

SOME ALGEBRAIC PROPERTIES OF NON-COMMUTATIVE FUZZY STRUCTURES

Laurențiu Leuştean

National Institute for Research and Development in Informatics-ICI, Bucharest
E-mail: leo@u3.ici.ro

Abstract: Pseudo-BL algebras [2, 5] are non-commutative generalizations of BL-algebras, the algebraic structures for Basic Logic, the fuzzy logic introduced by Hájek [7]. In this paper we study some algebraic properties concerning co-annihilators and minimal prime filters on pseudo-BL algebras, extending some results obtained in [1] for distributive lattices.

Keywords Non-commutative fuzzy logic, pseudo-BL algebra, co-annihilators, minimal prime filters

1 Introduction

BL-algebras are the algebraic structures for Hájek's Basic Logic [7]. The main example of a BL-algebra is the interval $[0, 1]$ endowed with the structure induced by a t -norm. MV-algebras, Gödel algebras and product algebras are the most known classes of BL-algebras. Recent investigations are concerned with non-commutative generalizations for these structures.

In [2, 5], pseudo-BL algebras were defined as non-commutative generalizations of BL-algebras. In [4], there was introduced a notion of pseudo- t -norm in order to recapture some of the properties of pseudo-BL algebras. For the interval $[0, 1]$, this notion induces more general algebras named weak pseudo-BL algebras.

In [1], Davey studies the interrelation between minimal prime ideals conditions and annihilators conditions on distributive lattices.

In this paper we extend some of Davey's

results to pseudo-BL algebras. The same results hold in BL-algebras too, since they are a particular case.

2 Definitions and first properties

A *pseudo-BL algebra* ([2, 5]) is an algebra $\mathbf{A} = (A, \wedge, \vee, \odot, \rightsquigarrow, \rightarrow, 0, 1)$ with five binary operations $\wedge, \vee, \odot, \rightsquigarrow, \rightarrow$ and two constants $0, 1$ such that:

- (A1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;
- (A2) $(A, \odot, 1)$ is a monoid;
- (A3) $a \odot b \leq c$ iff $a \leq b \rightsquigarrow c$ iff $b \leq a \rightarrow c$;
- (A4) $a \wedge b = (a \rightsquigarrow b) \odot a = a \odot (a \rightarrow b)$;
- (A5) $(a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1$;
 $(a \rightarrow b) \vee (b \rightarrow a) = 1$.

In the sequel, we shall agree that the operations \wedge, \vee, \odot have priority towards the operations $\rightsquigarrow, \rightarrow$. Sometimes, for the sake of clearness, we shall put parentheses even if this is not necessary.

It is proved in [2] that commutative pseudo-BL algebras are BL-algebras. For details on BL-algebras see [7, 8].

A pseudo-BL algebra \mathbf{A} is non-trivial iff $0 \neq 1$. For any pseudo-BL algebra \mathbf{A} , the reduct $L(\mathbf{A}) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. A *pseudo-BL chain* is a linear pseudo-BL algebra, i.e. a pseudo-BL algebra such that its lattice order is total.

For any $a \in A$, we define $a^{\sim} = a \rightsquigarrow 0$ and $a^{-} = a \rightarrow 0$. We shall write a^{\approx} instead of $(a^{\sim})^{\sim}$ and $a^{\bar{}}$ instead of $(a^{-})^{-}$.

We denote the set of natural numbers by ω . We define $a^0 = 1$ and $a^n = a^{n-1} \odot a$

for $n \in \omega - \{0\}$. The *order* of $a \in A$, in symbols $ord(a)$, is the smallest $n \in \omega$ such that $a^n = 0$. If no such n exists, then $ord(a) = \infty$.

The following properties hold in any pseudo-BL algebra \mathbf{A} and will be used in the sequel. See [2] for details.

