ON RICCI SOLITONS AND TWISTORIAL HARMONIC MORPHISMS

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Abstract

We study the soliton flow on the domain of a twistorial harmonic morphism between Riemannian manifolds of dimensions four and three. Assuming real-analyticity, we prove that, for the Gibbons–Hawking construction, any soliton flow is uniquely determined by its restriction to any local section of the corresponding harmonic morphism. For the Beltrami fields construction, we identify a contour integral whose vanishing characterises the trivial soliton flows.

1. Introduction

A solution of the Ricci flow which evolves by scaling and diffeomorphism is called a *Ricci soliton*. Specifically, if $g_t = c_t\psi_t^* g$ satisfies the equation

$$\frac{\partial g_t}{\partial t} = -2\,{}^M\mathrm{Ric}(g_t)$$

on some time interval $[0, \delta)$, where c_t is a family of positive scalars such that $c_0 = 1$ and ψ_t is a family of diffeomorphisms satisfying $\psi_0 = \mathrm{id}$, then

$$ {}^M\mathrm{Ric} + ag + \tfrac{1}{2}\,\mathcal{L}_E g = 0\,; \tag{1.1}$$

where $2a = c_0'$ and the vector field E, called the *soliton flow*, is given at each $x \in M$ by $E_x = \frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x)|_{t=0}$. Conversely, a solution to (1.1) on a Riemannian manifold (M, g) gives a small time solution to the Ricci flow equation, under a completeness assumption. Ricci solitons occur as rescaled limits at singularity formation and as asymptotic limits of immortal solutions, that is solutions that exist for all future time [9]. Both of these limits are interpreted in terms of Cheeger-Gromov-Hamilton pointed convergence of Ricci flows [6]. In the case when $E = \mathrm{grad} f$ is the gradient of a function, then the soliton is said to be of *gradient type*. A soliton is called *shrinking, steady* or *expanding* according as the constant a is negative, zero or positive, respectively. See the reference [5] for an overview.

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Any soliton metric which is Einstein is called *trivial*. In dimension 3, any compact soliton is trivial [7]. On the other hand, in dimension 4, non-trivial soliton metrics in the form of Kähler metrics of cohomogeneity one on certain projective bundles on $\mathbb{C}P^n$ have been found by Koiso [8]. Also, on $\mathbb{R}^n$ there exist the well-known *Gaussian solitons* with flow given by $E = \operatorname{grad}\big(-\frac{a}{2}|x - x_0|^2\big)$, where $x_0 \in \mathbb{R}^n$ is some arbitrary point.

In this article we are particularly concerned with questions of existence and uniqueness in dimension 4. In order to address these issues, we make the additional assumption that the metric g supports a twistorial harmonic morphism onto a 3-manifold (see below). Up to homotheties, any such harmonic morphism $\varphi : (M, g) \to (N, h)$, with nonintegrable horizontal distribution, is locally given either by the so-called Gibbons–Hawking or by the Beltrami fields constructions. Both of these methods for constructing a metric involve specifying data on the codomain N from which the metric g is derived. A similar idea was applied in [2] to construct 3-dimensional solitons, by supposing the existence of a semi-conformal mapping onto a surface. Note that by a result in [3], any horizontally conformal conformal submersion from a 3-dimensional conformal manifold to a surface P can be extended to a unique twistorial map from its 4-dimensional heaven space to P.

In Section 3, we suppose that (M, g) is given by applying the Gibbons-Hawking construction to a real-analytic Riemannian 3-manifold (N, h), in particular, g is expressed in terms of a harmonic function on (N, h). As a consequence, this defines a twistorial harmonic morphism $\varphi : (M, g) \to (N, h)$. Then we show that any real-analytic soliton flow on (M, g) is uniquely determined by its restriction to any local section of φ (Theorem 3.1). We, also, obtain an ansatz for the construction of a soliton flow from a harmonic function and a solution to the monopole equation on N (Theorem 4.5).

In Section 5, we suppose that (M, g) is given by the Beltrami fields construction. Now, the metric g is given in terms of a 1-form on (N, h) satisfying the Beltrami fields equation. Theorem 5.1 shows the equivalence between the triviality of any real-analytic soliton flow E and the vanishing of a contour integral involving the complexification $(\mathcal{L}_E h)^{\mathbb{C}}$ to a local complexification of N.

2. Soliton flows on the domain of a twistorial harmonic morphism

A *harmonic morphism* between Riemannian manifolds is a map which, locally, pulls back harmonic functions to harmonic functions (see [4] for more information on harmonic morphisms).

