d \sum_{|\mathbf{j}h| \leq R} \int_{-\infty}^{\infty} |(-\Delta_h)^{1/4} e^{it\Delta_h}\varphi)_{\mathbf{j}}|^2 dt \leq C(\epsilon, d) R \|\varphi\|^2_{l^2(h\mathbb{Z}^d)}$$

holds for all $\varphi \in l^2(h\mathbb{Z}^d)$ with $\operatorname{supp}\widehat{\varphi} \in A^h_\epsilon$, uniformly on $h > 0$.

3.4. **Strichartz estimates for filtered data.** In this section we derive Strichartz-like estimates for the operator $S^h(t)$ when it acts on functions belonging to $I^h_{\epsilon,d}$, the class of functions defined in (3.29).

The main ingredient in obtaining Strichartz estimates is the following result due to Keel and Tao, [23].

Proposition 3.1. *([23], Theorem 1.2) Let H be a Hilbert space, (X, dx) be a measure space and $U(t) : H \to L^2(X)$ be a one parameter family of mappings, which obey the energy estimate*

$$\|U(t)f\|_{L^2(X)} \leq C\|f\|_H \tag{3.32}$$

and the decay estimate

$$\|U(t)U(s)^*g\|_{L^\infty(X)} \leq C|t-s|^{-\sigma}\|g\|_{L^1(X)} \tag{3.33}$$

for some $\sigma > 0$. Then

$$\|U(t)f\|_{L^q(\mathbb{R},\, L^r(X))} \leq C\|f\|_{L^2(X)}, \tag{3.34}$$

$$\Big\| \int_{\mathbb{R}} (U(s))^* F(s,\cdot)ds \Big\|_{L^2(X)} \leq C\|F\|_{L^{q'}(\mathbb{R},\, L^{r'}(X))}, \tag{3.35}$$

$$\Big\| \int_0^t U(t)(U(s))^* F(s)ds \Big\|_{L^q(\mathbb{R},\, L^r(X))} \leq C\|F\|_{L^{\tilde{q}'}(\mathbb{R},\, L^{\tilde{r}'}(X))} \tag{3.36}$$

for all (q,r) and $(\tilde{q},\tilde{r})$, σ-admissible pairs.

Remark 3.2. *With the same arguments as in [23], the following also holds for all (q,r) and $(\tilde{q},\tilde{r})$, σ-admissible pairs:*

$$\Big\| \int_0^t U(t-s)F(s)ds \Big\|_{L^q(\mathbb{R},\, L^r(X))} \leq C\|F\|_{L^{\tilde{q}'}(\mathbb{R},\, L^{\tilde{r}'}(X))}. \tag{3.37}$$

In the case of the Schrödinger semigroup, $S(t-s) = S(t)S(s)^*$, so (3.37) and (3.36) are the same. In our applications we will often deal with operators that do not satisfy $S(t-s) = S(t)S(s)^*$.

Let us choose positive $\epsilon < \pi/2$, $K^{1,\epsilon}_d$ as in Theorem 3.3 and $U(t)\varphi = K^{1,\epsilon}_d * \varphi$. We apply the above proposition to $U(t)$, $X = \mathbb{Z}^d$, dx the counter measure and $H = l^2(\mathbb{Z}^d)$. We obtain Strichartz estimates for the semigroup $S^1(t)$ when acts on function belonging to $I^1_{\epsilon,d}$, i.e. when $h = 1$. Thus, by a scaling argument, we obtain the following result for filtered initial data.

Theorem 3.5. *Let $0 < \epsilon < \pi/2$ and (q,r), $(\tilde{q},\tilde{r})$ two $d/2$-admissible pairs.*
i) There exists a positive constant $C(d,r,\epsilon)$ such that

$$\|S^h(\cdot)\varphi\|_{L^q(\mathbb{R},\, l^r(h\mathbb{Z}^d))} \leq C(d,r,\epsilon)\|\varphi\|_{l^2(h\mathbb{Z}^d)} \tag{3.38}$$

holds for all functions $\varphi \in \mathcal{I}^h_{\epsilon,d}$ and for all $h > 0$.
ii) There exists a positive constant $C(d,r,\tilde{r},\epsilon)$ such that

$$\Big\| \int_0^t S^h(t-s)f(s)ds \Big\|_{L^q(\mathbb{R},\, l^r(h\mathbb{Z}^d))} \leq C(d,r,\tilde{r},\epsilon)\|f\|_{L^{\tilde{q}'}(\mathbb{R},\, l^{\tilde{r}'}(h\mathbb{Z}^d))} \tag{3.39}$$

holds for all functions $f \in L^{\tilde{q}'}(\mathbb{R}, l^{\tilde{r}'}(h\mathbb{Z}^d))$ *with* $f(t) \in \mathcal{I}^h_{\epsilon,d}$ *for all* $t \in \mathbb{R}$, *and for all* $h > 0$.

4. A dissipative scheme

In the previous section we analyzed the Fourier filtering method for the conservative scheme (1.8). Another possible remedy to the lack of dispersive estimates is to introduce a scheme containing a numerical viscosity term in order to compensate the artificial numerical dispersion.

We propose the following viscous semidiscretization of (1.1):

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = ia(h)\,\mathrm{sgn}(t)\Delta_h u^h,\ t \neq 0, \\ \\ u^h(0) = \varphi^h, \end{cases} \tag{4.40}$$

where $a(h)$ is a positive function which tends to zero as h tends to zero.

We remark that the proposed scheme is a combination of the conservative approximation of the Schrödinger equation (1.8) and a semi-discretization of the heat equation in a suitable time-scale:

$$\frac{du^h}{dt} = a(h)\Delta_h u^h,\ t > 0,$$

which may be viewed as a discretization of

$$u_t = a(h)\Delta u,\ t > 0.$$

The scheme (4.40) generates a semigroup of contractions in $l^2(h\mathbb{Z}^d)$, $S^h_+(t)$, for $t > 0$. Similarly one may define $S^h_-(t)$, for $t < 0$. In the sequel we denote by $S^h(t)$ the two operators.

In this Section 4.1 we will obtain norm decay estimates for the operator $S^h(t)$. We first analyze the $l^1(h\mathbb{Z}^d) - l^\infty(h\mathbb{Z}^d)$ decay of $S^h(t)$. In contrast with the continuous case where $\|u(t)\|_{L^\infty(\mathbb{R}^d)} \lesssim t^{-d/2}$ for all $t \neq 0$, the behaviour of the l^∞-norm of the solutions will be different when $t \to 0$ and when $t \to \infty$. The low frequency components determine the behaviour for large time t, similar to the continuous one $t^{-d/2}$. For $t \sim 0$, the behaviour is given by the high frequency components. Once the $l^1(h\mathbb{Z}^d) - l^\infty(h\mathbb{Z}^d)$ analysis will have been done, we will prove in Section 4.2 Strichartz-like estimates for the linear operator $S^h(t)$. Section 4.3 is devoted to the analysis of the local smoothing properties of the operator $S^h(t)$. Finally, in Section 4.4 we give an application to a nonlinear problem. We will consider a numerical scheme for NSE based on the dissipative scheme (4.40).

4.1. Dispersive estimates on the operator $S^h(t)$.

The following Theorem gives an estimate for the $l^\infty(h\mathbb{Z}^d)$ norm of the solutions of equation (4.40).

Theorem 4.1. *Let $\alpha > d/2$ and $a(h)$ be a positive function such that*

$$\inf_{h>0} \frac{a(h)}{h^{2-\frac{d}{\alpha}}} > 0. \tag{4.41}$$

There exist positive constants $c(d,\alpha)$ such that

$$\|S^h(t)\varphi\|_{l^\infty(h\mathbb{Z})} \leq c(d,\alpha)\left[\frac{1}{|t|^{\frac{d}{2}}} + \frac{1}{|t|^\alpha}\right]\|\varphi\|_{l^1(h\mathbb{Z}^d)}. \tag{4.42}$$

holds for all $t \neq 0$, $\varphi \in l^1(h\mathbb{Z}^d)$ and $h > 0$.

The decay of $S^h(t)$ for large time t is the same as in the continuous case. However, when looking for global Strichartz estimates the behavior at $t = 0$ also plays a role. According to (4.42) the behaviour at $t \sim 0$ is more singular since $\alpha > d/2$. It is condition (4.41) imposed on $a(h)$ the one that guarantees that the high frequency components of the fundamental solutions of (4.40) behave in $l^\infty(h\mathbb{Z}^d)$-norm as $|t|^{-\alpha}$ as $t \sim 0$.

Proof. Taking SDTF in (4.40) we obtain that, in Fourier variable $S^h(t)\varphi$ is given by:

$$\widehat{S^h(t)\varphi}(\xi) = \exp(-itp_h(\xi) - |t|a(h)p_h(\xi))\widehat{\varphi}(\xi).$$

Let us define the operators $S^{h,j}(t)\varphi = K^{h,j}(t) * \varphi$, $j = 1, 2$, where

$$K^{h,1}(t,\mathbf{j}) = \int_{[-\frac{\pi}{4h},\frac{\pi}{4h}]^d} e^{-itp_h(\xi)}e^{-|t|a(h)p_h(\xi)}e^{i\mathbf{j}\cdot\xi h}d\xi,\ \mathbf{j} \in \mathbb{Z}^d$$

and

$$K^{h,2}(t,\mathbf{j}) = \int_{\Omega_h} e^{-itp_h(\xi)}e^{-|t|a(h)p_h(\xi)}e^{i\mathbf{j}\cdot\xi h}d\xi,\ \mathbf{j} \in \mathbb{Z}^d,$$

Ω_h being defined as $\Omega_h = [-\pi/h, \pi/h]^d \setminus [-\pi/4h, \pi/4h]^d$. $S^{h,1}$ (respectively $S^{h,2}$) take account of the low (respectively high) frequency components.

We will prove that for some constant $c(d,\alpha)$, independent of h, the two operators satisfy:

$$\|S^{h,1}(t)\varphi\|_{l^\infty(h\mathbb{Z})} \leq \frac{c(d,\alpha)}{|t|^{d/2}}\|\varphi\|_{l^1(h\mathbb{Z}^d)}$$

and

$$\|S^{h,2}(t)\varphi\|_{l^\infty(h\mathbb{Z})} \leq \frac{c(d,\alpha)}{|t|^{\alpha}}\|\varphi\|_{l^1(h\mathbb{Z}^d)}.$$

This immediately implies (4.42).