- (1) $a \odot b \leq a, b$;
- (2) $a \leq b$ implies $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$;
- (3) $a \odot b \leq a \wedge b$;
- (4) $(a \vee b) \odot (a \vee c) \leq a \vee (b \odot c)$

Let \mathbf{A} be a pseudo-BL algebra. According to [2], a *filter* of \mathbf{A} is a non-empty subset F of A such that for all $a, b \in A$,

- (i) if $a, b \in F$, then $a \odot b \in F$;
- (ii) if $a \in F$ and $a \leq b$, then $b \in F$.

By (3), it is obvious that any filter of \mathbf{A} is also a filter of the lattice $L(A)$.

A filter F of \mathbf{A} is *proper* if $F \neq A$. A proper filter P of \mathbf{A} is *prime* if for all $a, b \in A$, $a \vee b \in P$ implies $a \in P$ or $b \in P$. We shall denote by $Spec(A)$ the set of prime filters of the pseudo-BL algebra \mathbf{A} .

A proper filter U of \mathbf{A} is an *ultrafilter* (or a *maximal filter*) if it is not contained in any other proper filter. We shall denote by $\mathcal{M}(A)$ the intersection of all ultrafilters of \mathbf{A} . Obviously, $\mathcal{M}(A)$ is a proper filter of \mathbf{A} .

We remind some properties of filters that will be used in the sequel.

Proposition 2.1 ([2], Theorem 3.25)

Let F be a filter of the pseudo-BL algebra \mathbf{A} and let S be a \vee -closed subset of A (i.e. if $a, b \in S$, then $a \vee b \in S$) such that $F \cap S = \emptyset$. Then there exists a prime filter P of \mathbf{A} such that $F \subseteq P$ and $P \cap S = \emptyset$.

Proposition 2.2 Any proper filter of \mathbf{A} can be extended to a prime filter.

Proof: By [2], Corollary 3.26. \square

Proposition 2.3 ([2], Corollary 3.32)
Any ultrafilter of \mathbf{A} is a prime filter of \mathbf{A} .

Proposition 2.4 ([2], Remark 3.33)

Any proper filter of \mathbf{A} can be extended to an ultrafilter.

Let $X \subseteq A$. The filter of \mathbf{A} generated by X will be denoted by $\langle X \rangle$. We have that $\langle \emptyset \rangle = \{1\}$ and $\langle X \rangle = \{a \in A \mid x_1 \odot \cdots \odot x_n \leq a \text{ for some } n \in \omega - \{0\} \text{ and some } x_1, \dots, x_n \in X\}$ if $\emptyset \neq X \subseteq A$. For any $a \in A$, $\langle a \rangle$ denotes the principal filter of \mathbf{A} generated by $\{a\}$. It follows that $\langle a \rangle = \{b \in A \mid a^n \leq b \text{ for some } n \in \omega - \{0\}\}$.

Proposition 2.5 ([2], Lemma 3.11)

For any $a, b \in A$,

$$\langle a \vee b \rangle = \langle a \rangle \cap \langle b \rangle.$$

Let us denote by $\mathcal{F}(\mathbf{A})$ the set of all filters of \mathbf{A} . Then

Proposition 2.6 ([2], Proposition 3.8)

$(\mathcal{F}(\mathbf{A}), \subseteq)$ is a complete lattice. For every family $\{F_i\}_{i \in I}$ of filters of A , we have that $\bigwedge_{i \in I} F_i = \bigcap_{i \in I} F_i$ and $\bigvee_{i \in I} F_i = \langle \bigcup_{i \in I} F_i \rangle$.

3 Co-annihilators and co-annihilator filters

Let \mathbf{A} be a pseudo-BL algebra, F be a filter of \mathbf{A} and $a \in A$. The *co-annihilator of a relative to F* is the set $(F, a) = \{x \in A \mid x \vee a \in F\}$. To indicate the relevant pseudo-BL algebra, (F, a) is sometimes written $(F, a)_A$. The co-annihilator $\langle b \rangle, a$ is abbreviated to (b, a) .

Proposition 3.1 Let F be a filter of \mathbf{A} and $a \in A$. Then (F, a) is a filter of \mathbf{A} .

