# ASYMPTOTIC BEHAVIOUR FOR A NONLOCAL DIFFUSION EQUATION ON A LATTICE

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Abstract. In this paper we study the asymptotic behaviour as $t \to \infty$ of solutions to a nonlocal diffusion problem on a lattice, namely, $u_n'(t) = \sum_{j\in\mathbb{Z}^d} J_{n-j}u_j(t) - u_n(t)$ with $t \geq 0$ and $n \in \mathbb{Z}^d$. We assume that $J$ is nonnegative and verifies $\sum_{n\in\mathbb{Z}^d} J_n = 1$. We find that solutions decay to zero as $t \to \infty$ and prove an optimal decay rate using, as our main tool, the discrete Fourier transform.

## 1. Introduction

In this paper our main concern is the study of the asymptotic behaviour of the following nonlocal equation on a lattice

$$\begin{cases} {u_n}'(t) = (J * u)_n(t) - u_n(t), & t \geq 0,\, n \in \mathbb{Z}^d, \\ u_n(0) = \varphi_n, & n \in \mathbb{Z}^d, \end{cases} \tag{1.1}$$

where by $(J * u)$ we denote the discrete convolution,

$$(J * u)_n = \sum_{j\in\mathbb{Z}^d} J_{n-j}u_j.$$

Trough the paper we assume that the kernel $J$ is nonnegative and satisfies,

$$\sum_{n\in\mathbb{Z}^d} J_n = 1. \tag{1.2}$$

Equation (1.1), is called *nonlocal diffusion equation.* Continuous analogous to (1.1), like $u_t(x,t) = J * u(x,t) - u(x,t)$, have been recently widely used to model diffusion processes, see, for example, [2], [3], [5], [6], [8], [9], [10], [16] and [17]. In particular, let us mention that these equations are also used in models of neuronal activity, see [7], [11], [13] and [14]. Also there is a discrete counterpart for nonlocal models, see [1], [3] and references therein. In all these models the asymptotic behaviour of the solution (see [4]) is relevant, both from its pure mathematical and its applied point of view. Concerning (1.1), as stated in [9] (see also [3]), if $u_i(t)$ is thought of as the density of a single population at the point $i$ at time $t$, and $J_{i-j}$ is thought of as the probability distribution of jumping from location $i$ to location $j$, then $(J * u)(t)$ is the rate at which individuals are arriving to position $i$ from all other places and $-u_i(t)$ is the rate at which they are leaving location $i$ to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density $u$ satisfies equation (1.1).

To study the asymptotic beahviour of solutions to (1.1) let us introduce the discrete Laplacian given by

$$(\Delta_d u)_n = \sum_{k=1}^{d} (u_{n+e_k} - 2u_n + u_{n-e_k}),$$

where $\{e_k\}_{k=1}^d$ is the canonical basis on $\mathbb{R}^d$. Note that this is a *local* diffusion operator.

Our first result says that the asymptotic behaviour as $t \to \infty$ of solutions to (1.1) is the same as the one for the evolution equation associated to a fractional power of the discrete Laplacian.

**Theorem 1.1.** *Let $u$ be a solution of equation (1.1) with $\varphi \in l^1(\mathbb{Z}^d)$. If there exist positive constants $\alpha$ and $A$ such that*

$$\widehat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha), \qquad as\ \xi \to 0,$$

*then the asymptotic behaviour of $u(t)$ is given by*

$$\lim_{t\to\infty} t^{d/\alpha}\|u(t) - v(t)\|_{l^\infty(\mathbb{Z}^d)} = 0,$$

*where $v$ is solution of $v' = -A(-\Delta_d)^{\alpha/2}v$ with initial datum $v_n(0) = \varphi_n$, $n \in \mathbb{Z}^d$.*

In view of this result, we analyze the asymptotic profile of the solutions to $v' = -A(-\Delta_d)^{\alpha/2}v$.