Let $\varphi : (M, g) \to (N, h)$ be a submersive harmonic morphism, with M and N oriented, $\dim M = 4$, $\dim N = 3$. Denote by λ the dilation of φ and let $\mathscr{V} = \ker \mathrm{d}\varphi$ and $\mathscr{H} = \mathscr{V}^{\perp}$ be the vertical and horizontal distributions of φ, respectively. We orient $\mathscr{V}$ and $\mathscr{H}$ such that the isomorphisms $TM = \mathscr{V} \oplus \mathscr{H}$ and $\mathscr{H} = \varphi^*(TN)$ be orientation preserving.

Let V be the (fundamental) vertical vector field, characterised by the fact that it is vertical, positive and $g(V,V)=\lambda^2$. Then, locally, $g=\lambda^{-2}\,\varphi^*(h)+\lambda^2\,\theta^2$, where θ is the vertical dual of V; that is, $\theta(V)=1$ and $\theta|_{\mathscr{H}}=0$.

Then φ is twistorial (with respect to the opposite orientation on M) if and only if the following relation holds [12]:

$$\mathrm{d}^{\mathscr{H}}\big(\lambda^{-2}\big)=*_{\mathscr{H}}\Omega\,, \tag{2.1}$$

where $\Omega=\mathrm{d}\theta$. It follows that, at least outside the set where $\Omega=0$, we have $c=V(\lambda^{-2})$ is constant (on each component) and the Ricci tensors ${}^{M}\mathrm{Ric}$ and ${}^{N}\mathrm{Ric}$ of (M,g) and (N,h), respectively, are given by the following relations [11]:

$$\begin{aligned} &{}^{M}\mathrm{Ric}\,|_{\mathscr{V}}=0\,,\quad {}^{M}\mathrm{Ric}\,|_{\mathscr{V}\otimes\mathscr{H}}=0\,,\\ &{}^{M}\mathrm{Ric}\,|_{\mathscr{H}}=\varphi^*\big({}^{N}\mathrm{Ric}\big)-\frac{c^2}{2}\,\varphi^*(h)\,. \end{aligned} \tag{2.2}$$

Furthermore, we have the following facts (see [11], [12] and the references therein):

- (N,h) has constant sectional curvature if and only if (M,g) is self-dual;
- (N,h) has constant sectional curvature equal to $\frac{c^2}{4}$ if and only if (M,g) is Einstein; moreover, if (M,g) is Einstein then it is Ricci-flat self-dual.

From now on, we shall assume $\Omega_x\neq 0$, at each $x\in M$ (if $\Omega=0$ then, locally, g is a warped-product; see [1] and the references therein for results on soliton flows on such metrics). Then there exists a unique basic vector field Z such that $\iota_Z\Omega=0$ and which is projected onto a unit vector field on (N,h).

Proposition 2.1. *Let $\varphi:(M,g)\to(N,h)$ be a twistorial harmonic morphism, and let E be a vector field on M; denote by f and F the function and the horizontal vector field, respectively, such that $E=fV+F$.*

Then E is a soliton flow, with the corresponding constant a, if and only if, locally, the following five relations hold:

$$\begin{aligned} &V(f)+a-\tfrac{1}{2}\lambda^2\big(fc+z\,\Omega(X,Y)\big)=0\,,\\ &X(f)-y\,\Omega(X,Y)+\lambda^{-4}V(x)=0\,,\\ &Y(f)+x\,\Omega(X,Y)+\lambda^{-4}V(y)=0\,,\\ &Z(f)+\lambda^{-4}V(z)=0\,,\\ &\mathcal{L}_F\big(\varphi^*(h)\big)+\lambda^2\big(-c^2+fc+z\,\Omega(X,Y)+2a\lambda^{-2}\big)\varphi^*(h)\\ &\qquad\qquad +2\lambda^2\varphi^*\big({}^{N}\mathrm{Ric}\big)=0 \text{ on } \mathscr{H}\,, \end{aligned} \tag{2.3}$$

where (X,Y,Z) is a horizontal frame, projected by φ onto a positive orthonormal frame on (N,h), and x, y, z are functions characterised by $F=xX+yY+zZ$.

Proof. Firstly, note that (2.3) does not dependent of the pair (X,Y), having the stated properties. Also, (2.1) is (locally) equivalent to the conditions $X(\lambda)=Y(\lambda)=0$ and

$Z(\lambda^{-2}) = \Omega(X,Y)$. Thus, by also using $V(\lambda^{-2}) = c$, we obtain

$$E(\ln\lambda) = -\tfrac{1}{2}\, fc\lambda^2 - \tfrac{1}{2}\, z\lambda^2\Omega(X,Y)\,, \tag{2.4}$$

which implies

$$E(\ln\lambda) + V(f) + a = V(f) + a - \tfrac{1}{2}\lambda^2\big(fc + z\,\Omega(X,Y)\big)\,. \tag{2.5}$$

Also, we have

$$\begin{aligned}(\mathcal{L}_E g)(V,V) &= (\mathcal{L}_{fV} g)(V,V) + (\mathcal{L}_F g)(V,V)\\ &= fV(\lambda^2) + 2g([V,fV],V) + F(\lambda^2) + 2g([V,F],V)\\ &= fV(\lambda^2) + F(\lambda^2) + 2\,V(f)\lambda^2\\ &= E(\lambda^2) + 2\lambda^2 V(f)\\ &= 2\lambda^2\big(E(\ln\lambda) + V(f)\big)\,.\end{aligned} \tag{2.6}$$

Now, (2.2), (2.5) and (2.6) show that the first relation of (2.3) is equivalent to (1.1), restricted to $\mathscr{V}$.