Proof: We have that $a \vee 1 = 1 \in F$, hence $1 \in (F, a)$. If $x \leq y$ and $x \in (F, a)$, then $x \vee a \in F$ and $x \vee a \leq y \vee a$, so $y \vee a \in F$, that is $y \in (F, a)$. Suppose now that $x, y \in (F, a)$, i.e. $x \vee a, y \vee a \in F$. It follows that $(x \vee a) \odot (y \vee a) \in F$. But, by (4), we

have that $(x \vee a) \odot (y \vee a) \leq (x \odot y) \vee a$. Hence, $(x \odot y) \vee a \in F$, so $x \vee y \in (F, a)$. \square

Proposition 3.2 Let F, G be filters of \mathbf{A} and $a, b \in A$. Then

- (i) $F \subseteq (F, a)$;
- (ii) $a \leq b$ implies $(F, a) \subseteq (F, b)$;
- (iii) $F \subseteq G$ implies $(F, a) \subseteq (G, a)$;
- (iv) $(F, a) = A$ iff $a \in F$;
- (v) $(F, a \wedge b) = (F, a) \cap (F, b)$;
- (vi) $(F \cap G, a) = (F, a) \cap (G, a)$;
- (vii) $(b, a) = (b, a \wedge b) = (a \vee b, a) = (b, a \odot b)$.

Proof: (i) Let $x \in F$. Then $x \vee a \geq x \in F$, hence $x \vee a \in F$. That is, $x \in (F, a)$.

(ii) Let $x \in (F, a)$. Then $x \vee a \in F$ and $x \vee a \leq x \vee b$, since $a \leq b$. It follows that $x \vee b \in F$, that is $x \in (F, b)$.

(iii) Let $x \in (F, a)$. Then $x \vee a \in F \subseteq G$, hence $x \in (G, a)$.

(iv) If $(F, a) = A$, then $0 \in (F, a)$, hence $a = a \vee 0 \in F$. Conversely, if $a \in F$, then for any $x \in A$, $a \leq x \vee a$, so $x \vee a \in F$. That is, for any $x \in A$, $x \in (F, a)$.

(v) Since $a \wedge b \leq a, b$, by (ii), it follows that $(F, a \wedge b) \subseteq (F, a) \cap (F, b)$. Conversely, let $x \in (F, a) \cap (F, b)$, i.e. $x \vee a \in F$ and $x \vee b \in F$. Since F is also a filter of the distributive lattice $L(A)$, we have that $x \vee (a \wedge b) = (x \vee a) \wedge (x \vee b) \in F$. That is, $x \in (F, a \wedge b)$.

(vi) Applying (iii) and the fact that $F \cap G \subseteq F, G$, it follows that $(F \cap G, a) \subseteq (F, a) \cap (G, a)$. The converse inclusion is obvious.

(vii) Since $a \odot b \leq a \wedge b \leq a$, by (ii), we have that $(b, a \odot b) \subseteq (b, a \wedge b) \subseteq (b, a)$. We shall prove now that $(b, a) \subseteq (b, a \odot b)$. Let $x \in (b, a)$, so $x \vee a \in \langle b \rangle$. Then, there is $n \in \omega - \{0\}$ such that $b^n \leq x \vee a$. Applying twice (2) and (4), it follows that $b^{n+1} \leq (x \vee a) \odot b \leq (x \vee a) \odot (x \vee b) \leq x \vee (a \odot b)$. Hence, $x \vee (a \odot b) \in \langle b \rangle$, that is $x \in (b, a \odot b)$. Thus, we have proved that $(b, a) = (b, a \wedge b) = (b, a \odot b)$. Applying Proposition 2.5, (v) and (iv), we get that

$$(a \vee b, a) = (\langle a \vee b \rangle, a) = (\langle a \rangle \cap \langle b \rangle, a) = (\langle a \rangle, a) \cap (\langle b \rangle, a) = A \cap (b, a) = (b, a). \quad \square$$

If X is a non-empty subset of A , then ${}^\perp X = \{a \in A \mid x \vee a = 1 \text{ for any } x \in X\}$ is a filter of \mathbf{A} called the *co-annihilator filter* of X (see [2]). It is easy to see that for any $a \in A$, ${}^\perp a = (\{1\}, a) = \{x \in A \mid x \vee a = 1\}$.