**Theorem 1.2.** *Let us consider $\varphi \in l^1(\mathbb{Z}^d)$. Then the solution to*

$$\begin{cases} v'(t) = -A(-\Delta_d)^{\alpha/2}v, & t > 0,\\ v(0) = \varphi, \end{cases}$$

*satisfies*

$$\lim_{t\to\infty} \sup_{j\in\mathbb{Z}^d} \Big| t^{d/\alpha} v([jt^{1/\alpha}], t) - \Big(\sum_{n\in\mathbb{Z}^d} \varphi_n\Big) G^A(j)\Big| = 0,$$

*where $G^A$ is defined by*

$$G^A(x) = \int_{\mathbb{R}^d} e^{ix\xi} e^{-A|\xi|^\alpha} d\xi,$$

*and $[\cdot]$ is the floor function.*

## 2. Proofs of the results

In our analysis, we make use of the semidiscrete Fourier transform (SDFT) (we refer to [12] and [15] for the mains properties of the SDTF). For any $v \in l^2(\mathbb{Z}^d)$ we define its SDFT by:

$$\widehat{v}(\xi) = \sum_{j\in\mathbb{Z}^d} e^{-i\xi\cdot j} v_j, \qquad \xi \in [-\pi, \pi]^d.$$

In view of property (1.2), $\widehat{J}$ belongs to $L^\infty([-\pi, \pi])$ and $\widehat{J}(0) = 1$.

*Proof of Theorem 1.1.* Applying the SDFT to the solutions of equation (1.1) we get

$$\widehat{u}\,'(t, \xi) = \widehat{J}(\xi)\widehat{u}(t, \xi) - \widehat{u}(t, \xi), \qquad \xi \in [-\pi, \pi]^d,\ t > 0.$$

Solving this ODE we find that

$$\widehat{u}(t, \xi) = e^{t(\widehat{J}(\xi)-1)}\widehat{\varphi}(\xi), \qquad \xi \in [-\pi, \pi]^d,\ t > 0. \tag{2.1}$$

In the same way, $v$, the solution to $v' = -A(-\Delta_d)^{\alpha/2}v$ satisfies

$$\widehat{v}(t, \xi) = e^{-Atp^\alpha(\xi)}\widehat{\varphi}(\xi), \qquad \xi \in [-\pi, \pi]^d,\ t > 0, \tag{2.2}$$

where

$$p(\xi)=\Big(4\sum_{k=1}^{d}\sin^2\big(\frac{\xi_k}{2}\big)\Big)^{1/2}.$$

Using the Fourier representation of $u$ and $v$ given by (2.1) and (2.2) we find that

$$\begin{aligned}\|u(t)-v(t)\|_{l^\infty(\mathbb{Z}^d)} &\leq \int_{[-\pi,\pi]^d}|\widehat{u}(\xi,t)-\widehat{v}(\xi,t)|d\xi\\ &= \int_{[-\pi,\pi]^d}|\exp(t(\widehat{J}(\xi)-1))-\exp(-Atp^\alpha(\xi))||\widehat{\varphi}(\xi)|d\xi.\end{aligned}$$

By our hypothesis there exists a positive $R<\pi$ such that

$$|\widehat{J}(\xi)|\leq 1-\frac{|\xi|^\alpha}{2},\ |\xi|\leq R.$$

Once $R$ has been fixed, there exists $\delta>0$ such that

$$|\widehat{J}(\xi)|\leq 1-\delta \qquad \text{for all } \xi\in\Omega_R=\{\xi\in[-\pi,\pi]^d,|\xi|>R\}.$$

Hence, it is easy to see that

$$\begin{aligned}\int_{\xi\in\Omega_R}|e^{t(\widehat{J}(\xi)-1)}-e^{-Atp^\alpha(\xi)}||\widehat{\varphi}(\xi)|d\xi &\leq \|\widehat{\varphi}\|_{L^\infty([-\pi,\pi]^d)}\int_{\xi\in\Omega_R}(e^{t(|\widehat{J}(\xi)|-1)}+e^{-Atp^\alpha(\xi)})d\xi\\ &\leq \|\widehat{\varphi}\|_{L^\infty([-\pi,\pi]^d)}\int_{\xi\in\Omega_R}(e^{-t\delta}+\exp(-At\inf_{\xi\in\Omega_R}p^\alpha(\xi))d\xi.\end{aligned}$$

