Scientific Report for Phase I of the Project

1 Scientific results

The following lines give a short description of the research results obtained by the members of the team during the first phase of the project.

- In the paper *Classification of Pseudo-Riemannian submersions with totally geodesic fibres from pseudo-hyperbolic spaces* (by *Gabriel Baditoiu*; paper published in **Proc. London Math. Soc.**) we classified pseudo-Riemannian submersions with connected totally geodesic fibres from a real pseudo-hyperbolic space onto a pseudo-Riemannian manifold. In section 3, we exhibited the construction of the Hopf pseudo-Riemannian submersions from pseudo-hyperbolic spaces, which ensures the existence of at least one pseudo-Riemannian submersion in each class (a)-(g) of Theorem 1.1. In section 4, we saw that the base space is isometric to either a pseudo-hyperbolic space or a complete, simply connected, special Osserman pseudo-Riemannian manifold, which was classified in A. Bonome, R. Castro, E. Garcia-Rio, L. Hervella and R. Vazquez-Lorenzo, Pseudo-Riemannian manifolds with simple Jacobi operators, J. Math. Soc. Japan 54 (2002), no. 4, 847-875. To exclude the Cayley planes of octonions, and of para-octonions from the list of possible base spaces, we proved that the curvature tensor of base space has a Clifford structure. For the remaining cases, we established that the dimension and the index of the total space are, in fact, those claimed in Theorem 1.1. This reduced the equivalence problem of two pseudo-Riemannian submersions to the one of the same base space, which we resolved in section 5. Section 6 features consequences of Theorem 1.1: (a) the classification of the pseudo-Riemannian submersions with totally geodesic fibres from complex pseudo-hyperbolic spaces or from para-complex projective spaces under the assumption that the fibres are, respectively, complex or para-complex submanifolds and (b) the non-existence of the pseudo-Riemannian submersions with quaternionic or para-quaternionic fibres from quaternionic pseudo-hyperbolic spaces and para-quaternionic projective spaces, respectively.

• In the paper *Spectral geometry of Riemannian Legendre foliations* (by *Gabriel Baditoiu*, Stere Ianus, Anna Maria Pastore; paper published in **Bull. Math. Soc. Sci. Math. Roumanie**) we computed the first spectral invariants of a Riemannian Legendre foliation with minimal leaves on a Sasakian manifold M of constant φ-sectional curvature and we obtained certain geometric properties of two such isospectral Riemannian foliations. In the final section we provided an example on the 3-dimensional sphere, and we made some remarks on the non-existence of such foliations on some higher dimensional spheres.

• Frobenius manifolds represent a geometrization of the WDVV-equations (Witten-Dijkgraaf-Verlinde-Verlinde equations) and appear in various areas of mathematics (singularity theory, Gromow-Witten invariants, integrable systems). A Frobenius manifold is a manifold M together with an (associative, commutative, with unit field) multiplication $\circ_M$ on TM and a (non-degenerate) multiplication invariant flat metric g_M, which satisfy certain compatibility conditions. The associativity of $\circ_M$ reduces, in flat coordinates for the metric g_M, to the WDVV-equations. In the paper *On adding a variable to a Frobenius manifold and generalizations* (by *Liana David*; paper published in **Geom. Dedicata**), we considered a vector bundle $\pi : V \to M$ whose base M has a multiplication $\circ_M$ and a metric g_M on TM which make $(M, \circ_M, g_M)$ a Frobenius manifold, and whose typical fiber has the structure of a Frobenius algebra. Using a connection D on the bundle $\pi : V \to M$, we constructed a multiplication $\circ_V$ and a metric g_V on the manifold V and we determined necessary and sufficient conditions for $(V, \circ_V, g_V)$ to be Frobenius. When $\pi : V \to M$ is trivial of rank one and the connection D is the standard flat canonical connection, we reobtained the well-known construction of adding a variable to a Frobenius manifold, from C. Sabbah's book *Isomonodromic Deformations and Frobenius manifolds* (2005), Chapter VII. Similar type of constructions for tt^* were also developed.