Young's inequality reduces the proof of (4.42) to the following estimates on the two kernels:

$$|K^{h,1}(t)|_{l^\infty(h\mathbb{Z}^d)} \leq \frac{c(d,\alpha)}{|t|^{d/2}};\ \ |K^{h,2}(t)|_{l^\infty(h\mathbb{Z}^d)} \leq \frac{c(d,\alpha)}{|t|^{\alpha}}\ t \neq 0. \tag{4.43}$$

The kernel $K^{h,1}(t)$ behaves as the conservative kernel associated with (1.8) since the Hessian matrix $H_{p_h}(\xi) = (\partial_{ij} p_h(\xi))_{i,j=1}^d$ always has the rank d in $[-\pi/4h, \pi/4h]^d$. In other words, no artificial viscosity is needed for this low frequency range to ensure the optimal decay. To estimate the second kernel $K^{h,2}(t)$ we use in an essential way the dissipative effect introduced by the term $\exp(-|t|p_h(\xi))$ away from the origin $\xi \sim 0$. Note however that, for the second one, we obtain the decay rate $|t|^{-\alpha}$ instead of $|t|^{-d/2}$. The critical exponent $\alpha = d/2$ can not be reached by a viscous approximation of the LSE as in (4.40). Indeed, in view of (4.41) if one takes $\alpha = d/2$, (4.40) becomes an approximation of the viscous Schrödinger equation $iu_t + \Delta u = i\Delta u$.

The kernel $K^{h,2}$ satisfies for all $t \neq 0$ the following rough estimate:

$$\begin{aligned} |K^{h,2}(t)|_{l^\infty(h\mathbb{Z}^d)} &\leq \int_{\Omega_h} e^{-|t|a(h)p_h(\xi)} d\xi \leq \int_{\Omega_h} \exp\left(-4d\sin^2(\frac{\pi}{8})\frac{|t|a(h)}{h^2}\right) d\xi \\ &\leq \frac{c(\alpha,d)}{h^d}\left(\frac{h^2}{|t|a(h)}\right)^\alpha \leq \frac{c(\alpha,d)}{|t|^\alpha}\left[\inf_{h>0}\frac{a(h)}{h^{2-d/\alpha}}\right]^{-\alpha} \leq \frac{c(\alpha,d)}{|t|^\alpha}. \end{aligned}$$

Note that in the last inequality assumption (4.41) plays a key role.

Going back to $K^{h,1}$, it is convenient to rewrite it as:

$$K^{h,1}(t) = K^{h,3}(t) * \exp(|t|a(h)\Delta_h)$$

where $K^{h,3}(t,\cdot)$ is given by:

$$K^{h,3}(t,\mathbf{j}) = \int_{[-\frac{\pi}{4h},\frac{\pi}{4h}]^d} e^{-itp_h(\xi)} e^{i\mathbf{j}\cdot\xi h} d\xi,\ \mathbf{j} \in \mathbb{Z}^d, \tag{4.44}$$

i.e. the conservative semidiscrete kernel restricted to the frequency set $[-\pi/4h, \pi/4h]^d$.

We recall that the operator $\exp(|t|\Delta_h)$ is a contraction in $l^1(h\mathbb{Z}^d)$ (see for instance [12], Theorem 1.3.3, p. 14):

$$\|\exp(|t|\Delta_h)\|_{l^1(h\mathbb{Z}^d)-l^1(h\mathbb{Z}^d)} \leq 1. \tag{4.45}$$

Applying Theorem 3.3 and (4.45) we obtain that $\|K^{h,1}(t)\|_{l^\infty(h\mathbb{Z}^d)} \leq c(d)|t|^{-d/2}$, which finishes the proof. □

4.2. **Strichartz like Estimates.** In this section we derive space-time estimates for the linear operator $S^h(t)$. The estimates are different from the ones obtained in the continuous case, the behaviour of the semigroup as $t \to \infty$ and $t \to 0$ being different, as we have seen above. As a consequence, our estimates for $S^h(t)$ will be given in spaces of the form $L^q(\mathbb{R}, l^r(h\mathbb{Z}^d)) + L^{q_1}(\mathbb{R}, l^r(h\mathbb{Z}^d))$. More precisely the term $|t|^{-d/2}$ in the $l^\infty(h\mathbb{Z}^d)$-norm of $S^h(t)$ gives us estimates in the space $L^q(\mathbb{R}, l^r(h\mathbb{Z}^d))$ with (q,r) an $d/2$-admissible pair. The second one, $|t|^{-\alpha}$, provides estimates in the space $L^{q_1}(\mathbb{R}, l^r(h\mathbb{Z}^d))$ with (q_1,r) an α-admissible pair.

We recall that the Strichartz estimates are used to prove the local existence for the nonlinear problem. So, a local version of them suffices to prove the local well possedness of the nonlinear problem. Using that $\alpha > d/2$ and $q > q_1$, the global ones will provide local estimates in the space $L^{q_1}(I, l^r(h\mathbb{Z}^d))$, I being a bounded interval. This fails on unbounded time intervals, where the L^q-norm cannot be compared to the L^{q_1} one.

Theorem 4.2. *Let $\alpha \in (d/2, d]$ and $a(h)$ be satisfying (4.41). Also let us consider $d/2$-admissible pairs (q, r), $(\tilde{q}, \tilde{r})$ and α-admissible pairs (q_1, r), $(\tilde{q}_1, \tilde{r})$.*
i) There exists a positive constant $C(d, \alpha, r)$ such that

$$\|S^h(\cdot)\varphi\|_{L^q(\mathbb{R}, l^r(h\mathbb{Z}^d))+L^{q_1}(\mathbb{R}, l^r(h\mathbb{Z}^d))} \leq C(d, \alpha, r)\|\varphi\|_{l^2(h\mathbb{Z}^d)} \tag{4.46}$$

holds for all $\varphi \in l^2(h\mathbb{Z}^d)$, uniformly on $h > 0$.
ii) There exists a positive constant $C = C(d, \alpha, r, r_1)$ such that

$$\Big\| \int_0^t S^h(t-s)f(s)ds\Big\|_{L^q(\mathbb{R},l^r(h\mathbb{Z}^d))+L^{q_1}(\mathbb{R},l^r(h\mathbb{Z}^d))} \leq C\|f\|_{L^{\tilde{q}'}(\mathbb{R},l^{\tilde{r}'}(h\mathbb{Z}^d))\cap L^{\tilde{q}_1{}'}(\mathbb{R},l^{\tilde{r}'}(h\mathbb{Z}^d))}$$

holds for all $f \in L^{\tilde{q}'}(\mathbb{R}, l^{\tilde{r}'}(h\mathbb{Z}^d)) \cap L^{\tilde{q}_1{}'}(\mathbb{R}, l^{\tilde{r}'}(h\mathbb{Z}^d))$, uniformly on $h > 0$.

The following Corollary represents a simple consequence of the above Theorem.

Corollary 4.1. *Let I be a bounded interval, (q, r) and $(\tilde{q}, \tilde{r})$, $1/2$-admissible pairs and (q_1, r) and $(\tilde{q}_1, \tilde{r})$ α-admissible ones. Then*

i) There exists a positive constant $C = C(I, d, \alpha, r)$ such that

$$\|S^h(t)\varphi\|_{L^{q_1}(I, l^r(h\mathbb{Z}^d))} \leq C\|\varphi\|_{l^2(h\mathbb{Z}^d)}. \tag{4.47}$$

ii) There exists a positive constant $C = C(I, d, \alpha, r, r_1)$ such that

$$\Big\| \int_0^t S^h(t-s)f(s)ds\Big\|_{L^{q_1}(I, l^r(h\mathbb{Z}^d))} \leq C\|f\|_{L^{\tilde{q}_1{}'}(I, l^{\tilde{r}'}(h\mathbb{Z}^d))}. \tag{4.48}$$

Proof of Theorem 4.2. We write the semigroup $S^h(t)$ as in the proof of Theorem 4.1: $S^h(t) = S^{h,1}(t) + S^{h,2}(t)$. Observe that $S^{h,1}(t)$ and $S^{h,2}(t)$ satisfy the hypothesis of Proposition 3.1 with $\sigma = d/2$ and $\sigma = \alpha$ respectively. Applying Proposition 3.1 to each of the operators $S^{h,1}(t)$ and $S^{h,2}(t)$ we obtain the desired result on $S^h(t)$. □

Once Theorem 4.2 has been proved, Corollary 4.1 follows by using only the definition of the sum spaces involved in Theorem 4.2 and Hölder inequality.

4.3. **Local smoothing effect.** As we mentioned in the introduction the local smoothing property is very useful in proving the convergence in the nonlinear context. In this section we consider the piecewise linear and continuous interpolator $\mathbf{P}_1^h$ and we analyze the local smoothing property of $\mathbf{P}_1^h S^h(t)$. This result will be applied later in Section 4.5 to provie the convergence of a numerical scheme for NSE.

We use the piecewise lineal interpolator instead of a piecewise constant one since the last one does not belong to $H^{1/2}_{loc}(\mathbb{R}^d)$, having less regularity than the continuous Schrödinger semigroup. The following Theorem concerns the local smoothing property of $\mathbf{P}_1^h S^h(t)$.

Theorem 4.3. *Let I be a bounded interval, $\alpha \in (d/2, d]$ and $\chi \in C_c^\infty(\mathbb{R}^d)$. Then*
i) There exists a positive constant $C(I,\chi)$ such that the following

$$\int_I \int_{\mathbb{R}^d} \chi^2 |(\mathrm{I}-\Delta)^{1/4} \mathbf{P}_1^h S^h(t)\varphi|^2 ds dt \leq C(I,\chi) \|\varphi\|_{l^2(h\mathbb{Z}^d)} \tag{4.49}$$

holds for all $\varphi \in l^2(h\mathbb{Z}^d)$ and $h>0$.
ii)There exists a positive constant $C(I,\chi)$ such that the following

$$\int_I \int_{\mathbb{R}^d} \chi^2 \Big|(\mathrm{I}-\Delta)^{1/4} \mathbf{P}_1^h \Big(\int_0^t S^h(t-s) f(s) ds\Big)\Big|^2 dx dt \leq C(I,\chi) \|f\|^2_{L^1(I,\, l^2(h\mathbb{Z}^d))} \tag{4.50}$$

holds for all $f \in L^1(I,\, l^2(h\mathbb{Z}^d))$ and $h>0$.
iii) Let (q,r) be an α-admissible pair such that $(q,r) \neq (2, 2\alpha/(\alpha-1))$. Then there is a positive constant $s = s(r,d)$ such that the following

$$\int_I \int_{\mathbb{R}^d} \chi^2 \Big|(\mathrm{I}-\Delta)^{s} \mathbf{P}_1^h \Big(\int_0^t S^h(t-s) f(s) ds\Big)\Big|^2 dx dt \leq C(I,\chi) \|f\|^2_{L^{q'}(I,\, l^{r'}(h\mathbb{Z}^d))} \tag{4.51}$$

holds for all $f \in L^{q'}(I,\, l^{r'}(h\mathbb{Z}^d))$ and $h>0$.

In the continuous case, estimate (4.51) holds for $s(r) = 1/4$. The homogenous case has been proved by Kenig, Ponce and Vega [24]. The inhomogeneous case is reduced to the homogenous one by using the results of Christ and Kiselev [9] and Strichartz estimates.