Proposition 3.3 ([2], Proposition 3.37)

Let $\emptyset \neq X, Y \subseteq A$. Then,

- (i) If $X \subseteq Y$, then ${}^\perp Y \subseteq {}^\perp X$;
- (ii) $X \subseteq {}^{\perp\perp} X$;
- (iii) ${}^\perp X = {}^{\perp\perp\perp} X$;
- (iv) ${}^\perp X = {}^\perp \langle X \rangle$;
- (v) $\langle X \rangle \cap {}^\perp X = \{1\}$.

Now, let us recall some facts from lattice theory (see [6]). Let $(L, \vee, \wedge, 0)$ be a lattice with 0. An element $a^* \in L$ is a *pseudocomplement* of $a \in L$ iff $a \wedge a^* = 0$ and $a \wedge x = 0$ implies that $x \leq a^*$. The lattice L is called *pseudocomplemented* iff every element has a pseudocomplement.

Proposition 3.4 Let \mathbf{A} be a pseudo-BL algebra. Then the lattice $\mathcal{F}(A)$ is pseudocomplemented. For any filter F , its pseudocomplement is ${}^\perp F$.

Proof: By Proposition 3.3(v), we have that $F \cap {}^\perp F = \{1\}$. Let G be a filter of \mathbf{A} such that $F \cap G = \{1\}$. We shall prove that $G \subseteq {}^\perp F$. Let $a \in G$. For any $x \in F$, we have that $x \vee a \in F \cap G = \{1\}$, since $x \vee a \geq x \in F$ and $a \vee x \geq a \in G$. Hence, $x \vee a = 1$ for any $x \in F$, so $a \in {}^\perp F$. It follows that ${}^\perp F$ is the pseudocomplement of F . \square

Let $Co - An(A) = \{{}^\perp X \mid X \subseteq A\}$ be the set of co-annihilator filters of \mathbf{A} . Applying Proposition 3.3(iv), we get that $Co - An(A) = \{{}^\perp F \mid F \in \mathcal{F}(A)\}$. Hence, $Co - An(A)$ is the set of pseudocomplements of the pseudocomplemented lattice $\mathcal{F}(A)$. Applying known results from lattice theory, the following proposition follows.

Proposition 3.5 Let \mathbf{a} be a pseudo-BL algebra and F, G filters of \mathbf{A} . Then

- (i) ${}^{\perp}1 = A$ and ${}^{\perp}A = \{1\}$;
- (ii) $\{1\}, A \in Co - An(A)$;
- (iii) $F \in Co - An(A)$ iff ${}^{\perp\perp}F = F$;
- (iv) if $F, G \in Co - An(A)$, then $F \cap G \in Co - An(A)$;
- (v) if $F, G \in Co - An(A)$, then $F \vee_{Co - An(A)} G = {}^{\perp}({}^{\perp}F \cap {}^{\perp}G)$;
- (vi) $(Co - An(A), \cap, \vee_{Co - An(A)}, \perp, \{1\}, A)$ is a Boolean algebra;
- (vii) ${}^{\perp\perp}(F \cap G) = {}^{\perp\perp}F \cap {}^{\perp\perp}G$.

Proof: See [6], Theorem 6.4, p. 58 and Theorem 15.1, p. 166 \square

Proposition 3.6 Let $a, b \in A$. Then

- (i) $a \leq b$ implies ${}^{\perp}a \subseteq {}^{\perp}b$ and ${}^{\perp\perp}b \subseteq {}^{\perp\perp}a$;
- (ii) ${}^{\perp\perp}a = \{x \in A \mid x \vee y = 1 \text{ for any } y \in A \text{ such that } y \vee a = 1\}$;
- (iii) $a \in {}^{\perp\perp}a$;
- (iv) ${}^{\perp\perp}a \cap {}^{\perp\perp}b = {}^{\perp\perp}(a \vee b)$.

