Scientific report for the project "Hermitian and quaternionic structures on manifolds and applications"

This report describes the activity of the team members of the project *Hermitian and quaternionic structures on manifolds and applications*, during the entire implementation of the project.

In the first part we describe the activity during 2012-2015; in the second part the activity during 2016.

The team members are: Liana David (Director of the project); Monica Aprodu; Gabriel Baditoiu; Radu Pantilie.

1 The activity during 2012-2015

1.1 Articles reported during 2012-2015

All articles below mention the financial support of grant. The journals **Proceedings of the London Mathematical Society**, **Annales de l'Institut Fourier**, **Communications in Mathematical Physics**, **Annali della Scuola Normale Superiore di Pisa** belong to the highest region (the 'red region') in the ISI classification.

1. D. Alekseevsky, L. David: *Tanaka structures (non-holonomic G-structures) and Cartan connections*, Conformal and Complex Geometry, volume in the Honour of Paul Gauduchon, Editors: A. Moroianu, L. Ornea, **Journal of Geometry and Physics**, vol. 91 (2015) p. 88-100

2. D. V. Alekseevsky, L. David: *A note about invariant SKT-structures and generalized Kähler structures on flag manifolds*, **Proceedings of the Edinburgh Mathematical Society**, vol. 55, no. 3, 2012, p. 543-549

3. Monica Aprodu, Marian Aprodu: *Holomorphic vector bundles on Kähler manifolds and totally geodesic foliations on Euclidian open domains*, **Differential Geometry and Its Applications**, 39 (2015) , p. 10-19

4. G. Baditoiu: *Classification of pseudo-Riemannian submersions with totally geodesic fibers from pseudo-hyperbolic spaces*, **Proceedings of the London Mathematical Society**, vol. 105, no 6, (2012) p. 1315-1338

5. G. Baditoiu, S. Ianus, A. M. Pastore: *Spectral properties of Riemannian Legendre foliations*, **Bulletin de la Societe Scientifique Mathematique de Roumanie**, vol. 56(104), no. 2, 2013, p. 135-150

6. P. Baird, R. Pantilie: *On Ricci solitons and twistorial harmonic morphisms*, sent for publication, arxiv:1210.4688

7. Liana David: *On cotangent manifolds, complex structures and generalized geometry*, **Annales de l'Institut Fourier**, vol. 66 (1) (2016), p. 1-28

8. L. David: *On adding a variable to a Frobenbius manifold and generalizations*, **Geometria Dedicata**, vol. 167 (2013), p. 189-214

9. L. David: *F-manifolds and eventual identities*, **Oberwolfach Reports**, vol. 9 (2), 2012, p. 1389-1392

10. L. David, C. Hertling: *Regular F-manifolds: initial conditions and Frobenius metrics*, accepted at **Annali della Scuola Normale Superiore di Pisa**, 35 pages

11. L. David, I. A. B. Strachan: *Symmetries of F-manifolds with eventual identities and special families of connections* , **Annali della Scuola Normale Superiore di Pisa**, Vol. XIII (2014), issue 3, p. 641-674

12. Liana David, I. A. B. Strachan: *tt* Geometry on the Big Phase Space*, **Communications in Mathematical Physics**, vol. 329 (1) (2014), p. 295-323

13. S. Marchiafava, R. Pantilie: *Twistor Theory for co-CR quaternionic manifolds and related structures*, **Israel Journal of Mathematics**, 195 (2013) p. 347-371

14. S. Marchiafava, R. Pantilie: *A note on CR quaternionic maps*, **Advances in Geometry**, 13 (4) (2013), p. 605-612

15. R. Pantilie: *Generalized quaternionic manifolds*, **Annali di Matematica Pura ed Applicata**, 193(2014), p. 633-641

16. R. Pantilie: *On the classification of real vector spaces of a quaternionic vector space*, **Proceedings of the Edinburgh Mathematical Society**, vol. 56, no. 2 (2013), p. 615-622

17. R. Pantilie: *On the twistor space of a (co-)CR quaternionic manifold*, **New-York Journal of Mathematics** 20 (2014), p. 959-971

18. R. Pantilie: *On the quaternionic manifolds whose twistor spaces are Fano manifolds*, **Tohoku Mathematical Journal** (2) 67 (2015) no 4, p. 507-511

1.2 Presentations as "Invited Speakers" at workshops

Monica Aprodu gave a presentation at: *Workshop for Young Researchers in Mathematics*, Constanta (Romania), May 2013.

Gabriel Baditoiu gave presentations at: *Workshop for Young Researchers in Mathematics*, Constanta (Romania), May 2013; *Real and Complex Differential Geometry*, Bucharest, September 2014.