In our case the arguments of [9] can not be applied. The key point in their proof is that the Schrödinger semigroup satisfies $S(t-s) = S(t)S(s)^*$ for all reals t and s, identity which does not hold in our case, the operator $S^h(t)S^h(s)^*$ being more dissipative than $S^h(t-s)$.

Proof. We divide the proof in three steps, each one corresponds to the one of estimates (4.49), (4.50) and (4.51).

Step I. Proof of (4.49)**.** Let us write $\mathbf{P}_1^h S^h(t)\varphi$ as $\mathbf{P}_1^h S^h(t)\varphi = I_1\varphi(t) + I_2\varphi(t)$, where

$$\widehat{I_1\varphi(t)}(\xi) = \mathbf{1}_{\{|\xi| \leq \pi/2h\}} (\mathbf{P}_1^h S^h(t)\varphi)\widehat{\ }(\xi)$$

and

$$\widehat{I_1\varphi(t)}(\xi) = \mathbf{1}_{\{|\xi| > \pi/2h\}} (\mathbf{P}_1^h S^h(t)\varphi)\widehat{\ }(\xi).$$

We then define $I_1\varphi(t)$ and $I_2\varphi(t)$ by inverting the Fourier transform.

We will prove that, for any $R>0$, the two terms satisfy the following inequalities

$$\int_{|x|<R}\int_{-\infty}^{\infty}|(-\Delta)^{1/4}I_1\varphi|^2dtdx\le C(R)\|\varphi\|^2_{l^2(h\mathbb{Z}^d)}$$

and

$$\int_{\mathbb{R}^d}\int_{-\infty}^{\infty}|(-\Delta)^{d/4\alpha}I_2\varphi|^2dtdx\le C(R)\|\varphi\|^2_{l^2(h\mathbb{Z}^d)}.$$

Finally taking into account that $\alpha\le d$ we obtain (4.49).

Case a). Estimates on $I_1\varphi$. By definition

$$(I_1\varphi)(t,x)=\int_{|\xi|\le\pi/2h}e^{-itp_h(\xi)}e^{-|t|a(h)p_h(\xi)}e^{ix\xi}\widehat{\mathbf{P}_1^h\varphi}(\xi)d\xi.$$

We will reduce the estimates on $I_1\varphi$ to those of $J\varphi$, where $J\varphi$ is defined by

$$(J\varphi)(t,x)=\int_{|\xi|\le\pi/2h}e^{-itp_h(\xi)}e^{ix\xi}\widehat{\mathbf{P}_1^h\varphi}(\xi)d\xi.$$

Defining $\Psi(t,x)$ as follows

$$\Psi(t,x)=\sup_{s\ge 0}\left|\int_{|\xi|\le\pi/2h}|\xi|^{1/2}e^{-itp_h(\xi)}e^{-sp_h(\xi)}e^{ix\xi}\widehat{\mathbf{P}_1^h\varphi}(\xi)d\xi\right|$$

the following

$$|(-\Delta)^{1/4}(I_1\varphi)(t,x)|\le\Psi(t,x)$$

holds for any t and x. Classical properties of Poisson's integrals ([35], Th. 1, p. 62, Ch. III) shows that the function Ψ satisfies

$$\|\Psi(\cdot,x)\|_{L^2(\mathbb{R}_t)}\lesssim\|(-\Delta)^{1/4}J\varphi(\cdot,x)\|_{L^2(\mathbb{R}_t)}.$$

It remains to prove that $J\varphi$ satisfies

$$\int_{|x|<R}\int_{-\infty}^{\infty}|(-\Delta)^{1/4}J\varphi(t,x)|^2dtdx\lesssim C(R)\|\mathbf{P}_1^h\varphi\|_{L^2(\mathbb{R}^d)}. \tag{4.52}$$

To prove the last inequality we make use of the following Lemma.

Lemma 4.1. *(Theorem 4.1, [24]) Let $\mathcal{O}$ be an open set in $\mathbb{R}^d_\xi$, and ψ be a $C^1(\mathcal{O})$ function such that $\nabla\psi(\xi)\neq 0$ for any $\xi\in\mathcal{O}$. Assume that there is $N\in\mathbb{N}$ such that for any $(\xi_1,\dots,\xi_{d-1})\in\mathbb{R}^{d-1}_\xi$ and $r\in\mathbb{R}$ the equations*

$$\psi(\xi_1,\dots,\xi_k,\underline{\xi},\xi_{k+1},\dots,\xi_{d-1})=r,\ \ k=0,\dots,d-1,$$

have at most N solutions $\underline{\xi}\in\mathbb{R}$. For $a\in L^\infty(\mathbb{R}^d\times\mathbb{R})$ and $f\in\mathcal{S}(\mathbb{R}^d)$ define

$$W(t)f(x)=\int_{\mathcal{O}}e^{i(t\psi(\xi)+x\xi)}a(x,\psi(\xi))\widehat{f}(\xi)d\xi;$$

then for any $R>0$

$$\int_{|x|\leq R}\int_{-\infty}^{\infty}|W(t)f(x)|^2dtdx\leq cRN\int_{\mathcal{O}}\frac{|\widehat{f}(\xi)|^2}{|\nabla\psi(\xi)|}d\xi$$

where c is independent of R and N and f.

Applying this lemma with $\mathcal{O}=\{\xi\in\mathbb{R}^d:|\xi|\leq\pi/4h\}$, $W=J$ and $\psi=p_h$ we obtain that

$$\int_{|x|<R}\int_{-\infty}^{\infty}|J\varphi(t,x)|^2dtdx\leq CR\int_{|\xi|\leq\pi/2h}\frac{|\widehat{\mathbf{P}_1^h\varphi}(\xi)|^2}{|\nabla p_h(\xi)|}d\xi\lesssim R\int_{|\xi|\leq\pi/2h}\frac{|\widehat{\mathbf{P}_1^h\varphi}(\xi)|^2}{|\xi|}d\xi,$$

which proves (4.52).

Case b). Estimates on $I_2\varphi$. Using in essential manner the assumption (4.41) on the function $a(h)$, the term $I_2\varphi$ satisfies

$$\begin{aligned}\int_{\mathbb{R}^d}|(-\Delta)^{d/4\alpha}I_2\varphi(t,x)|^2dx&\lesssim h^{2-d/\alpha}\int_{|\xi|\geq\frac{\pi}{2h}}|\xi|^2|\widehat{\mathbf{P}_1^hS^h(t)}\varphi|^2d\xi\\&\lesssim a(h)\|\nabla(\mathbf{P}_1^hS^h(t)\varphi)\|^2_{L^2(\mathbb{R}^d)}=a(h)\|\nabla_hS^h(t)\varphi\|^2_{l^2(h\mathbb{Z}^d)}\\&=a(h)\int_{[-\pi/h,\pi/h]^d}p_h(\xi)e^{-2ta(h)p_h(\xi)}|\widehat{\varphi}(\xi)|^2d\xi.\end{aligned}$$

Integrating the last inequality on time we get

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d}|(-\Delta)^{d/4\alpha}I_2\varphi|^2dxdt\lesssim\|\varphi\|^2_{l^2(h\mathbb{Z}^d)}.$$

Step II. Proof of (4.50)**.** Let us denote

$$\Psi_f=\int_0^tIS^h(t-s)f(s)ds.$$

Without loss of generality we consider $I=(0,T)$. For any $\chi\in C_c^\infty(\mathbb{R}^d)$ we have

$$\|\chi\Psi_f(t)\|_{H^{1/2}(\mathbb{R}^d)}\leq\int_0^tg(t,s)ds,\tag{4.53}$$

where $g(t,s)=\|\chi\mathbf{P}_1^hS^h(t-s)f(s)\|_{H^{1/2}(\mathbb{R}^d)}$.

Integrating inequality (4.53) on time variable t we obtain

$$\begin{aligned}\|\chi\Psi_f\|_{L^2((0,T),\,H^{1/2}(\mathbb{R}^d)}&=\Big\|\int_0^T\mathbf{1}_{(0,t)}(s)g(t,s)ds\Big\|_{L^2_t((0,T))}\\&\leq\int_0^T\|\mathbf{1}_{(s,T)}(t)g(t,s)\|_{L^2_t((0,T))}ds.\end{aligned}$$

Using (4.49) on the homogenous term we have

$$\begin{aligned}\|\mathbf{1}_{(s,T)}(t)g(t,s)\|^2_{L^2_t((0,T))} &= \int_s^T |g(t,s)|^2 dt = \int_s^T \|\chi \mathbf{P}_1^h S^h(t-s)f(s)\|^2_{H^{1/2}(\mathbb{R}^d)}dt\\ &\leq C(T,\chi)\|f(s)\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Integrating on $t\in(0,T)$ the last inequality we obtain

$$\|\chi\Psi_f\|_{L^2((0,T),H^{1/2}(\mathbb{R}^d))}\leq C(T,\chi)\|f\|_{L^1((0,T),\,L^2(\mathbb{R}^d))}.$$

Step III. Proof of (4.51)**.** Estimate (4.51) follows by interpolation of (4.50) and the Strichartz estimate (4.48) applied for a suitable α-admissible pair (q_1,r_1). More precisely, by (4.48)

$$\int_I\int_{\mathbb{R}^d}\chi^2\Big|\mathbf{P}_1^h\Big(\int_0^t S^h(t-s)f(s)ds\Big)\Big|^2dxdt\leq C(I,\chi)\|f\|^2_{L^{q_1'}(I,l^{r_1'}(h\mathbb{Z}^d))}. \tag{4.54}$$

for any α-admisible pair (q_1,r_1). Using the fact that our estimates do not involve the endpoint $(2,2\alpha/(\alpha-1))$ we apply (4.54) with $(q_1,r_1)=(2,2\alpha/(\alpha-1))$, α-admissible pair. An interpolation between (4.50) and (4.54) gives us the existence of a positive constant $s(r,d)$, independent of h, such that (4.51) is satisfied. □

4.4. **Application to a Nonlinear Problem.** We concentrate on the semilinear NSE equation in $\mathbb{R}^d$:

$$\begin{cases} iu_t+\Delta u=|u|^pu,\ t>0,\\ u(0,x)=\varphi(x),\ x\in\mathbb{R}^d,\end{cases} \tag{4.55}$$

the case when nonlinearity is given by $f(u)=-|u|^pu$ being the same. In fact, the key point in the global existence of the solutions is that the L^2-scalar product $(f(u),u)$ is a real number. All the results extend to more general nonlinearities $f(u)$ (see [6], Ch. 4.6, p. 109, for L^2-solutions).

The first existence and uniqueness result for (4.55) for L^2 initial data is as follows.