Proof: (i) $a \leq b$ implies $\langle b \rangle \subseteq \langle a \rangle$. Hence, applying Proposition 3.3(v), (ii), we have that ${}^{\perp}a = {}^{\perp}\langle a \rangle \subseteq {}^{\perp}\langle b \rangle = {}^{\perp}b$. Applying again Proposition 3.3(ii), we get ${}^{\perp\perp}b \subseteq {}^{\perp\perp}a$.
(ii) By the definition.
(iii) It follows from (ii).
(iv) Applying Proposition 3.3(iv), Proposition 3.5(vii) and Proposition 2.5 we get that ${}^{\perp\perp}a \cap {}^{\perp\perp}b = {}^{\perp\perp}\langle a \rangle \cap {}^{\perp\perp}\langle b \rangle = {}^{\perp\perp}(\langle a \rangle \cap \langle b \rangle) = {}^{\perp\perp}\langle a \vee b \rangle = {}^{\perp\perp}(a \vee b)$. \square

4 Minimal prime filters belonging to a filter F

A prime filter M which is minimal in the poset of prime filters containing a filter F is called a *minimal prime filter belonging to F* . A minimal prime filter belonging to $\{1\}$ is called simply a *minimal prime filter*. Hence, a minimal prime filter of \mathbf{A} is a

minimal element of the poset $(Spec(A), \subseteq)$.

In the sequel, we shall present some results concerning minimal prime filters belonging to a filter. All these propositions are inspired by [1].

Proposition 4.1 If S is \vee -closed and F is a proper filter of \mathbf{A} , then there is a minimal prime filter M belonging to F and disjoint from S .

Proof: If F is a proper filter, then the set $\{P \in Spec(A) \mid F \subseteq P \text{ and } F \cap S = \emptyset\}$ is non-empty, by Proposition 2.1. Apply Zorn's Lemma to get a minimal element of this set. \square

Proposition 4.2 Let F be a filter of \mathbf{A} and M be a prime filter including F . The following are equivalent:

- (i) M is a minimal prime filter belonging to F ;
- (ii) for all $a \in M$, there is $b \notin M$ such that $a \vee b \in F$.

Proof: (i) \Rightarrow (ii) Let $a \in M$ and let $S = \{a \vee b \mid b \in A - M\}$. If $a \vee b_1, a \vee b_2 \in S$, then $(a \vee b_1) \vee (a \vee b_2) = a \vee (b_1 \vee b_2) \in S$, since $A - M$ is \vee -closed, so $b_1, b_2 \in A - M$ implies $b_1 \vee b_2 \in A - M$. Hence, S is \vee -closed. Let us suppose that $F \cap S = \emptyset$. Applying Proposition 2.1, there exist a prime filter P such that $F \subseteq P$ and $P \cap S = \emptyset$. Since $a = a \vee 0$ and $0 \in A - M$, we have that $a \in S$, so $a \notin P$. It follows that $M \neq P$, since $a \in M$ and $a \notin P$. If there is $x \in P$ such that $x \notin M$, then $x \in A - M$, so $a \vee x \in S$. But $x \leq a \vee x$ and $x \in P$, hence $a \vee x \in P$. We have got that $a \vee x \in P \cap S = \emptyset$, that is a contradiction. Hence, $P \subseteq M$ and $P \neq M$, which contradicts the fact that M is a minimal prime filter belonging to F . It follows that $F \cap S \neq \emptyset$. Hence, there is $b \in A - M$ such that $a \vee b \in F$.

(ii) \Rightarrow (i) Let P be a prime filter of \mathbf{A} such that $F \subseteq P \subseteq M$. We shall prove that

$M \subseteq P$ also. Let $a \in M$. Then there is $b \notin M$ such that $a \vee b \in F \subseteq P$. Since P is a prime filter of \mathbf{A} , from $a \vee b \in P$ it follows that $a \in P$ or $b \in P$. But $b \notin M$ and $P \subseteq M$, so $b \notin P$. We get that $a \in P$. \square

A simple induction gives the following corollary for $n \in \omega - \{0\}$.