Liana David gave presentations at: *Singularity Theory and Integrable Systems*, Oberwolfach (Germany), April 2012; *The Central European Mathematical Seminar*, University of Masaryk (Brno, Cech Republic), December 2012; *Real and Complex Differential Geometry*, University of Bucharest (Romania), September 2014; *Geometry and PDEs*, West University of Timisoara (Romania), May 2015.

Radu Pantilie gave presentations at: *Aniversary Conference, Faculty of Science 150 Years*, University of Bucharest, September 2013; *The 10th Differential Geometry Day and The Differential Geometry Workshop*, Lund (Sweden), May 2014; *Harmonic maps, biharmonic maps, harmonic morphisms and related topics*, Calgari (Italy), June 2015.

1.3 Other participations at workshops

Monica Aprodu participated at *The 8th Congress of Romanian Mathematicians*, University Al. I. Cuza (Iasi, Romania), June 2015 and at *Workshop for Young Researchers in Mathematics*, Ovidius University (Constanta, Romania), Mai 2015.

1.4 Prezentations in seminars

The team members participated at the differential geometry seminar from IMAR (the host institution of the project), where they presented their research results.

1.5 Visits for collaboration and scientific documentation

Gabriel Baditoiu visited: *The Erwin Schroedinger International Institute for Mathematical Physics* (Vienna, Austria), in 2012.

Liana David visited: *Institut des Hautes Etudes Scientifiques* in 2012; *University of Mazaryk* (Brno, Cech Republic) in 2012 and 2013; *Universitaty of Mannheim* (Germany) in 2013 and 2015; *University of Hamburg* (Germany) in 2015.

Radu Pantilie visited *University of Brest* (France) in 2012; *University La Sapienza* (Rome, Italy) in 2013.

1.6 Foreign visitors

Prof. *Dmitri Alekseevsky* (Institute of Transmission Problems, Moscova) and Prof. *Ian Strachan* (University of Glasgow, Great Britain) visited IMAR, for collaboration on the topics of the project. They also gave presentations in the geometry seminar from IMAR.

1.7 Workshop

The Director of the project co-organized a workshop *The 11-th International Workshop on Differential Geometry and its Applications* Ploiesti (Romania), September 19-22, 2013. From the budget of the grant it was partially supported the participation of Prof. Dmitri Alekseevsky from Moscow (Russia).

1.8 Research results

We present briefly the research results obtained during 2012-2015. More details can be found in the anual reports from 2012, 2013, 2014 and 2014.

1.8.1 Monica Aprodu

P. Baird and J. C. Wood have solved completely a problem of classification of harmonic morphisms with one-dimensional fibres from open domains in $\mathbb{R}^3$, raised by Jacobi. This problem is to describe (locally) foliations in lines on open domains in $\mathbb{R}^3$ which verify a geometric extra-condition, horizontal weak conformality (HWC). In the paper *Holomorphic vector bundles on Kähler manifolds and totally geodesic foliations on Euclidian domains* (by Monica Aprodu and Marian Aprodu), there were extended the results of Baird and Wood to foliations of higher codimension. It was established a relation between holomorphic maps to an orthogonal Grassmannian, together with a section in the universal bundle, and certain foliations with totally geodesic leaves on Euclidean open domains. It was found a geometric condition for a totally geodesic foliation to originate in a holomorphic vector bundle. For codimension-two foliations, this description recovers the results of Baird and Wood. The universal objects that play a key role are the orthogonal Grassmannians. Monica Aprodu studied the laplacians on stratifolds and their connection with harmonic maps on Riemann stratifolds. She found an explicit description of the laplacians in some special cases. This represents a step towards the development of a theory of harmonic maps on singular spaces, other than polyhedra. These results will be gathered in a future work.

1.8.2 Gabriel Baditoiu

Between 2012-2015, Gabriel Baditoiu worked on the following topics:

• Classification of pseudo-Riemannian submersions with totally geodesic fibres on pseudo-hyperbolic spaces. In the paper *Classification of Pseudo-Riemannian submersions with totally geodesic fibres from pseudo-hyperbolic spaces* (author: G. Baditoiu) it is shown that any pseudo-Riemannian submersion with totally geodesic fibres from a real pseudo-hyperbolic space is equivalent to a Hopf pseudo-Riemannian submersion. As a consequence, it is obtained the classification of pseudo-Riemannian submersions from pseudo-hyperbolic (para-)complex spaces and with fibres (para-)complex totally geodesic submanifolds . It is also proven that there are no pseudo-Riemannian submersions with fibres quaternionic submanifolds from quaternionic pseudo-hyperbolic spaces.