Theorem 4.4. *(Global existence in $L^2(\mathbb{R}^d)$, Tsutsumi, [40]). For $0\leq p<4/d$ and $\varphi\in L^2(\mathbb{R}^d)$, there exists a unique solution u in $C(\mathbb{R},\,L^2(\mathbb{R}^d))\cap L^q_{loc}(\mathbb{R},\,L^{p+2}(\mathbb{R}^d))$ with $q=4(p+1)/pd$ that satisfies the L^2-norm conservation property and depends continuously on the initial condition in $L^2(\mathbb{R}^d)$.*

The proof uss standard arguments, the key ingredient been to work in the space $C(\mathbb{R},L^2\mathbb{R}^d)\cap L^q_{loc}(\mathbb{R},L^{p+2}(\mathbb{R}^d))$ which requires and is intimately related to the Strichartz estimates.

Local existence is proved by applying a fixed point argument to the integral formulation of (4.55). Global existence holds because of the $L^2(\mathbb{R}^d)$-conservation property which excludes finite-time blow-up.

In order to introduce a numerical approximation of equation (4.55) it is convenient to give the definition of the weak solution of equation (4.55).

Definition 4.1. *We say that u is a weak solution of (4.55) if*
i) $u \in C(\mathbb{R},\, L^2(\mathbb{R}^d)) \cap L^q_{loc}(\mathbb{R},\, L^{p+2}(\mathbb{R}^d))$
ii) $u(0) = \varphi$ *a.e. and*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} u(-i\psi_t + \Delta\psi)dxdt = \int_{\mathbb{R}}\int_{\mathbb{R}^d} |u|^p u\psi dxdt \tag{4.56}$$

for all $\psi \in \mathcal{D}(\mathbb{R},\, H^2(\mathbb{R}^d))$*, where p and q are as in the statement of Theorem 4.4.*

In this section we consider the following viscous numerical approximation scheme of (4.55):

$$\left\{\begin{array}{rcl} i\dfrac{du^h}{dt} + \Delta_h u^h & = & i\,\mathrm{sgn}(t)a(h)\Delta_h u^h + |u^h|^p u^h, \quad t \neq 0, \\ u^h(0) & = & \varphi^h, \end{array}\right. \tag{4.57}$$

with $0 < p < 4/d$ and $a(h) = h^{2-d/\alpha(h)}$ such that $\alpha(h) \downarrow d/2$ and $a(h) \to 0$ as $h \downarrow 0$. The critical case $p = 4/d$ will be analyzed in Section 4.6.

The main result on the convergence of (4.57) in the subcritical case $p < 4/d$ is the following:

Theorem 4.5. *Let $p \in (0, 4/d)$ and $\alpha(h) \in (d/2, 2/p)$. Set*

$$\frac{1}{q(h)} = \alpha(h)\left(\frac{1}{2} - \frac{1}{p+2}\right)$$

so that $(q(h), p+2)$ is an $\alpha(h)$-admissible pair. Then for any $h > 0$ and every $\varphi^h \in l^2(h\mathbb{Z}^d)$, there exists a unique global solution $u^h \in C(\mathbb{R},\, l^2(h\mathbb{Z}^d)) \cap L^{q(h)}_{loc}(\mathbb{R},\, l^{p+2}(h\mathbb{Z}^d))$ of (4.57). Moreover, u^h satisfies

$$\|u^h\|_{L^\infty(\mathbb{R},\, l^2(h\mathbb{Z}^d))} \leq \|\varphi^h\|_{l^2(h\mathbb{Z}^d)} \tag{4.58}$$

and for any finite interval I

$$\|u^h\|_{L^{q(h)}(I,\, l^{p+2}(h\mathbb{Z}^d))} \leq c(I)\|\varphi^h\|_{l^2(h\mathbb{Z}^d)} \tag{4.59}$$

where the above constant is independent of h.

The restriction $\alpha(h) < 2/p$, guarantees that $q(h) > p + 2$. The condition $q(h) > p + 2$ is always satisfied in the subcritical case $p < 4/d$ and allows us to apply Banach's fix point theorem for small time T. In the critical case $p = 4/d$, this condition is not fulfilled and additional hypotheses on the initial data have to be imposed (see Section 4.6).

Proof. Let us choose T and M positive. We consider the metric space

$$\begin{aligned} E_h = \{u \in L^\infty((-T,T),\, l^2(h\mathbb{Z}^d)) \cap L^{q(h)}((-T,T),\, l^{p+2}(h\mathbb{Z}^d)), \\ \|u\|_{L^\infty((-T,T),\, l^2(h\mathbb{Z}^d))} + \|u\|_{L^{q(h)}((-T,T),\, l^{p+2}(h\mathbb{Z}^d))} \leq M\}, \end{aligned}$$

equipped with the distance

$$d(u,v)=\|u-v\|_{L^\infty((-T,T),\,l^2(h\mathbb{Z}^d))}+\|u-v\|_{L^{q(h)}((-T,T),\,l^{p+2}(h\mathbb{Z}^d))}.$$

We also consider the nonlinear map

$$\mathcal{H}^h(u)(t)=S^h(t)\varphi^h+i\int_0^t S^h(t-s)|u|^pu(s)ds.$$

The Strichartz-like estimates for the $\alpha(h)$-admissible pair $(q(h),p+2)$, given by Corollary 4.1, allow us proving that for small enough T, independent of h, and $M=2\|\varphi^h\|_{l^2(h\mathbb{Z}^d)}$, $\mathcal{H}^h(u)$ is a contraction on E_h. Thus we obtain the local existence and uniqueness of the solutions and estimates (4.58) and (4.59). To prove the global existence of the solutions we observe that the $l^2(h\mathbb{Z}^d)$-norm of the solutions remains uniformly bounded:

$$\frac{d}{dt}\|u^h(t)\|^2_{l^2(h\mathbb{Z}^d)}=2a(h)\mathrm{Re}\Big(\sum_{\mathbf{j}\in\mathbb{Z}^d}(\Delta_hu^h)_{\mathbf{j}}\overline{u}^h_{\mathbf{j}}\Big)\le 0.$$

Here and in the sequel Re denotes the real part of a complex number. □

4.5. **Convergence of the method.** Let us consider the piecewise constant interpolator $\mathbf{P}_0^hu^h$. This choice is motivated by the fact that it commutes with the nonlinearity. Let $\varphi\in L^2(\mathbb{R}^d)$ and φ^h such that $\mathbf{P}_0^h\varphi^h\to\varphi$ strongly in $L^2(\mathbb{R}^d)$. Clearly $\|\mathbf{P}_0^h\varphi^h\|_{L^2(\mathbb{R}^d)}\le C(\|\varphi\|_{L^2(\mathbb{R}^d)})$. Theorem 4.5 shows that $\|\mathbf{P}_0^hu^h\|_{L^\infty(\mathbb{R},\,L^2(\mathbb{R}^d))}\le C$. Moreover for any finite interval I:

$$\|\mathbf{P}_0^hu^h\|_{L^{q(h)}(I,\,L^{p+2}(\mathbb{R}^d))}\le C(I,\|\varphi\|_{L^2(\mathbb{R}^d)}), \tag{4.60}$$

and

$$\||\mathbf{P}_0^hu^h|^pEu^h\|_{L^{q'}(I,\,L^{(p+2)'}(\mathbb{R}^d))}\le C(I,\|\varphi\|_{L^2(\mathbb{R}^d)}). \tag{4.61}$$

Multiplying (4.57) by a test function $\psi\in C_c^\infty(\mathbb{R}^{d+1})$ we obtain that Eu^h satisfies

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{P}_0^hu^h(-i\psi_t+\Delta_h\psi)dxdt=&\int_{\mathbb{R}}\int_{\mathbb{R}^d}|\mathbf{P}_0^hu^h|^p\mathbf{P}_0^hu^h\psi dxdt\\ &+a(h)\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathrm{sgn}(t)\mathbf{P}_0^hu^h\Delta^h\psi dxdt.\end{aligned} \tag{4.62}$$

These uniform estimates and the regularity property proved in the previous section allow us proving the following result on the convergence of the scheme.

Theorem 4.6. *The sequence* $\mathbf{P}_0^hu^h$ *satisfies*

$$\mathbf{P}_0^hu^h\overset{\star}{\rightharpoonup}u \text{ in } L^\infty(\mathbb{R},L^2(\mathbb{R}^d)),\ \mathbf{P}_0^hu^h\rightharpoonup u\text{ in } L^s_{loc}(\mathbb{R},L^{p+2}(\mathbb{R}^d)),\forall\, s<q,$$

$$\mathbf{P}_0^hu^h\to u\text{ in } L^2_{loc}(\mathbb{R}\times\mathbb{R}^d),\ |\mathbf{P}_0^hu^h|^p|\mathbf{P}_0^hu^h|\rightharpoonup|u|^pu\text{ in } L^{q'}_{loc}(\mathbb{R},L^{(p+2)'}(\mathbb{R}^d))$$

where u *is the unique weak solution of NSE.*

Proof. In view of estimate (4.58) there exists a function $u \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))$ such that $\mathbf{P}_0^h u^h \overset{*}{\rightharpoonup} u$ in $L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))$.

Let us choose an $s < q$. For h sufficiently small, $s \le q(h) < q$. Estimate (4.60) and Hölder's inequality show that $\mathbf{P}_0^h u^h$ is uniformly bounded in $L^s(I, L^{p+2}(\mathbb{R}^d))$. This implies that $u \in L^s(I, L^{p+2}(\mathbb{R}^d))$, $\mathbf{P}_0^h u^h \rightharpoonup u$ in $L^s(I, L^{p+2}(\mathbb{R}^d))$ and

$$\|u\|_{L^s(I,\, L^{p+2}(\mathbb{R}^d))} \le \liminf_h \|\mathbf{P}_0^h u^h\|_{L^{q(h)}(I,\, L^{p+2}(\mathbb{R}^d))} \le C(I, \|\varphi\|_{L^2(\mathbb{R}^d)}).$$

Fatou's Lemma shows that $u \in L^q(\mathbb{R}, L^{p+2}(\mathbb{R}^d))$.

In the following we prove the existence of a function v such that $\mathbf{P}_0^h u^h \to v$ in $L^2_{loc}(\mathbb{R}^{1+d})$ and then $\mathbf{P}_0^h u^h \to v$ almost everywhere. This allows us to pass to the limit in the nonlinear term. To do that we consider the piecewise linear interpolator I and prove that Iu^h converges strongly in $L^2_{loc}(\mathbb{R}^{d+1})$ to a function v. Finally, we will transfer the strong convergence of $\mathbf{P}_1^h u^h$ to $\mathbf{P}_0^h u^h$ by proving that $\mathbf{P}_1^h u^h - \mathbf{P}_0^h u^h$ tends to zero in $L^2(\mathbb{R}^{d+1})$.