Further, for any horizontal vector field S, we have

$$\begin{aligned}(\mathcal{L}_E g)(V,S) &= g([V,E],S) + g(V,[S,E])\\ &= g\big(V(f)\,V + [V,F],S\big) + g\big(V,S(f)V + [S,F]\big)\\ &= g([V,F],S) + \lambda^2 S(f) - \lambda^2\,\Omega(S,F)\\ &= \lambda^2\big(S(f) - \Omega(S,F) + \lambda^{-4}\varphi^*(h)([V,F],S)\big)\,.\end{aligned} \tag{2.7}$$

From (2.7) we deduce that the second, third and fourth relations of (2.3) are equivalent to (1.1), restricted to $\mathscr{V}\otimes\mathscr{H}$.

Next, we have

$$\begin{aligned}(\mathcal{L}_E g)|_{\mathscr{H}} &= \big(\mathcal{L}_E(\lambda^{-2}\varphi^*(h))\big)|_{\mathscr{H}}\\ &= E(\lambda^{-2})\varphi^*(h)|_{\mathscr{H}} + \lambda^{-2}\big(\mathcal{L}_E\,\varphi^*(h)\big)|_{\mathscr{H}}\\ &= E(\lambda^{-2})\varphi^*(h)|_{\mathscr{H}} + \lambda^{-2}(\mathcal{L}_{fV}\varphi^*(h))|_{\mathscr{H}} + \lambda^{-2}(\mathcal{L}_F\varphi^*(h))|_{\mathscr{H}}\\ &= E(\lambda^{-2})\varphi^*(h)|_{\mathscr{H}} + \lambda^{-2}(\mathcal{L}_F\varphi^*(h))|_{\mathscr{H}}\\ &= -2\lambda^{-2}E(\ln\lambda)\varphi^*(h)|_{\mathscr{H}} + \lambda^{-2}\big(\mathcal{L}_F\varphi^*(h)\big)|_{\mathscr{H}}\,.\end{aligned} \tag{2.8}$$

From (2.4) and (2.8) we obtain

$$(\mathcal{L}_E g)|_{\mathscr{H}} = \big(fc + z\,\Omega(X,Y)\big)\varphi^*(h)|_{\mathscr{H}} + \lambda^{-2}\big(\mathcal{L}_F\varphi^*(h)\big)|_{\mathscr{H}}\,. \tag{2.9}$$

From (2.2) and (2.9) we deduce that the fifth relation of (2.3) is equivalent to (1.1), restricted to $\mathscr{H}$. □

3. Ricci solitons and the Gibbons–Hawking construction

Let $\varphi : (M, g) \to (N, h)$ be a twistorial harmonic morphism with $V(\lambda^{-2}) = 0$. This is equivalent to the fact that, locally, there exists a function u and a one-form A on N satisfying the monopole equation $\mathrm{d}u = *\,\mathrm{d}A$ and such that

$$g = u\,h + u^{-1}(\mathrm{d}t + A)^2 \,,$$

with $\varphi : M = N \times \mathbb{R} \to N$ the projection; in particular, $\lambda^{-2} = u$, $V = \frac{\partial}{\partial t}$ is a Killing vector field, and $\theta = \mathrm{d}t + A$.

Note that, if (N, h) is real-analytic then, as u and A satisfy $\Delta u = 0$ and $\Delta A = 0$, we have that, also, g is real-analytic.

Next, we state the main result of this section.

Theorem 3.1. *Let (M, g) be given by the Gibbons–Hawking construction, with (N, h) real-analytic.*

Then any real-analytic soliton flow on (M, g) is uniquely determined by its restriction to any local section of φ.

The proof of Theorem 3.1 will be given below.

From now on, in this section, *we shall denote by X, Y and Z the projections onto N of the corresponding vector fields appearing in* (2.3). Then, as $\Omega = \mathrm{d}A$, we have $Z = \frac{1}{|\mathrm{d}u|}\,\mathrm{grad}\,u$, $X(u) = Y(u) = 0$. Hence, the horizontal lift of X is $\widetilde{X} = -A(X)\frac{\partial}{\partial t} + X$ and, similarly, for Y and Z. Consequently, $\Omega(X, Y) = Z(u) = |\mathrm{d}u|$.