• Spectral properties of Riemannian Legendre foliations. In the paper *Spectral Geometry of Riemannian Legendre foliations* (authors: G. Baditoiu, S. Ianus and A. M. Pastore) are obtained geometric characterizations of isospectral minimal Riemannian Legendre foliations on compact Sasakian manifolds of constant ϕ-sectional curvature. In particular, it is studied the existence of minimal Riemannian Legendre foliations on 3, 5 and 7-dimensional spheres and on sphere tangent bundles. Classification of pseudo-Riemannian Einstein G-homogeneous metrics on real and complex pseudo-hyperbolic spaces under the hypothesis that G is a connected closed Lie subgroup of $SO_0(n-r,r+1)$ acting completely reducible on $\mathbb{R}^{n+1}$. Under this hypothesis, it is proven that the pseudo-Riemannian Einstein $G-$homogeneous metrics on real and complex pseudo-hyperbolic spaces are homothetic to either canonical ones, or the Einstein metrics of the canonical variation of a Hopf pseudo-Riemannian submersion. This classification and the complete classification of the connected closed Lie subgroups of $SO_0(n-r,r+1)$ acting effectively and transitively on a real pseudo-hyperbolic space were included in the 2016 preprint submitted for publication with title *Classification of homogeneous Einstein metrics on pseudo-hyperbolic spaces.*

1.8.3 Liana David

A brief description of the main results obtained by Liana David:

• It was introduced and studied the notion of symmetric generalized complex structure. Given such a structure $\mathcal{J}$ and an arbitrary connection D on

a manifold M, it was defined a notion of integrability for the pair $(\mathcal{J}, D)$. Such integrable pairs on semisimple Lie groups were described. In this way, it was obtained a large class of complex structures on the cotangent manifold of a semisimple Lie group.

• Frobenius manifolds are a geometrization of the so called WDVV-equations. It was generalized the construction of adding a variable to a Frobenius manifold (Sabbah, *Isomonodromic deformations and Frobenius manifolds, 2007*), this giving a new method to construct Frobenius manifolds.

• The notion of F-manifold was introduced by C. Hertling si Y. Manin, in 1998, as a weaker notion than the notion of a Frobenius manifold. It was developed a duality for F-manifolds with eventual identities. This question was raised by Y. Manin (*Adv. Math., 2004*). The relation between this duality and other constructions from the theory of Frobenius manifolds was studied: Dubrovin's "almost duality" (*Geometry, Topology and Mathematical Physics, 1997*) and tt^* geometry (collaboration with I. Strachan).

• It was shown that every regular F-manifold (i.e. an F-manifold for which the multiplication by the Euler field is a regular endomorphism, at any point) admits a prefered coordinate system. Compatible Frobenius metrics in this coordinate system were described. It was also shown that any regular F-manifold is locally isomorphic to the parameter space of a universal Malgrange connection. It was proven an initial conditions theorem for Frobenius metrics on regular F-manifolds. To a large extend, these results generalize the results of Dubrovin (*Geometry of 2D topological field theory, 1996*) on regular semisimple F-manifolds (collaboration with C. Hertling).

• Solutions of the tt^*-equations (defined by S. Cecotti, C. Vafa in *Nuclear Physics B, 1991*) on fibrations of infinite rank, were constructed (collaboration with I. Strachan).

1.9 Radu Pantilie

A brief description of the mai results obtained by Radu Pantilie:

• The introduction and the initiation of the study of the following notions (also, providing significant applications): CR quaternionic maps (jointly with S. Marchiafava); co-CR quaternionic manifolds (jointly with S. Marchiafava); generalized quaternionic manifolds, quaternionic-like manifolds (also, it is given a natural construction of such manifolds having homogeneous twistor

spaces).

• The introduction of a natural functor from the pairs formed of a quaternionic vector space and a real subspace, to the coherent analytic sheaves over the Riemann sphere, thus, providing a new approach to the classification of the real subspaces of a quaternionic vector space.

• The characterisation, in the setting of the Kodaira-Spencer deformation theory, of the twistor spaces of (co-)CR quaternionic manifolds.

• The generalization, to quaternionic manifolds whose twistor spaces are Fano, of two results of S. Salamon and C.R. LeBrun, respectively, regarding positive quaternionic-Khler manifolds.

• The study of the moment maps of (tri)holomorphic isometric actions, on (hyper-)Kähler manifolds, from the perspective of harmonic morphims (jointly with M. Benyounes and E. Loubeau)

• The study of the Ricci soliton flow on the domain (of dimension four) of a twistorial harmonic morphism (jointly with P. Baird).