Corollary 4.3 Let F be a filter of \mathbf{A} and M_0, \dots, M_n be $n+1$ distinct minimal prime filters belonging to F . Then, there are $a_0, \dots, a_n \in A$ such that $a_i \vee a_j \in F$ ($i \neq j$) and $a_i \notin M_i$ ($i = 0, \dots, n$).

Proof: If $n = 1$, let $x_0 \in M_1 - M_0$ and $x_1 \in M_0 - M_1$. By the above proposition, it follows that there is $y_1 \notin M_0$ such that $x_1 \vee y_1 \in F$. Since M_0 is prime, from $x_0 \in /M_0$ and $y_1 \notin M_0$, we get that $x_0 \vee y_1 \notin M_0$. It follows that $a_0 = x_0 \vee y_1$ and $a_1 = x_1$ are the required elements. Assume the result is true for $n = k$ and let us prove it for $n = k + 1$. Let M_0, \dots, M_{k+1} be $k + 2$ distinct minimal prime filters belonging to F . Let x_i ($i = 0, \dots, k$) satisfy $x_i \vee x_j \in F$ ($i \neq j$) and $x_i \notin M_i$ ($i = 0, \dots, k$). For any $i = 0, \dots, k$, there is $y_i \in M_{k+1}$ such that $y_i \notin M_i$. If we take $y = y_1 \wedge \dots \wedge y_k$, then $y \in M_{k+1} - \bigcup_{i=0}^k M_i$. Applying Proposition 4.2, we get $z \notin M_{k+1}$ such that $y \vee z \in F$. It follows that $a_i = x_i \vee y$ ($i = 0, \dots, k$) and $a_{k+1} = z$ establish the result. \square

From now on all indexed joins and meets range from 0 to n , where $n \in \omega - \{0\}$, and all joins of filters are taken in the lattice $\mathcal{F}(A)$ of filters of \mathbf{A} . A family of filters is *comaximal* if its join is A .

Proposition 4.4 Let F be a filter of \mathbf{A} . Then for $n \in \omega - \{0\}$ the following are equivalent:

- (i) any $n + 1$ distinct minimal prime filters belonging to F are comaximal;
- (ii) any prime filter containing F contains at most n distinct minimal prime filters belonging to F ;

- (iii) if $a_0, \dots, a_n \in A$ with $a_i \vee a_j \in F$ ($i \neq j$), then $\bigvee_i (F, a_i) = A$;
- (iv) $(F, \bigvee_j a_j) = \bigvee_i (F, \bigvee_{j \neq i} a_j)$ holds identically in \mathbf{A} .

Proof: (i) \Leftrightarrow (ii) Trivial.

(ii) \Rightarrow (iii) Let $a_0, \dots, a_n \in A$ such that $a_i \vee a_j \in F$ ($i \neq j$) and $\bigvee_i (F, a_i) \neq A$. It follows that $(F, a_i) \neq A$ for all $i = 0, \dots, n$ and, by Proposition 3.2(iv), we get that $a_i \notin F$ for all i . Since $\bigvee_i (F, a_i)$ is a proper filter of \mathbf{A} , applying Proposition 2.2 we obtain a prime filter P of \mathbf{A} such that $\bigvee_i (F, a_i) \subseteq P$. For $i = 0, \dots, n$, let $S_i = \{x \vee y \mid x \leq a_i, y \notin P\}$. If $x_1 \vee y_1, x_2 \vee y_2 \in S_i$, then $x_1 \leq a_i, x_2 \leq a_i$ and $y_1, y_2 \notin P$. Hence $x_1 \vee x_2 \leq a_i$ and $y_1 \vee y_2 \notin P$, since P is prime. It follows that $(x_1 \vee y_1) \vee (x_2 \vee y_2) = (x_1 \vee x_2) \vee (y_1 \vee y_2) \in S_i$. Hence, S_i is \vee -closed. If $F \cap S_i \neq \emptyset$, then there are $x \leq a_i$ and $y \notin P$ such that $x \vee y \in F$. It follows that $a_i \vee y \in F$, hence $y \in (F, a_i)$. But $y \notin P$, so $(F, a_i) \not\subseteq P$, which is a contradiction. Hence $F \cap S_i = \emptyset$ for all i . Applying Proposition 4.1, for each i there is M_i a minimal prime filter belonging to F such that $M_i \cap S_i = \emptyset$. Suppose that $M_i \not\subseteq P$, so there is $x \in M_i$ such that $x \notin P$. Then $x = x \vee 0 \in S_i$, hence $x \in M_i \cap S_i = \emptyset$, that is a contradiction. Hence, $M_i \subseteq P$ for all i , so $\bigvee_i M_i \subseteq P \neq A$. Since $a_i \notin P$, we get that $a_i \notin M_i$ for any i . But, $a_i \vee a_j \in F \subseteq M_i$ ($i \neq j$) and M_i is prime, hence $a_j \in M_i$ ($i \neq j$). It follows that $M_i \neq M_j$ ($i \neq j$). Thus, we have obtained $n + 1$ minimal prime filters belonging to F such that their join is not A , that is a contradiction with (ii).