We proceed with the proof of the strong convergence of $\mathbf{P}_1^h u^h$. Let us consider a bounded interval $I \subset \mathbb{R}$ and a bounded domain $\Omega \subset \mathbb{R}^d$. Theorem 4.3 gives us the existence of a positive s, independent of h, such that $\|\mathbf{P}_1^h u^h\|_{L^2(I,\, H^s(\Omega))} \le C(I, \Omega, \|\varphi\|_{L^2(\mathbb{R}^d)})$. We also have the uniform boundedness of its time derivative:

$$\begin{aligned}
\left\| \frac{d\mathbf{P}_1^h u^h}{dt} \right\|_{L^1(I,H^{-2}(\mathbb{R}^d))} &\le \|\Delta_h \mathbf{P}_1^h u^h\|_{L^1(I,H^{-2}(\mathbb{R}^d))} + \|\mathbf{P}_1^h(|u^h|^p u^h)\|_{L^1(I,H^{-2}(\mathbb{R}^d))} \\
&\le \|\mathbf{P}_1^h u^h\|_{L^1(I,\, L^2(\mathbb{R}^d))} + \|\mathbf{P}_1^h(|u^h|^p u^h)\|_{L^1(I,\, L^{(p+2)'}(\mathbb{R}^d))} \\
&\le C(I, \|\varphi\|_{L^2(\mathbb{R}^d)}).
\end{aligned}$$

Using the embeddings $H^s(\Omega) \underset{comp}{\hookrightarrow} L^2(\Omega) \hookrightarrow H^{-2}(\Omega)$ and the compactness results of [31] we obtain the existence of a function v such that $\mathbf{P}_1^h u^h \to v$ in $L^2_{loc}(\mathbb{R} \times \mathbb{R}^d)$.

In the following we prove that $\mathbf{P}_1^h u^h - \mathbf{P}_0^h u^h \to 0$ in $L^2(\mathbb{R}^{d+1})$. Classical result on interpolation ([30], Th. 3.1.5, p. 122) give us that

$$\begin{aligned}
\int_{\mathbb{R}^d} |\mathbf{P}_1^h u^h(t) - \mathbf{P}_0^h u^h(t)|^2 dx &\le h^2 \|\nabla_h u^h(t)\|^2_{l^2(h\mathbb{Z}^d)} \\
&= h^2 \int_{[-\pi/h,\pi/h]^d} p_h(\xi) e^{-2|t|p_h(\xi)a(h)} |\widehat{\varphi}^h(\xi)|^2 d\xi.
\end{aligned}$$

Integrating the last inequality on time and using that $\alpha(h) \to 1/2$ as $h \to 0$ we obtain

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |\mathbf{P}_1^h u^h(t) - \mathbf{P}_0^h u^h(t)|^2 dx dt \le \frac{h^2}{a(h)} C(\|\varphi\|_{L^2(\mathbb{R}^d)}) = h^{1/\alpha(h)} C(\|\varphi\|_{L^2(\mathbb{R}^d)}) \to 0.$$

The strong convergence $\mathbf{P}_1^h u^h - \mathbf{P}_0^h u^h \to 0$ in $L^2(\mathbb{R}^{d+1})$ shows that $u = v$ and $\mathbf{P}_0^h u^h \to u$ in $L^2_{loc}(\mathbb{R}\times\mathbb{R}^d)$. Moreover, up to a subsequence $\mathbf{P}_0^h u^h \to u$ a.e. in $\mathbb{R}\times\mathbb{R}^d$ and thus $|\mathbf{P}_0^h u^h|^p \mathbf{P}_0^h u^h \to |u|^p u$ a.e. in $\mathbb{R}\times\mathbb{R}^d$. Using (4.61) we obtain that $|\mathbf{P}_0^h u^h|^p \mathbf{P}_0^h u^h \rightharpoonup |u|^p u$ in $L^{q'}(\mathbb{R},\, L^{p+2}(\mathbb{R}^d))$. All the above weak convergence of $\mathbf{P}_0^h u^h$ and (4.62) show that u satisfies (4.56).

It remains to prove that $u \in C(\mathbb{R},\, L^2(\mathbb{R}^d))$ and $u(0) = \varphi$. To prove that $u \in C(\mathbb{R},\, L^2(\mathbb{R}^d))$ it is sufficient to prove its continuity at $t = 0$.

We remark that for any positive $0 \le t \le T$:

$$\begin{aligned}\|u^h(t) - S^h(t)\varphi^h\|_{l^2(h\mathbb{Z}^d)} &\le \Big\| \int_0^t S^h(t-s)|u^h|^p u^h ds \Big\|_{L^\infty([0,T],l^2(\mathbb{Z}^d))} \\ &\le \| |u^h|^p u^h \|_{L^{q'}([0,T],\, l^{(p+2)'}(h\mathbb{Z}^d))} \le T^\alpha C(\|\varphi\|_{L^2(\mathbb{R}^d)}) \le CT^\alpha\end{aligned}$$

for some positive α and C independent of h. Using the weak convergence $\mathbf{P}_0^h u^h - \mathbf{P}_0^h S^h(\cdot)\varphi^h \overset{*}{\rightharpoonup} u - S(\cdot)\varphi$ in $L^\infty([0,T], L^2(\mathbb{R}^d))$ we get

$$\|u(t) - S(t)\varphi\|_{L^2(\mathbb{R}^d)} \le \liminf_{h\to 0} \|\mathbf{P}_0^h u^h - \mathbf{P}_0^h S^h(\cdot)\widetilde{\Pi}\varphi^h\|_{L^\infty([0,T],\, L^2(\mathbb{R}^d))} \le T^\alpha$$

which proves that $u(t) \to \varphi$ in $L^2(\mathbb{R}^d)$ as $t \to 0$. $\square$

4.6. **The Critical Case $p = 4/d$.** Our method works similarly in the critical case $p = 4/d$ for small initial data. It suffices to modify the approximation scheme by taking a nonlinear term of the form $|u^h|^{2/\alpha(h)}u^h$ in the semidiscrete equation (4.57) with $a(h) = h^{2-d/\alpha(h)}$ and $\alpha(h) \downarrow d/2$, $a(h) \downarrow 0$, so that, asymptotically, it approximates the critical nonlinearity of the continuous Schrödinger equation. In this way the critical continuous exponent $p = 4/d$ is approximated by semidiscrete critical problems. The critical semidiscrete problem presents the same difficulties as the continuous one. Thus, the initial datum needs to be assumed to be small. But the smallness condition is independent of the mesh-size $h > 0$. More precisely, the following holds.

Theorem 4.7. *Let $\alpha(h) > d/2$ and $p(h) = 2/\alpha(h)$. There exists a constant ϵ, independent of h, such that for all $\|\varphi^h\|_{l^2(h\mathbb{Z}^d)} < \epsilon$, the semidiscrete critical equation has a unique global solution $u^h \in C(\mathbb{R},\, l^2(h\mathbb{Z}^d)) \cap L^{p(h)+2}_{loc}(\mathbb{R},\, l^{p(h)+2}(h\mathbb{Z}^d))$. Moreover, for any $\alpha(h)$-admissible pairs (q,r)*

$$\|u^h\|_{L^q(I,l^r(h\mathbb{Z}^d)} \le C(q,I)\|\varphi^h\|_{l^2(h\mathbb{Z}^d)}$$

for all finite interval I.

Observe that, in particular, $(d+2)/\alpha(h), 4/d+2)$ is an $\alpha(h)$-admissible pair. This allows us to bound the solutions u^h in any space $L^s_{loc}(\mathbb{R},\, L^{4/d+2}(\mathbb{R}))$ with $s < 4/d+2$. With the same notation as in the subcritical case the following convergence result holds.

Theorem 4.8. *When $p = 4/d$ and under the smallness assumption on the initial datum u_0, the sequence $\mathbf{P}_0^h u^h$ satisfies*

$\mathbf{P}_0^h u^h \overset{\star}{\rightharpoonup} u$ *in* $L^\infty(\mathbb{R},\, L^2(\mathbb{R}^d))$, $\mathbf{P}_0^h u^h \rightharpoonup u$ *in* $L^s_{loc}(\mathbb{R},\, L^{4/d+2}(\mathbb{R}^d))\ \forall\, s < 4/d+2$,

$\mathbf{P}_0^h u^h \to u$ *in* $L^2_{loc}(\mathbb{R}\times\mathbb{R}^d)$, $|\mathbf{P}_0^h u^h|^{p(h)}|\mathbf{P}_0^h u^h| \rightharpoonup |u|^{4/d}u$ *in* $L^{(4/d+2)'}_{loc}(\mathbb{R},\, L^{(4/d+2)'}(\mathbb{R}^d))$ *where u is the unique weak solution of critical NSE.*

5. A two-grid algorithm

In this section to compensate the lack of dispersion proved in Section 3 we propose a two-grid algorithm (inspired by [17]) and that, to some extent, acts as a filter for those unwanted high frequency components.

The method is roughly as follows. We consider two meshes: the coarse one of size $4h$, $h > 0$, $4h\mathbb{Z}^d$, and the finer one, $h\mathbb{Z}^d$, of size $h > 0$. The method relies basically on solving the finite-difference semi-discretization (1.8) on the fine mesh $h\mathbb{Z}^d$, but only for slow data, interpolated from the coarse grid $4h\mathbb{Z}^d$. As we shall see, the $1/4$ ratio between the two meshes is important to guarantee the convergence of the method. This particular structure of the data cancels the two pathologies of the discrete symbol mentioned above. Indeed, a careful Fourier analysis of those initial data (we refer to [42] for the theory of multi-grid methods) shows that their discrete Fourier transform vanishes quadratically in each variable at the points $\xi = (\pm\pi/2h)^d$ and $\xi = (\pm\pi/h)^d$. As we shall see, this suffices to recover the dispersive properties of the continuous model.

Once we get the discrete version of the dispersive properties we are able to apply it to a semi-discretization of the NSE with nonlinearity $f(u) = |u|^p u$. The nonlinear term is approximated in a such way that it allows to apply the dispersive estimates of the linear semigroup. We recall that such estimates are valid only in a subspace of $l^2(h\mathbb{Z}^n)$ of data interpolated from the coarse grid. In the subcritical case we prove the global existence of the solutions for initial data in $l^2(h\mathbb{Z}^d)$. We also consider the critical case $p = 4/d$ for small initial data.

We introduce the space of the slowly oscillating sequences (SOS). The SOS on the fine grid $h\mathbb{Z}^d$ are those which are obtained from the coarse grid $4h\mathbb{Z}^d$ by an interpolation process. Any function defined on the lattice $h\mathbb{Z}^d$ can be viewed as a function on the lattice $\mathbb{Z}^d$. This is the way we will proceed in the definition of the projection operator $\widetilde{\Pi}$ and its adjoint.