2 Activity during 2016

2.1 Articles reported in 2016

1. M. A. Aprodu: *On the exponential map on Riemannian polyhedra*, **sent for publication**.

2. D. V. Alekseevsky, L. David: *Prolongation of Tanaka structures: an alternative approach*, accepted in **Annali di Matematica Pura ed Applicata**.

3. G. Baditoiu: *Classification of homogeneous Einstein metrics on pseudo-hyperbolic spaces*, **sent for publication**.

4. L. David, C. Hertling: *Hermitian metrics on F-manifolds*, **Journal of Geometry and Physics**, 107 (2016), p. 60-78.

5. M. Benyounes, E. Loubeau, R. Pantilie: *Harmonic morphisms and moment maps on hyper-Kähler manifolds*, accepted in **Manuscripa Mathematica**

6. R. Pantilie: *Quaternionic-like manifolds and Homogeneous Twistor Spaces*, accepted in **Proceedings of the Royal Society A**.

2.2 Presentations as "Invited Speaker" at workshops

Gabriel Baditoiu gave a presentation at: *Workshop for Young Researchers in Mathematics*, Constanta, May 2016.

Liana David gave a presentation at *Geometric structures related to Hermitian and almost Hermitian manifolds*, Leibniz University, Hannover (Germany), May 2016 and a presentation at *Monthly IMAR-Lectures*.

2.3 Other participations at workshops

Monica Aprodu participated at *Workshop for Young Researchers in Mathematics*, Ovidius University (Constanta, Romania), May 2016.

Gabriel Baditoiu participated at *The XXIVth International Conference on Integrable Systems and Quantum Symmetries*, *Cech Technical University* (Prague, Cech Republic), June 2016.

2.4 Prezentations in seminars

The **team members** participated at the geometry seminar from IMAR, where they gave presentations on their research results obtained in the project. **Liana David** gave a presentation in the differential geometry seminar from *University of Hamburg* (Germany), on generalized complex geometry. **Radu Pantilie** gav two presentations on quaternionic geometry at *University of Ferrara* (Italy).

2.5 Visits for collaboration/scientific research

Liana David visited *University of Hamburg* (Germany). **Radu Pantilie** visited *Universitatea din Ferrara* (Italia).

2.6 Foreign visitors

Prof. *Claus Hertling* (University of Mannheim, Germany) and Prof. *Andrei Moroianu* (University of Versailles and CNRS, France) visited IMAR, for collaboration on the topics of the project. They gave presentations at IMAR, on their research results.

2.7 Research results

We describe the research results obtained the team members, during 2016.

2.7.1 Monica Aprodu

Riemannian polyhedra are typical examples of relevant singular spaces and they provide the appropriate framework for the study of harmonicity. It is natural to believe that, since Riemannian polyhedra are generalizations of Riemannian differentiable manifolds, the geometric properties of manifolds extend canonically to polyhedra. One major geometric ingredient in the geometry of Riemannian manifolds is the exponential map, defined around a given point, which gives a local diffeomorphism between the tangent space at that point and the manifold. The exponential map can also be defined for manifolds with boundary. Starting from this fact, we have proved that Riemannian polyhedra also admit exponential maps at points in codimension–one strata that behave similarly to the classical case. The main difficulty is to replace the tangent space, which is of little use at singular points, with the tangent cone. Locally, around a point in a codimension–one stratum, the tangent cone looks like a union of semi–spaces and the exponential map can be defined on each of these components. This result is the content of the 2016 preprint *On the exponential map on Riemannian polyhedra.*

2.7.2 Gabriel Baditoiu

In 2016, Gabriel Baditoiu worked on the preprint with the title *Classification of homogeneous Einstein metrics on pseudo-hyperbolic spaces*, where the following classifications are obtained:

• Classification of closed connected Lie subgroups G of $SO_0(n-r, r+1)$ acting effectively and transitively on a non-degenerate n-dimensional hyperboloid (also called pseudo-hyperbolic space)

• Classification of pseudo-Riemannian Einstein G-homogeneous metrics on real and complex pseudo-hyperbolic spaces under the hypothesis that G acts completely reducible on $\mathbb{R}^{n+1}$. Under these conditions, it is shown that the G-homogeneous Einstein pseudo-Riemannian metric is homothetic

to either the canonical metric or the Einstein metric of the canonical variation of a Hopf pseudo-Riemannian submersion.