Tacking into account that the right hand side in the last inequality is exponentially small, it remains to analyze the term

$$I(t)=\int_{|\xi|\leq R}|\widehat{u}(\xi,t)-\widehat{v}(\xi,t)|d\xi.$$

Let us choose a function $r(t)\to 0$ such that $r(t)t^{1/\alpha}\to\infty$ as $t\to\infty$. The remaining term $I(t)$ satisfies:

$$I(t)=\int_{|\xi|\leq R}|e^{t(\widehat{J}(\xi)-1)}-e^{-Atp^\alpha(\xi)}||\widehat{\varphi}(\xi)|d\xi\leq I_1(t)+I_2(t)$$

where

$$I_1(t)=\int_{|\xi|\leq r(t)}|e^{t(\widehat{J}(\xi)-1)}-e^{-Atp^\alpha(\xi)}||\widehat{\varphi}(\xi)|d\xi$$

and

$$I_2(t)=\int_{r(t)\leq|\xi|\leq R}|e^{t(\widehat{J}(\xi)-1)}-e^{-Atp^\alpha(\xi)}||\widehat{\varphi}(\xi)|d\xi.$$

Using that, for some positive constant $c$, the following holds

$$c|\xi|\leq p(\xi)\leq|\xi| \qquad \text{for all } \xi\in[-\pi,\pi]^d,$$

the term $I_2(t)$ can be estimated as follows:

$$\begin{aligned} t^{d/\alpha} I_2(t) &\leq t^{d/\alpha}\|\widehat{\varphi}\|_{L^\infty([-\pi,\pi]^d)} \int_{r(t)\leq|\xi|\leq R} (e^{-Atp^\alpha(\xi)} + e^{t(|\widehat{J}(\xi)|-1)})d\xi \\ &\leq t^{d/\alpha}\|\varphi\|_{L^1(\mathbb{Z}^d)} \int_{r(t)\leq|\xi|\leq R} (e^{-Atp^\alpha(\xi)} + e^{-t|\xi|^\alpha/2})d\xi \\ &\leq t^{d/\alpha}\|\varphi\|_{L^1(\mathbb{Z}^d)} \int_{r(t)\leq|\xi|\leq R} e^{-Bt|\xi|^\alpha} d\xi \\ &= \|\varphi\|_{L^1(\mathbb{Z}^d)} \int_{t^{1/\alpha}r(t)|\xi|\leq t^{1/\alpha}} e^{-B|\xi|^\alpha} d\xi \\ &\leq \|\varphi\|_{L^1(\mathbb{Z}^d)} t^{d/\alpha} e^{-Btr^\alpha(t)} \to 0. \end{aligned}$$

To estimate $I_1(t)$ we first observe that there exists a function $h(\xi)$ with $h(\xi) \to 0$ as $|\xi| \to 0$ and such that

$$|\widehat{J}(\xi) - 1 - A|\xi|^\alpha| \leq |\xi| h(\xi)$$

for all $\xi$ in a sufficiently small ball centered at the origin. Thus for all such $\xi$

$$|\widehat{J}(\xi) - 1 - Ap^\alpha(\xi)| \leq |\xi|^\alpha h(\xi) + ||\xi|^\alpha - p^\alpha(\xi)| \lesssim |\xi|^\alpha h(\xi) + |\xi|^{3\alpha}.$$

In view of this property we get

$$\begin{aligned} I_1(t) &\leq t^{d/\alpha}\|\widehat{\varphi}\|_{L^\infty([-\pi,\pi]^d)} \int_{|\xi|\leq r(t)} e^{-Atp^\alpha(\xi)} |e^{t(\widehat{J}(\xi)-1-Atp^\alpha(\xi))} - 1| d\xi \\ &\leq t^{d/\alpha}\|\varphi\|_{L^1(\mathbb{Z}^d)} \int_{|\xi|\leq r(t)} e^{-Atp^\alpha(\xi)} t|\xi|^\alpha (h(\xi) + |\xi|^{3\alpha}) d\xi \\ &\leq t^{d/\alpha}\|\varphi\|_{L^1(\mathbb{Z}^d)} \int_{|\xi|\leq r(t)} e^{-Bt|\xi|^\alpha} (t|\xi|^\alpha h(\xi) + t|\xi|^{4\alpha}) d\xi. \end{aligned}$$