Let us consider the multilinear interpolator I acting on the coarse grid $4\mathbb{Z}^d$. We define the operator $\widetilde{\Pi} : l^2(4\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$ by

$$(\widetilde{\Pi} f)_{\mathbf{j}} = (If)_{\mathbf{j}}, \quad \mathbf{j} \in \mathbb{Z}^d \tag{5.63}$$

and its adjoint $\widetilde{\Pi}^* : l^2(\mathbb{Z}^d) \to l^2(4\mathbb{Z}^d)$:

$$(\widetilde{\Pi} f, g)_{l^2(\mathbb{Z}^d)} = (f, \widetilde{\Pi}^* g)_{l^2(4\mathbb{Z}^d)}, \tag{5.64}$$

where $(\cdot,\cdot)_{l^2(\mathbb{Z}^d)}$ and $(\cdot,\cdot)_{l^2(4\mathbb{Z}^d)}$ are the inner products on $l^2(\mathbb{Z}^d)$ respectively $l^2(4\mathbb{Z}^d)$.

In Section 3, we proved that there is no gain (uniformly in h) of integrability or local smoothing effect of the linear semigroup $S^h(t)$ generated by the conservative scheme (1.8). However, there are subspaces of $l^2(h\mathbb{Z}^d)$, namely $\widetilde{\Pi}(4h\mathbb{Z}^d)$, where $S^h(t)$ has appropriate decay properties, uniformly on $h>0$. The main results concerning the gain of integrability are given in the following Theorem.

Theorem 5.1. *Let $p\geq 2$ and (q,r), $(\tilde q,\tilde r)$ two 1/2-admissible pairs. The following properties hold*

i) There exists a positive constant $C(d,p)$ such that

$$\|S^h(t)\widetilde{\Pi}\varphi\|_{l^p(h\mathbb{Z}^d)}\leq C(d,p)|t|^{-d(\frac12-\frac1p)}\|\widetilde{\Pi}\varphi\|_{l^{p'}(h\mathbb{Z}^d)} \tag{5.65}$$

for all $\varphi\in l^{p'}(4h\mathbb{Z}^d)$, $h>0$ and $t\neq 0$.

ii) There exists a positive constant $C(d,r)$ such that

$$\|S^h(t)\widetilde{\Pi}\varphi\|_{L^q(\mathbb{R},\,l^r(h\mathbb{Z}^d))}\leq C(d,r)\|\widetilde{\Pi}\varphi\|_{l^2(h\mathbb{Z}^d)} \tag{5.66}$$

for all $\varphi\in l^2(4h\mathbb{Z}^d)$ and $h>0$.

iii) There exists a positive constant $C(d,r)$ such that

$$\Big\|\int_{-\infty}^{\infty}S^h(t)^*\widetilde{\Pi}f(s)ds\Big\|_{l^2(h\mathbb{Z}^d)}\leq C(d,r)\|\widetilde{\Pi}f\|_{L^{q'}(\mathbb{R},l^{r'}(h\mathbb{Z}^d))} \tag{5.67}$$

for all $f\in L^{q'}(\mathbb{R},\,l^{\tilde r'}(4h\mathbb{Z}^d))$ and $h>0$.

iv) There exists a positive constant $C(d,r,\tilde r)$ such that

$$\Big\|\int_0^t S^h(t-s)\widetilde{\Pi}f(s)ds\Big\|_{L^q(\mathbb{R},\,l^r(h\mathbb{Z}^d))}\leq C(d,r,\tilde r)\|\widetilde{\Pi}f\|_{L^{\tilde q'}(\mathbb{R},\,l^{\tilde r'}(h\mathbb{Z}^d))} \tag{5.68}$$

for all $f\in L^{\tilde q'}(\mathbb{R},\,l^{\tilde r'}(4h\mathbb{Z}^d))$ and $h>0$.

The following lemma gives a characterization of data that are obtained by a two-grid algorithm involving the meshes $4h\mathbb{Z}^d$ and $h\mathbb{Z}^d$. Its proof uses only the definition of the discrete Fourier transform and for that we omit it.

Lemma 5.1. *Let $\psi\in l^2(4h\mathbb{Z}^d)$. Then for all $\xi\in[-\pi/h,\pi/h]^d$*

$$\widehat{\widetilde{\Pi}\psi}(\xi)=4^d\widehat{\Pi\psi}(\xi)\prod_{k=1}^d\cos^2(\xi_k h)\cos^2\left(\frac{\xi_k h}{2}\right),$$

where $(\Pi\psi)_{\mathbf{j}}=\psi_{\mathbf{j}}$ if $\mathbf{j}\in 4\mathbb{Z}^d$ and vanishes elsewhere.

Remark 5.1. *A simpler construction may be done by interpolating $2h\mathbb{Z}^d$ sequences. We then get for all $\psi\in l^2(2h\mathbb{Z}^d)$ and $\xi\in[-\pi/h,\pi/h]^d$*

$$\widehat{\widetilde{\Pi}\psi}(\xi)=2^d\widehat{\Pi\psi}(\xi)\prod_{k=1}^d\cos^2\left(\frac{\xi_k h}{2}\right),$$

with $(\Pi\psi)_{\mathbf{j}} = \psi_{\mathbf{j}}$ *if* $\mathbf{j} \in 2\mathbb{Z}^d$ *and vanishes elsewhere. This cancels the spurious numerical solutions at the frequencies* $\{\pm\pi/h\}^d$*, but not at* $\{\pm\pi/2h\}^d$*. In this case, as we proved in Section 3, the Strichartz estimates fail to be uniform on* h*. Thus we rather choose* $1/4$ *as the ratio between the grids for the two-grid algorithm.*

Proof. Let us define the family of weighted operators $A^h_\alpha(t) : l^2(h\mathbb{Z}^d) \to l^2(h\mathbb{Z}^d)$ by

$$(\widehat{A^h_\alpha(t)f})(\xi) = e^{-itp_h(\xi)}|g(\xi h)|^\alpha \widehat{f}(\xi),\ \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right],$$

where

$$g(\xi) = \prod_{k=1}^d \cos(\xi_k)\cos\left(\frac{\xi_k}{2}\right).$$

We will prove that for any $\alpha \geq 1/4$, $A^h_\alpha(t)$ satisfies the hypothesis of Proposition 3.1. Then, observing that $S^h(t)\widetilde{\Pi}\varphi = A^h_2(t)\Pi\varphi$, we obtain (5.66), (5.67) and (5.68).

Is easy to see that $\|A^h_\alpha(t)\varphi\|_{l^2(h\mathbb{Z}^d)} \leq \|\varphi\|_{l^2(h\mathbb{Z}^d)}$. It remains to prove that for any $\alpha \geq 1/4$ and $t \neq s$ the following holds:

$$\|A^h_\alpha(t)A^h_\alpha(s)^*\psi\|_{l^\infty(h\mathbb{Z}^d)} \leq c(\alpha,d)|t-s|^{-d/2}\|\psi\|_{l^1(h\mathbb{Z}^d)}. \tag{5.69}$$

A scaling argument reduces the proof to the case $h = 1$. We claim that (5.69) holds once

$$\|A^1_\beta(t)\|_{l^\infty(h\mathbb{Z}^d)} \leq c(\beta,d)|t|^{-d/2}\|\psi\|_{l^1(h\mathbb{Z}^d)} \tag{5.70}$$

holds for all $\beta \geq 1/2$. Indeed, using that the operator $A^1_\alpha(t)$ satisfies $A^1_\alpha(t)^* = A^1_\alpha(-t)$ for all real t, we obtain

$$\begin{aligned}\|A^1_\alpha(t)A^1_\alpha(s)^*\psi\|_{l^\infty(\mathbb{Z}^d)} &= \|A^1_\alpha(t)A^1_\alpha(-s)\psi\|_{l^\infty(\mathbb{Z}^d)}\\ &= \|A^1_{2\alpha}(t-s)\psi\|_{l^\infty(\mathbb{Z}^d)} \lesssim |t-s|^{-d/2}\|\psi\|_{l^1(\mathbb{Z}^d)},\end{aligned}$$

for all $t \neq s$ and $\psi \in l^1(\mathbb{Z}^d)$.

In the following we prove (5.70). We write $A^1_\beta(t)$ as a convolution $A^1_\beta(t)\psi = K^t_\beta * \psi$ where $\widehat{K^t_\beta}(\xi) = e^{-itp_1(\xi)}|g(\xi)|^\beta$. By Young's inequality it is sufficient to prove that for any $\beta \geq 1/2$ and $t \neq 0$ the following holds:

$$\|K^t_\beta\|_{l^\infty(\mathbb{Z}^d)} \leq c(\beta,d)|t|^{-d/2}.$$

We observe that K^t_β can be written by separation of variables as

$$\widehat{K^t_\beta}(\xi) = \prod_{k=1}^d e^{-4it\sin^2(\frac{\xi_k}{2})}\left|\cos(\xi_k)\cos\left(\frac{\xi_k}{2}\right)\right|^\beta = \prod_{j=1}^d \widehat{K^t_{1,\beta}}(\xi_j).$$

It remains to prove that $\|K^t_{1,\beta}\|_{l^\infty(\mathbb{Z})} \leq c(\beta)|t|^{-1/2}$. We make use of the following Lemma:

Lemma 5.2. *(Corollary 2.9, [24]) Let $(a,b)\subset\mathbb{R}$ and $\psi\in C^3(a,b)$ be such that ψ'' has a finite number of changes of monotonicity. Then*

$$\left|\int_a^b e^{i(t\psi(\xi)-x\xi)}|\psi''(\xi)|^{1/2}\phi(\xi)d\xi\right|\leq c_\psi|t|^{-1/2}\left\{\|\phi\|_{L^\infty(a,b)}+\int_a^b|\phi'(\xi)|d\xi\right\}.$$

holds for all real numbers x and t.

Applying the above Lemma with $\phi(\xi)=|\cos\xi|^{\beta-1/2}|\cos(\xi/2)|^\beta$, $\beta\geq 1/2$, and $\psi(\xi)=-4\sin^2(\xi/2)$, we obtain that $\|K^t_{1,\beta}\|_{l^\infty(\mathbb{Z})}\leq c(\beta)|t|^{-1/2}$, which finishes the proof. □

5.1. **A conservative approximation of the NSE.** We concentrate on the semilinear NSE equation (4.55). We consider the following semi-discretization

$$i\frac{du^h}{dt}+\Delta_h u^h=\widetilde{\Pi}f(\widetilde{\Pi}^*u^h),\ t\in\mathbb{R};\qquad u^h(0)=\widetilde{\Pi}\varphi^h, \tag{5.71}$$

where $f(u)=|u|^pu$. In order to prove the global well-posedness of (5.71), it is sufficient to guarantee the conservation of the $l^2(h\mathbb{Z}^d)$-norm of solutions, a property that the solutions of NSE satisfy. The choice $\widetilde{\Pi}f(\widetilde{\Pi}^*u^h)$ as an approximation of the nonlinear term $f(u)$ is motivated by the following identity:

$$(\widetilde{\Pi}f(\widetilde{\Pi}^*u^h),u^h)_{l^2(h\mathbb{Z}^d)}=(f(\widetilde{\Pi}^*u^h),\widetilde{\Pi}^*u^h)_{l^2(4h\mathbb{Z}^d)}\in\mathbb{R}. \tag{5.72}$$

This will allow us to prove the conservation of the $l^2(h\mathbb{Z}^d)$-norm of the solutions and their global existence.