From a paper due to Ziller from 1982, it is known that the homogeneous Riemannian Einstein on sphere and complex projective spaces are the Riemannian Einstein metrics of the canonical variation of Hopf Riemannian submersions. We show that, under certain conditions, the same holds in the pseudo-Riemannian case. Unlike the Riemannian case investigated by Ziller, the difficulty of the pseudo-Riemannian case is due to the lack of a previous known classification of the Lie subgroup G, as above, acting transitively and effectively on the named spaces. Only certain particular cases were known before. From a paper due to Onishchik from 1969 are known the semisimple such Lie groups G, and from a paper due to Wolf from 1964 are known the Lie groups G simultaneously acting transitively on a non-degenerate hyperboloid and a certain degenerate one. The problem of classifying all closed connected subgroups G of $SO_0(n-r, r+1)$ acting transitively and effectively on real pseudo-hyperbolic space is completely solved in the preprint. One can distinguish two cases: (i) G acts completely reducible on $\mathbb{R}^{n+1}$ and (ii) G do not acts completely reducible on $\mathbb{R}^{n+1}$. In case (i) we establish a T-duality between our spaces and their compact correspondents and then, by a result due to Kath, there exists a bijective correspondence between homogeneous pseudo-Riemannian Einstein metrics on our space and their compact Riemannian duals. Under the assumption that G is a closed connected Lie subgroup of $SO_0(n-r, r+1)$, by Ziller's classification, it also follows that in the pseudo-Riemannian case, any G-homogeneous Einstein metric is homothetic to either the canonical one or the Einstein metric of the canonical variation of a Hopf pseudo-Riemannian submersion.

2.7.3 Liana David

Liana David worked on the article *Prolongation of Tanaka structures: an alternative approach* (autors D. V. Alekseevsky, L. David) and at the article *Hermitian metrics on F-manifolds* (autors L. David, C. Hertling).

Together with Dmitri Alekseevsky, in the first article mentioned above she developed a new point of view on the prolongation procedure of Tanaka structures. The classical theory of prolongations of G-structures was generalized by N. Tanaka around 1970 to a large class of geometrical structures, the so called Tanaka structures. Examples of Tanaka structures include CR-structures, sub-Riemannian structures, sub-conformal structures. The prolongation procedure of Tanaka associates (by an induction process) to a Tanaka structure of finite order an absolute parallelism. Together wih the

well-known local classification of absolute parallelisms (described for example in Sternberg's book *Lectures on Differential Geometry*), the prolongation procedure is extremely useful to determine local invariants for Tanaka structures and for the study of their automorphism group. In the first article mentioned above, it was developed a new view-point on the Tanaka prolongation procedure, based on the theory of quasi-gradations in vector spaces (which is developed in the preprint), on the theory of G-structures and their torsion functions.

In the second article mentioned above it was considered the construction of Hermitian metrics on F-manifolds, from tt^*-geometry. It was clarified the relation between various notions and it was shown the existence of a canonical Hermitian metric on a large class of F-manifolds. In a neighborhood of irreducible points, this metric makes the manifold almost hyperbolic. Possible applications in singularity theory are described. This is joint work with Claus Hertling.

2.7.4 Radu Pantilie

R. Pantilie elaborated and contributed to the elaboration, respectively, of the following two papers: *Quaternionic-like manifolds and homogeneous twistor spaces* (author: Radu Pantilie) and *Harmonic morphisms and moment maps on hyper-Kähler manifolds* (authors: M. Benyounes, E. Loubeau, R. Pantilie).

In the first article mentioned above, motivated by the quaternionic geometry corresponding to the homogeneous complex manifolds endowed with (holomorphically) embedded Riemann spheres, R. Pantilie introduced and initiated the study of the 'quaternionic-like manifolds', thus, unifying the CR quaternionic and the ρ-quaternionic manifolds. In this paper it is, also, given a natural construction of homogeneous complex manifolds endowed with embedded spheres, thus, emphasizing the abundance of the quaternionic-like manifolds.

In the second paper mentioned above, R. Pantilie and his collaborators M. Benyounes and E. Loubeau, characterised the actions by holomorphic isometries on a Khler manifold, of an abelian Lie group, admitting a moment map which is horizontally weakly conformal (with respect to some Euclidean structure on the Lie algebra of the group). Furthermore, let ϕ be a hyper-Kähler moment map of an abelian Lie group T acting by triholomorphic isometries on a hyper-Kähler manifold M. If $\dim T = 1$ then ϕ is a harmonic

morphism. If $\dim T > 1$ and either ϕ has critical points, or M is nonflat and $\dim M = 4\dim T$ then ϕ cannot be horizontally weakly conformal.