The last term in the right hand side verifies

$$t^{d/\alpha} \int_{|\xi|\leq r(t)} e^{-Bt|\xi|^\alpha} t|\xi|^{4\alpha} d\xi \leq t^{-3} \int_{|\eta|\leq r(t)t^{1/\alpha}} e^{-B|\eta|^\alpha} |\eta|^{4\alpha} d\eta \to 0.$$

Hence we have to analyze the first one. In this case, by the same change of variables, we get

$$t^{d/\alpha} \int_{|\xi|\leq r(t)} e^{-Bt|\xi|^\alpha} t|\xi|^\alpha h(\xi) = \int_{|\eta|<r(t)t^{1/\alpha}} |\eta|^\alpha e^{-Bt|\eta|^\alpha} h(\eta t^{-1/\alpha}).$$

Applying Lebesgue convergence theorem we obtain that also this term converges to zero as $t \to \infty$. This ends the proof. □

Now we prove our second result, Theorem 1.2, that describes the asymptotic profile of solutions to $v' = -A(-\Delta_d)^{\alpha/2} v$.

*Proof of Theorem 1.2.* Using the Fourier representation of $v$ we have

$$v(j,t) = \int_{[-\pi,\pi]^d} e^{-Atp^\alpha(\xi)} e^{ij\xi} \widehat{\varphi}(\xi)\, d\xi, \qquad j \in \mathbb{Z}^d,\ t > 0.$$

Thus

$$\begin{aligned} t^{d/\alpha}v([jt^{1/\alpha}],t) &= t^{d/\alpha}\int_{[-\pi,\pi]^d} e^{-Atp^\alpha(\xi)}e^{i[jt^{1/\alpha}]\xi}\widehat{\varphi}(\xi)\,d\xi \\ &= \int_{[-\pi t^{1/\alpha},\pi t^{1/\alpha}]^d} e^{-Atp^\alpha(\xi t^{-1/\alpha})}\exp\Big(i\xi\frac{[jt^{1/\alpha}]}{t^{1/\alpha}}\Big)\widehat{\varphi}(\xi t^{-1/\alpha})\,d\xi \end{aligned}$$

and

$$G^A(j)=\int_{\mathbb{R}^d} e^{-A|\xi|^\alpha}e^{i\xi j}\,d\xi.$$

Denoting

$$I(j,t)=t^{d/\alpha}v([jt^{1/\alpha}],t)-\widehat{\varphi}(0)G^A(j)$$

we obtain

$$\begin{aligned} |I(j,t)| &\leq \Big|\int_{[-\pi t^{1/\alpha},\pi t^{1/\alpha}]^d} e^{-Atp^\alpha(\xi t^{-1/\alpha})}e^{ij\xi}-\widehat{\varphi}(0)\int_{\mathbb{R}^d} e^{-A|\xi|^\alpha}e^{i\xi j}d\xi\Big| \\ &\quad+\int_{[-\pi t^{1/\alpha},\pi t^{1/\alpha}]^d} e^{-Atp^\alpha(\xi t^{-1/\alpha})}\big|e^{ij\xi}-e^{i\xi[jt^{1/\alpha}]t^{-1/\alpha}}\big||\widehat{\varphi}(\xi t^{-1/\alpha})|d\xi \\ &= I_1(j,t)+I_2(j,t). \end{aligned}$$

Therefore we have to get bounds for $I_1(j,t)$ and $I_2(j,t)$.