The following holds:

Theorem 5.2. *Let $p\in(0,4/d)$ and $q=4(p+2)/dp$. Then for all $h>0$ and for every $\varphi^h\in l^2(4h\mathbb{Z}^d)$, there exists a unique global solution $u^h\in C(\mathbb{R},\,l^2(h\mathbb{Z}^d))\cap L^q_{loc}(\mathbb{R},\,l^{p+2}(h\mathbb{Z}^d))$ of (5.71). Moreover, u^h satisfies*

$$\|u^h\|_{L^\infty(\mathbb{R},\,l^2(h\mathbb{Z}^d))}\leq\|\widetilde{\Pi}\varphi^h\|_{l^2(h\mathbb{Z}^d)}\ \text{and}\ \|u^h\|_{L^q(I,\,l^{p+2}(h\mathbb{Z}^d))}\leq c(I)\|\widetilde{\Pi}\varphi^h\|_{l^2(h\mathbb{Z}^d)}$$

for all finite intervals I, where the above constants are independent of h.

Proof of Theorem 5.2. The local existence and uniqueness are consequences of the Strichartz-like estimates given in Theorem 5.1 and of a fixed point argument in the space $L^\infty((-T,T),l^2(h\mathbb{Z}^d))\cap L^q((-T,T),\,l^{p+2}(h\mathbb{Z}^d))$ where T has to be assumed small. Identity (5.72) proves the global existence of the solution. □

5.2. **Convergence of the method.** In the sequel we consider the piecewise constant interpolator $\mathbf{P}_0^h$. We choose $(\varphi^h_{\mathbf{j}})_{\mathbf{j}\in\mathbb{Z}^d}$, an approximation of the initial datum $\varphi\in L^2(\mathbb{R}^d)$, such that $\mathbf{P}_0^h\widetilde{\Pi}\varphi^h$ converges strongly to φ in $L^2(\mathbb{R}^d)$. Thus, in particular, $\|\mathbf{P}_0^h\widetilde{\Pi}\varphi^h\|_{L^2(\mathbb{R}^d)}\leq C(\|\varphi\|_{L^2(\mathbb{R}^d)})$.

The main convergence result is the following:

Theorem 5.3. *Let p and q be as in Theorem 5.2 and u^h be the unique solution of (5.71). Then the sequence $\mathbf{P}_0^h u^h$ satisfies*

$$\mathbf{P}_0^h u^h \overset{\star}{\rightharpoonup} u \text{ in } L^\infty(\mathbb{R},\, L^2(\mathbb{R}^d)),\ \mathbf{P}_0^h u^h \rightharpoonup u \text{ in } L^q_{loc}(\mathbb{R},\, L^{p+2}(\mathbb{R}^d)), \tag{5.73}$$

$$\mathbf{P}_0^h u^h \to u \text{ in } L^2_{loc}(\mathbb{R}^{d+1}),\ \mathbf{P}_0^h \widetilde{\Pi} f(\widetilde{\Pi}^* u^h) \rightharpoonup |u|^p u \text{ in } L^{q'}_{loc}(\mathbb{R},\, L^{(p+2)'}(\mathbb{R}^d)) \tag{5.74}$$

where u is the unique solution of NSE.

The main difficulty in the proof of Theorem 5.3 is the strong convergence $\mathbf{P}_0^h u^h \to u$ in $L^2_{loc}(\mathbb{R}^{d+1})$. Once it is obtained, the second convergence in (5.74) easily follows. Without the strong convergence of $\mathbf{P}_0^h u^h$ towards u we are not able to pass to the limit in the nonlinear term. Another difficulty comes from the fact that the interpolator E has no compact support in the Fourier space. To simplify the proof we consider the band-limited interpolator $\mathbf{P}_*^h$ (cf. [41], Ch. II) and prove the compactness for $\mathbf{P}_*^h$. Once this is obtained we transfer the L^2-strong convergence of $\mathbf{P}_*^h u^h$ to $\mathbf{P}_0^h u^h$. This is a consequence of the following property of the piecewise constant interpolator $\mathbf{P}_0^h u^h$ (cf. [30], Th. 1.3.5, p. 122):

$$\|\mathbf{P}_0^h u^h(t) - \mathbf{P}_*^h u^h(t)\|_{L^2(\Omega)} \le h \|\mathbf{P}_*^h u^h(t)\|_{H^1(\Omega)}, \tag{5.75}$$

which holds for all real t and $\Omega \subset \mathbb{R}^d$.

We will prove that $\mathbf{P}_*^h u^h$ is uniformly bounded in $L^2_{loc}(\mathbb{R},\, H^{1/2}_{loc}(\mathbb{R}^d))$. Also we will obtain estimates on the $L^2_{loc}(\mathbb{R},\, H^1_{loc}(\mathbb{R}^d))$-norm. The last ones are not uniform on h but give us sufficient information to ensure that $\mathbf{P}_0^h u^h - \mathbf{P}_*^h u^h$ strongly converges to zero in $L^2_{loc}(\mathbb{R}^{d+1})$. The following lemma gives estimates of the local H^s-norm of $\mathbf{P}_*^h u^h$.

Lemma 5.3. *Let be $s \ge 1/2$, $I \subset \mathbb{R}$ a bounded interval and $\chi \in C_c^\infty(\mathbb{R}^d)$. Then there is a constant $C(I,\chi)$ such that*

$$\|\chi \mathbf{P}_*^h (S^h(t)\widetilde{\Pi}\varphi^h)\|_{L^2(I,H^s(\mathbb{R}^d))} \le \frac{C(I,\chi)}{h^{s-1/2}} \|\widetilde{\Pi}\varphi^h\|_{l^2(h\mathbb{Z}^d)} \tag{5.76}$$

holds for all functions $\varphi^h \in l^r(4h\mathbb{Z}^d)$. Moreover for any $1/2$-admissible pair (q,r)

$$\left\|\chi \mathbf{P}_*^h \left(\int_0^t S^h(t-\tau)\widetilde{\Pi} f^h(\tau)d\tau\right)\right\|_{L^2(I,H^s(\mathbb{R}^d))} \le \frac{C(I,\chi)}{h^{s-1/2}} \|\widetilde{\Pi} f^h\|_{L^{q'}(I,l^{r'}(h\mathbb{Z}^d))} \tag{5.77}$$

for all $f^h \in L^{q'}(I, l^{r'}(4h\mathbb{Z}^d))$.

Proof of Lemma 5.3. **Step I. Regularity of the homogenous term.** To prove (5.76) it is sufficient to prove for any $R>0$ the existence of a positive constant $C(I,R)$ such that

$$\int_I \int_{|x|<R} |(-\Delta)^{s/2} \mathbf{P}_*^h (S^h(t)\widetilde{\Pi}\varphi^h)|^2 dx dt \le \frac{C(I,R)}{h^{2s-1}} \int_{[-\pi/h,\pi/h]^d} |\widehat{\varphi}^h(\xi)|^2 d\xi.$$

Let us consider $\psi^h \in l^2(h\mathbb{Z}^d)$. Applying Lemma 4.1 to the function $I_*(S^h(t)\psi^h)$ we obtain

$$\begin{aligned}
\int_I\int_{|x|<R}|(-\Delta)^{s/2}\mathbf{P}^h_*(S^h(t)\psi^h)|^2dxdt &\leq C(I,R)\int_{[-\pi/h,\pi/h]^d}\frac{|\xi|^{2s}|\widehat{\mathbf{P}^h_*\psi^h}(\xi)|^2d\xi}{|\nabla p_h(\xi)|}\\
&\leq C(I,R)h^{1-2s}\int_{[-\pi/h,\pi/h]^d}\frac{(\sum_{j=1}^d\xi_j^2)^{1/2}|\widehat{\psi^h}(\xi)|^2d\xi}{(\sum_{j=1}^d\sin^2(\xi_jh)/h^2)^{1/2}}\\
&\lesssim C(I,R)h^{1-2s}\int_{[-\pi/h,\pi/h]^d}\frac{|\widehat{\psi}^h(\xi)|^2d\xi}{\prod_{j=1}^d|\cos(\xi_jh/2)|}, \qquad (5.78)
\end{aligned}$$

provided that all terms make sense. Now, we apply the last estimates to $\psi^h=\widetilde{\Pi}\varphi^h$. Thus

$$\begin{aligned}
\int_I\int_{|x|<R}|(-\Delta)^{s/2}\mathbf{P}^h_*(S^h(t)\widetilde{\Pi}\varphi^h)|^2dxdt &\leq \frac{C(I,R)}{h^{2s-1}}\int_{[-\pi/h,\pi/h]^d}\frac{|\widehat{\widetilde{\Pi}\varphi^h}(\xi)|^2d\xi}{\prod_{j=1}^d|\cos(\xi_jh/2)|}\\
&\leq \frac{C(I,R)}{h^{2s-1}}\int_{[-\pi/h,\pi/h]^d}|\widehat{\varphi}^h(\xi)|^2\prod_{j=1}^d|\cos(\xi_jh/2)|^3d\xi\\
&\leq \frac{C(I,R)}{h^{2s-1}}\|\varphi^h\|_{l^2(h\mathbb{Z}^d)}.
\end{aligned}$$

Step II. Regularity of the inhomogeneous part. In the following we prove (5.77). This estimate will be reduced to the homogenous one (5.76) by using the argument of Christ and Kiselev [9] (see also [4], [33] in the context of PDE). A simplified version, useful in PDE application is given in [33] :

Lemma 5.4. *Let X and Y be Banach spaces and assume that $K(t,s)$ is a continuous function taking its values in $B(X,Y)$, the space of bounded linear mappings from X to Y. Suppose that $-\infty\leq a<b\leq\infty$ and set*

$$Tf(t)=\int_a^bK(t,s)f(s)ds,\quad Wf(t)=\int_a^tK(t,s)f(s)ds.$$

Assume that $1\leq p<q\leq\infty$ and $\|Tf\|_{L^q([a,b],Y)}\leq\|f\|_{L^p([a,b],X)}$. Then

$$\|Wf\|_{L^q([a,b],Y)}\leq\|f\|_{L^p([a,b],X)}.$$

Without less generality we can consider $I=[0,T]$. In view of the above Lemma it is sufficient to prove that the operator

$$Tf(t)=\chi\mathbf{P}^h_*\Big(\int_0^TS^h(t-\tau)\widetilde{\Pi}f^h(\tau)d\tau\Big)$$

satisfies

$$\|Tf\|_{L^2([0,T],\,H^s(\mathbb{R}^d))}\leq C(T,\chi)\|\widetilde{\Pi}f^h\|_{L^{q'}([0,T],\,l^{r'}(h\mathbb{Z}^d))}.$$