(iii) \Rightarrow (iv) Applying Proposition 3.2(ii) it follows that $\bigvee_i (F, \bigvee_{j \neq i} a_j) \subseteq (F, \bigvee_j a_j)$. Thus, it remains to prove that $(F, \bigvee_j a_j) \subseteq \bigvee_i (F, \bigvee_{j \neq i} a_j)$. Let $x \in (F, \bigvee_j a_j)$, i.e. $x \vee \bigvee_j a_j \in F$. For any i , let $b_i = x \vee \bigvee_{j \neq i} a_j$. Then, $b_i \vee b_j = x \vee \bigvee_j a_j \in F$ for all $j \neq i$, hence, by (iii), we get that $\bigvee_i (F, b_i) = A$. From $x \in \bigvee_i (F, b_i)$, we get $k \in \omega - \{0\}$ and $y_1^0, \dots, y_k^0 \in (F, b_0), \dots, y_1^n, \dots, y_k^n \in$

(F, b_n) such that $y_1^0 \odot \cdots \odot y_1^n \odot \cdots \odot y_k^0 \odot \cdots \odot y_k^n \leq x$. Letting $t_p^i = x \vee y_p^i$ ($i = 0, \dots, n$, and $p = 1, \dots, k$), we have that $x \leq t_p^i$ and $t_p^i \in (F, b_i)$, since $y_p^i \leq t_p^i$ and $y_p^i \in (F, b_i)$. Since $x \leq t_p^i$, we have that $t_p^i \vee \bigvee_{j \neq i} a_j = t_p^i \vee x \vee \bigvee_{j \neq i} a_j = t_p^i \vee b_i \in F$, because $t_p^i \in (F, b_i)$. Hence, $t_p^i \in (F, \bigvee_{j \neq i} a_j)$ ($i = 0, \dots, n, p = 1, \dots, k$). Applying (4), we get that $(t_1^0 \odot \cdots \odot t_1^n \odot \cdots \odot t_k^0 \odot \cdots \odot t_k^n) = (x \vee y_1^0) \odot \cdots \odot (x \vee y_1^n) \odot \cdots \odot (x \vee y_k^0) \odot \cdots \odot (x \vee y_k^n) \leq x \vee (y_1^0 \odot \cdots \odot y_1^n \odot \cdots \odot y_k^0 \odot \cdots \odot y_k^n) = x$. Thus, we have got that $x \in \bigvee_i (F, \bigvee_{j \neq i} a_j)$