• In the paper *A note about invariant SKT-structures and generalized Kähler structures on flag manifolds* (by Dmitri Alekseevsky and *Liana David*; paper published in **Proc. Edinb. Math. Soc.**), we proved that any invariant SKT-structure (strong Kähler with torsion) defined on an homogeneous space of a compact semisimple Lie group is Kähler. (We recall that an almost Hermitian manifold (M, J, Ω) with Kähler form Ω and almost complex structure J is called SKT, if $d(Jd\Omega) = 0$). Applications to generalized complex geometry were developed. The methods which are used in this paper are algebraic, and use the structure theory and representation theory of semisimple Lie algebras.

• The paper *F-manifolds and eventual identities* (by *Liana David* and Ian Strachan; paper published in **Oberwolfach Reports**) is a survey of the results obtained by the two authors on the theory of Frobenius manifolds and their relation to tt^*-geometry (namely, a duality for F-manifolds with eventual identities, which generalizes Dubrovin's almost duality for Frobenius manifolds, and various of its applications related to the behaviour of the basic structures from tt^*-geometry under this duality).

• In the paper *A note on CR quaternionic maps* (by Stefano Marchiafava and *Radu Pantilie*; paper published in **Adv. Geom**), we introduced the notion of *CR quaternionic map* and we proved that any such real-analytic map, between CR quaternionic manifolds, is the restriction of a quaternionic map between quaternionic manifolds. As an application, we proved, for example, that for any submanifold M, of dimension $4k-1$, of a quaternionic manifold N, such that TM generates a quaternionic subbundle of $TN|_M$, of (real) rank $4k$, there exists, locally, a quaternionic submanifold of N, containing M as a hypersurface.

• In the paper *Twistor Theory for co-CR quaternionic manifolds and related structures* by (Stefano Marchiafava and *Radu Pantilie*; paper published in **Israel J. Math**), we introduced, in a general and non metrical framework, the class of co-CR quaternionic manifolds, which contains the class of quaternionic manifolds, whilst in dimension three it particularizes to give the Einstein–Weyl spaces. We showed that these manifolds have a rich natural Twistor Theory and, along the way, we obtain a heaven space construction for quaternionic-Kähler manifolds.

• In the paper *Generalized Quaternionic Manifolds* (by *Radu Pantilie*; paper published in **Ann. Mat. Pura Applic**) we initiated the study of the generalized quaternionic manifolds by classifying the generalized quaternionic vector spaces, and by giving two classes of non-classical examples of such manifolds. Thus, we showed that any complex symplectic manifold is endowed with a natural (non-classical) generalized quaternionic structure, and the same applies to the heaven space of any three-dimensional Einstein–Weyl space. In particular, on the product Z of any complex symplectic manifold M and the sphere there exists a natural generalized complex structure, with respect to which Z is the twistor space of M.

2 Presentations at international workshops

Liana David gave the following presentations as *Invited Speaker*: 1) "F-manifolds and eventual identities" at *Singularity theory and Integrable Systems* (Oberwolfach, Germany, April 2012); 2) " Invariant generalized complex structures on Lie groups" at *Symmetries in Geometry and Physics* (Luxembourg University); 3) " Conformal-Killing forms in quaternionic geometry", at *The Central European Mathematical Seminar* (Masaryk University, Brno, Cech Republic).

The *travel expenses* for these visits were supported by the budget of the project.

3 Visits Abroad

The activities below were partially supported by the budget of the grant.

Gabriel Baditoiu visited *Schroedinger Instiutute* (Vienna), where he participated at two worskhops from the program *K-theory and Quantic Fields* (June 2012). During the same visit, he also collaborated with Prof. Steven Rosenberg from Boston University (who visited the Schroedinger Institute by the same time).

Radu Pantilie visited, for three months, Brest University (France). During his visit he collaborated with Prof. P. Baird from Brest University. He also gave a talk at the *Seminar from Brest University.*

4 Participation at seminars

The members of the team attended on a regular basis the differential geometry seminar from IMAR (the host institution of the project), where they presented the research results obtained on the topics of the project.