Writing Tf in the following form

$$Tf(t) = \chi \mathbf{P}_*^h S^h(t) \widetilde{\Pi}\Big(\int_0^T S^h(-\tau) f^h(\tau) d\tau \Big)$$

and using estimate (5.76) we obtain that

$$\begin{aligned} \|Tf\|_{L^2([0,T],\, H^s(\mathbb{R}^d))} &\leq \frac{C(I,\chi)}{h^{s-1/2}} \Big\| \widetilde{\Pi}\Big(\int_0^T S^h(\tau)^* f^h(\tau) d\tau \Big) \Big\|_{l^2(h\mathbb{Z}^d)} \\ &= \frac{C(I,\chi)}{h^{s-1/2}} \Big\| \int_0^T S^h(\tau)^* \widetilde{\Pi} f^h(\tau) d\tau \Big\|_{l^2(h\mathbb{Z}^d)}. \end{aligned}$$

Estimate (5.67) given by Theorem 5.1 shows that the right hand side of the last estimate is uniformly bounded by the $L^{q'}([0,T], l^{r'}(h\mathbb{Z}^d))$-norm of $\widetilde{\Pi} f^h$, finishing the proof. □

Proof of Theorem 4.6. Theorem 5.2 shows that $\mathbf{P}_0^h u^h$ satisfies $\|\mathbf{P}_0^h u^h\|_{L^\infty(\mathbb{R},\, L^2(\mathbb{R}^d))} \leq C(\|\varphi\|_{L^2(\mathbb{R}^d)})$ and for any finite interval I the following:

$$\|\mathbf{P}_0^h u^h\|_{L^q(I,\, L^{p+2}(\mathbb{R}^d))} \leq C(I, \|\varphi\|_{L^2(\mathbb{R}^d)}), \tag{5.79}$$

and

$$\|\mathbf{P}_0^h \widetilde{\Pi} f(u^h)\|_{L^{q'}(I,\, L^{(p+2)'}(\mathbb{R}^d))} \leq C(I, \|\varphi\|_{L^2(\mathbb{R}^d)}) \tag{5.80}$$

hold, uniformly on h.

Moreover, multiplying (5.71) by a function $\psi \in C_c^\infty(\mathbb{R}^{d+1})$, $\mathbf{P}_0^h u^h$ satisfies

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbf{P}_0^h u^h (-i\psi_t + \Delta^h \psi) dx dt = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbf{P}_0^h \widetilde{\Pi} f(\widetilde{\Pi}^* u^h) \psi dx dt. \tag{5.81}$$

Step i). Weak convergence.
The uniform boundedness of $\mathbf{P}_0^h u^h$ in $L^\infty(\mathbb{R},\, L^2(\mathbb{R}^d))$, guarantees that there is a function $u \in L^\infty(\mathbb{R},\, L^2(\mathbb{R}^d))$ such that up to a subsequnce $\mathbf{P}_0^h u^h \overset{\star}{\rightharpoonup} u$ in $L^\infty(\mathbb{R},\, L^2(\mathbb{R}^d))$. By (5.79) we obtain that $u \in L^q(I,\, L^{p+2}(\mathbb{R}^d))$ and up to a subsequence

$$\mathbf{P}_0^h u^h \rightharpoonup u \text{ in } \ L^q(I,\, L^{p+2}(\mathbb{R}^d)). \tag{5.82}$$

Step ii). Strong Convergence of $\mathbf{P}_0^h u^h$.
Using Lemma 5.3 with $s = 1/2$ we obtain that for any smooth function χ, $\mathbf{P}_*^h u^h$ satisfies

$$\|\chi \mathbf{P}_*^h u^h\|_{L^2(I, H^{1/2}(\mathbb{R}^d))} \leq C(I, \chi, \|\varphi^h\|_{l^2(h\mathbb{Z}^d)}).$$

Let I be a finite interval and $\Omega \subset \mathbb{R}^d$ bounded. The same arguments as in Section 4.5 show the existence of a function v such that $\mathbf{P}_*^h u^h \to v$ in $L^2(I \times \Omega)$. By a diagonal process we get that $\mathbf{P}_*^h u^h \to v$ in $L^2_{loc}(\mathbb{R} \times \mathbb{R}^2)$.

In the following we transfer the strong convergence of $\mathbf{P}_*^h u^h$ to $\mathbf{P}_0^h u^h$. Classical properties of the interpolator $\mathbf{P}_0^h u^h$ (see [30], Th. 3.1.5, p. 122) give us

$$\int_\Omega |\mathbf{P}_0^h u^h - \mathbf{P}_*^h u^h|^2 dx \leq h^2 \|\mathbf{P}_*^h u^h\|^2_{H^1(\Omega)}.$$

Applying Lemma 5.3 with $s=1$ we obtain for any $\chi \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}\int_I\int_{\mathbb{R}^d} \chi^2 |\mathbf{P}_0^h u^h - \mathbf{P}_*^h u^h|^2 dxdt &\leq h^2 \int_I\int_{\mathbb{R}^d} \chi^2 |(I-\Delta)^{1/2}\mathbf{P}_*^h u^h|^2 dxdt\\ &\leq hC(I, \|\widetilde{\Pi}\varphi^h\|^2_{l^2(h\mathbb{Z}^d)}) \to 0,\ h\to 0.\end{aligned}$$

This shows that $\mathbf{P}_0^h u^h - \mathbf{P}_*^h u^h \to 0$ in $L^2_{loc}(\mathbb{R}\times\mathbb{R}^d)$. Using the strong convergence of $\mathbf{P}_*^h u^h$ towards v, we obtain that $v=u$ and

$$\mathbf{P}_0^h u^h \to u \text{ in } L^2_{loc}(\mathbb{R}\times\mathbb{R}^d).$$

Let $\Gamma\subset\mathbb{Z}^d$ be a finite set. Thus for any $s\in\Gamma$ we have $\mathbf{P}_0^h u^h(\cdot+sh)\to u$ in $L^2_{loc}(\mathbb{R}\times\mathbb{R}^d)$ and

$$\mathbf{P}_0^h u^h(\cdot+sh)\to u \text{ a.e. in } \mathbb{R}\times\mathbb{R}^d.$$

The operators $\widetilde{\Pi}$ and $\widetilde{\Pi}^*$ involve only a finite number of translations. Then

$$pc\widetilde{\Pi} f(\widetilde{\Pi}^* u^h) \to |u|^p u \text{ a.e. in } \mathbb{R}\times\mathbb{R}^d.$$

and

$$\mathbf{P}_0^h\widetilde{\Pi} f(\widetilde{\Pi}^* u^h) \rightharpoonup |u|^p u \text{ in } L^{q'}(I,\, L^{(p+2)'}(\mathbb{R}^d)). \tag{5.83}$$

All the above weak convergences of Eu^h and (5.81) show that u satisfies (4.56).

Step iii). Continuity of u in $L^2(\mathbb{R}^d)$ and identification of the initial datum. To prove that $u\in C(\mathbb{R},\, L^2(\mathbb{R}^d))$ it is sufficient to prove the continuity at $t=0$. We remark that for any positive $0\le t\le T$:

$$\begin{aligned}\|u^h(t) - S^h(t)\widetilde{\Pi}\varphi^h\|_{l^2(h\mathbb{Z}^d)} &\leq \left\|\int_0^t S^h(t-s)\widetilde{\Pi} f(\widetilde{\Pi}^* u^h(s))ds\right\|_{L^\infty([0,T],l^2(\mathbb{Z}^d))}\\ &\leq \||u^h|^{p+1}\|_{L^{q'}([0,T],\,l^{(p+2)'}(h\mathbb{Z}^d))} \leq T^\alpha \|u^h\|^\beta_{L^q(\mathbb{R},\,l^{p+2}(h\mathbb{Z}^d))} \leq CT^\alpha\end{aligned}$$

for some positive α, β and C independent of h. Using the weak convergence $\mathbf{P}_0^h u^h(t) - \mathbf{P}_0^h S^h(t)\widetilde{\Pi}\varphi^h \rightharpoonup v(t) - S(t)\varphi$ in $L^2(\mathbb{R}^d)$ we get

$$\|v(t)-S(t)\varphi\|_{L^2(\mathbb{R}^d)} \leq \liminf_h \|\mathbf{P}_0^h u^h(t) - \mathbf{P}_0^h S^h(t)\widetilde{\Pi}\varphi^h\|_{L^2(\mathbb{R}^d)} \leq T^\alpha$$

which prove that $v(t)\to\varphi$ in $L^2(\mathbb{R}^d)$ as $t\to 0$. This finishes the proof. $\square$

5.3. **The critical case $p = 4$.** Our method works similarly in the critical case $p = 4/d$ for small initial data. The initial datum needs to be assumed to be small, but the smallness condition is independent of the mesh-size $h > 0$. More precisely, the following holds.

Theorem 5.4. *There exists a constant ϵ, independent of h, such that for all initial data $\|\varphi^h\|_{l^2(h\mathbb{Z}^d)} < \epsilon$, the semidiscrete critical equation (5.71) with $p = 4/d$ has a unique global solution $u^h \in C(\mathbb{R},\, l^2(h\mathbb{Z}^d)) \cap L^{4/d+2}_{loc}(\mathbb{R},\, l^{4/d+2}(h\mathbb{Z}^d))$. Moreover for any 1/2-admissible pair (q, r), $u^h \in L^q_{loc}(\mathbb{R},\, l^r(h\mathbb{Z}^d))$ and*

$$\|u^h\|_{L^q(I,\, l^r(h\mathbb{Z}^d)} \leq C(q, I)\|\varphi^h\|_{l^2(h\mathbb{Z}^d)}$$

for all finite intervals I, uniformly on h.

With the same notation as in the subcritical case the following convergence result holds.

Theorem 5.5. *Let $p = 4/d$. Under the smallness assumption of Theorem 5.4, the sequence $\mathbf{P}^h_0 u^h$ satisfies*

$$\mathbf{P}^h_0 u^h \overset{\star}{\rightharpoonup} u \text{ in } L^\infty(\mathbb{R},\, L^2(\mathbb{R}^d)),\ \mathbf{P}^h_0 u^h \rightharpoonup u \text{ in } L^{4/d+2}_{loc}(\mathbb{R},\, L^{4/d+2}(\mathbb{R}^d)),$$

$$\mathbf{P}^h_0 u^h \to u \text{ in } L^2_{loc}(\mathbb{R}\times\mathbb{R}^d),\ \mathbf{P}^h_0 \widetilde{\Pi}(f(\widetilde{\Pi}^* u^h)) \rightharpoonup |u|^{4/d} u \text{ in } L^{(4/d+2)'}_{loc}(\mathbb{R},\, L^{(4/d+2)'}(\mathbb{R}^d))$$

where u is the unique weak solution of the critical NSE with $p = 4/d$.

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