(iv) \Rightarrow (i) Let M_0, \dots, M_n be $n+1$ distinct prime filters belonging to F . Then, by Corollary 4.3 there are $a_0, \dots, a_n \in A$ such that $a_i \vee a_j \in F$ ($i \neq j$) and $a_i \notin M_i$ ($i = 0, \dots, n$). We have that for $n \in \omega - \{0\}$, $\bigvee_j (\bigwedge_{k \neq j} a_k) = \bigwedge_{j < k} (a_j \vee a_k)$ is an identity in the class of distributive lattices, hence in the class of pseudo-BL algebras. We shall denote this identity by (I). Applying (I), it follows that $\bigvee_{j \neq i} (\bigwedge_{k \neq j} a_k) = \bigvee_{j \neq i} (a_i \wedge (\bigwedge_{k \notin \{i, j\}} a_k)) = a_i \wedge (\bigvee_{j \neq i} (\bigwedge_{k \notin \{i, j\}} a_k)) = a_i \wedge (\bigwedge_{j < k, i \notin \{j, k\}} (a_j \vee a_k))$. But $\bigwedge_{j < k, i \notin \{j, k\}} (a_j \vee a_k) \in F$, hence, by Proposition 3.2(iv) we get that $(F, \bigwedge_{j < k, i \notin \{j, k\}} (a_j \vee a_k)) = A$. Applying Proposition 3.2(v), it follows that $(F, \bigvee_{j \neq i} (\bigwedge_{k \neq j} a_k)) = (F, a_i \wedge (\bigwedge_{j < k, i \notin \{j, k\}} (a_j \vee a_k))) = (F, a_i) \cap (F, \bigwedge_{j < k, i \notin \{j, k\}} (a_j \vee a_k)) = (F, a_i) \cap A = (F, a_i)$. If $x \in (F, a_i)$, then $x \vee a_i \in F \subseteq M_i$. Since M_i is a prime filter and $a_i \notin M_i$, we get that $x \in M_i$. Hence, $M_i \supseteq (F, a_i)$ for all $i = 0, \dots, n$. It follows that $\bigvee_i M_i \supseteq \bigvee_i (F, a_i) = \bigvee_i (F, \bigvee_{j \neq i} (\bigwedge_{k \neq j} a_k)) = (F, \bigvee_j (\bigwedge_{k \neq j} a_k))$, by (iv). But, applying (I), we obtain that $\bigvee_j (\bigwedge_{k \neq j} a_k) = \bigwedge_{j < k} (a_j \vee a_k) \in F$, so, by Proposition 3.2(iv), $(F, \bigvee_j (\bigwedge_{k \neq j} a_k)) = A$. It follows that $\bigvee_i M_i \supseteq A$, that is $\bigvee_i M_i = A$, as required. \square

Corollary 4.5 Let $n \in \omega - \{0\}$. The following are equivalent:

(i) any $n+1$ distinct minimal prime filters are comaximal;

(ii) any prime filter contains at most n distinct minimal prime filters;

(iii) if $a_0, \dots, a_n \in A$ such that $a_i \vee a_j = 1$ ($i \neq j$), then $\bigvee_i {}^\perp a_i = A$;

(iv) ${}^\perp (\bigvee_j a_j) = \bigvee_i {}^\perp (\bigvee_{j \neq i} a_j)$ holds identically in \mathbf{A} .

Proof: Take $F = \{1\}$ in Proposition 4.4. \square

References

- [1] B. A. Davey, Some annihilator conditions on distributive lattices, *Algebra Universalis*, 4(1974), 316-322.
- [2] A. Di Nola, G. Georgescu, A. Iorgulescu, Pseudo-BL algebras: Part I, submitted.
- [3] A. Di Nola, G. Georgescu, A. Iorgulescu, Pseudo-BL algebras: Part II, manuscript.
- [4] P. Flondor, G. Georgescu, A. Iorgulescu, Pseudo t-norms and pseudo-BL algebras, submitted.
- [5] G. Georgescu, A. Iorgulescu, Pseudo-BL algebras: a noncommutative extension of BL-algebras (Abstract), *The Fifth International Conference FSTA 2000 on Fuzzy Sets Theory and Its Application*, February 2000, 90-92.
- [6] G. Grätzer, *Lattice Theory. First Concepts and Distributive Lattices*, W. H. Freeman and Company, San Francisco, 1972.
- [7] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, 1998.
- [8] E. Turunen, *Mathematics Behind Fuzzy Logic*, Physica-Verlag, Heidelberg, 1999.