**Step I. Estimates for $I_2(t)$.** For $I_2(t)$ we have the rough estimate

$$\begin{aligned} |I_2(j,t)| &\leq \|\widehat{\varphi}\|_{L^\infty([-\pi,\pi]^d)}\int_{[-\pi t^{1/\alpha},\pi t^{1/\alpha}]^d} e^{-Atp^\alpha(\xi t^{-1/\alpha})}\Big|\sin\big(\frac{jt^{1/\alpha}\xi-[jt^{1/\alpha}]\xi}{2t^{1/\alpha}}\big)\Big|d\xi \\ &\leq \|\varphi\|_{L^1(\mathbb{Z}^d)}\int_{[-\pi t^{1/\alpha},\pi t^{1/\alpha}]^d} e^{-Atp^\alpha(\xi t^{-1/\alpha})}\Big|\frac{jt^{1/\alpha}\xi-[jt^{1/\alpha}]\xi}{2t^{1/\alpha}}\Big|d\xi \\ &\leq \|\varphi\|_{L^1(\mathbb{Z}^d)}\int_{[-\pi t^{1/\alpha},\pi t^{1/\alpha}]^d} e^{-Atp^\alpha(\xi t^{-1/\alpha})}\frac{|\xi|}{t^{1/\alpha}}d\xi \\ &\lesssim t^{-1/\alpha}\|\varphi\|_{L^1(\mathbb{Z}^d)}\int_{[-\pi t^{1/\alpha},\pi t^{1/\alpha}]^d} e^{-c(\alpha)|\xi|^\alpha}|\xi|d\xi\to 0. \end{aligned}$$

**Step II. Estimates for $I_1(t)$.** Observe that $I_1$ satisfies:

$$\begin{aligned} I_1(j,t) &\leq \int_{[-\pi t^{1/\alpha},\pi t^{1/\alpha}]^d} |e^{-Atp^\alpha(\xi t^{-1/\alpha})}-e^{-A|\xi|^\alpha}||\widehat{\varphi}(\xi t^{-1/\alpha})|d\xi \\ &\quad+\int_{[-\pi t^{1/\alpha},\pi t^{1/\alpha}]^d} e^{-A|\xi|^\alpha}|\widehat{\varphi}(\xi t^{-1/\alpha})-\widehat{\varphi}(0)|d\xi \\ &\quad+|\widehat{\varphi}(0)|\int_{\xi\notin[-\pi t^{1/\alpha},\pi t^{1/\alpha}]^d} e^{-A|\xi|^\alpha}d\xi \\ &= I_3(t)+I_4(t)+I_5(t). \end{aligned}$$

In the case of the last integral, easily follows that $|\xi|\gtrsim t^{1/\alpha}$. Thus

$$I_5(t)\lesssim\int_{|\xi|\gtrsim t^{1/\alpha}} e^{-A|\xi|^\alpha}d\xi\to 0.$$

For $I_4$ we have the following estimate:

$$I_4(t) = \int_{\mathbb{R}^d} e^{-A|\xi|^\alpha} |\widehat{\varphi}(\xi t^{-1/\alpha}) - \widehat{\varphi}(0)| \chi_{[-\pi t^{1/\alpha}, \pi t^{1/\alpha}]^d}\, d\xi$$

and the Lebesgue dominated convergence theorem guarantees that $I_4(t) \to 0$ as $t \to \infty$.

Using that $p(\xi)$ satisfies $c|\xi| \leq p(\xi) \leq |\xi|$ for some positive $c$ and the mean value theorem we get:

$$\begin{aligned} I_3(t) &\leq \|\widehat{\varphi}\|_{L^\infty([-\pi,\pi]^d)} \int_{[-\pi t^{1/\alpha}, \pi t^{1/\alpha}]^d} e^{-Atp^\alpha(\xi t^{-1/\alpha})} |tp^\alpha(\xi t^{-1/\alpha}) - |\xi|^\alpha|\, d\xi \\ &\lesssim \|\varphi\|_{L^1(\mathbb{Z}^d)} \int_{[-\pi t^{1/\alpha}, \pi t^{1/\alpha}]^d} e^{-c|\xi|^\alpha} |tp^\alpha(\xi t^{-1/\alpha}) - |\xi|^\alpha|\, d\xi. \end{aligned}$$

Applying again the dominated convergence theorem we obtain that $I_3(t) \to 0$ as $t \to \infty$.

The proof is now complete. $\square$

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