Thus, Proposition 2.1 gives the following result.

Corollary 3.2. *Let (M, g) be given by the Gibbons–Hawking construction. Then a vector field $E = fV + F$, with F horizontal, is a soliton flow on (M, g), with the corresponding constant a, if and only if the following five relations hold:*

$$\begin{aligned}
&\frac{\partial f}{\partial t} + a - \frac{1}{2}\,z u^{-1}|\mathrm{d}u| = 0 \,,\\
&-A(X)\frac{\partial f}{\partial t} + X(f) - y\,|\mathrm{d}u| + u^2\,\frac{\partial x}{\partial t} = 0 \,,\\
&-A(Y)\frac{\partial f}{\partial t} + Y(f) + x\,|\mathrm{d}u| + u^2\,\frac{\partial y}{\partial t} = 0 \,,\\
&-A(Z)\frac{\partial f}{\partial t} + Z(f) + u^2\,\frac{\partial z}{\partial t} = 0 \,,\\
&\mathcal{L}_F h + 2u^{-1}\,{}^N\mathrm{Ric} + u^{-1}\big(z\,|\mathrm{d}u| + 2au\big)h = 0 \ \text{ on } \mathscr{H} \,.
\end{aligned} \tag{3.1}$$

Assuming real-analyticity, Corollary 3.2 gives the following result.

Corollary 3.3. *Let (M, g) be given by the Gibbons–Hawking construction, with (N, h) real-analytic. Let $E = fV + F$ be a real-analytic vector field on (M, g), where F is horizontal.*

On writing, locally, $f = \sum_{j=0}^{\infty} t^j f_j$ and $F = \sum_{j=0}^{\infty} t^j \widetilde{F_j}$, with f_j functions on N, and

$\widetilde{F_j}$ the horizontal lifts of vector fields $F_j = x_jX + y_jY + z_jZ$ on N, we have that E is a soliton flow on (M, g), with the corresponding constant a, if and only if the following relations hold:

$$
\begin{aligned}
&f_{j+1} = \frac{1}{2(j+1)}\, z_j u^{-1} |\mathrm{d}u| \ , \ \big(j \in \mathbb{N} \setminus \{0\}\big) \ , \\
&f_1 = \frac{1}{2}\, z_0 u^{-1} |\mathrm{d}u| - a \ , \\
&-(j+1)A(X)f_{j+1} + X(f_j) - y_j|\mathrm{d}u| + (j+1)u^2 x_{j+1} = 0 \ , \ \big(j \in \mathbb{N}\big) \ , \\
&-(j+1)A(Y)f_{j+1} + Y(f_j) + x_j|\mathrm{d}u| + (j+1)u^2 y_{j+1} = 0 \ , \ \big(j \in \mathbb{N}\big) \ , \\
&-(j+1)A(Z)f_{j+1} + Z(f_j) + (j+1)u^2 z_{j+1} = 0 \ , \ \big(j \in \mathbb{N}\big) \ , \\
&\mathcal{L}_{F_j} h = 2(j+1)A \odot F_{j+1}^\flat - z_j u^{-1} |\mathrm{d}u|\, h \ , \ \big(j \in \mathbb{N} \setminus \{0\}\big) \ , \\
&\mathcal{L}_{F_0} h + 2u^{-1}\, {}^N\mathrm{Ric} + \big(2a + z_0 u^{-1} |\mathrm{d}u|\big) h - 2A \odot F_1^\flat = 0 \ ,
\end{aligned}
\tag{3.2}
$$

where $\odot$ denotes he symmetric product.

Proof. It is easy to see that the first four relations of (3.1) are equivalent to the first five relations of (3.2).

Let S be a vector field on N and let $\widetilde{S}$ be its horizontal lift. As $\widetilde{S} = -A(S)\frac{\partial}{\partial t} + S$, we have $\widetilde{S}(t^j) = -jA(S)t^{j-1}$. Hence,

$$
(\mathcal{L}_{t^j \widetilde{F_j}} h)(\widetilde{S}, \widetilde{S}) = t^j (\mathcal{L}_{F_j} h)(S, S) - 2jt^{j-1} \big(A \odot F_j^\flat\big)(S, S) \ ,
$$

from which the last two relations follow quickly. □

Now, we can give the proof of Theorem 3.1.

Proof of Theorem 3.1. It is sufficient to prove that E is determined by its restriction to $N \times \{0\}$.

The first and the fifth equations of (3.2) give

$$
-\frac{1}{2}\, z_j u^{-1} |\mathrm{d}u| A(Z) + \frac{1}{2j}\, Z\big(z_{j-1} u^{-1} |\mathrm{d}u|\big) + (j+1)u^2 z_{j+1} = 0 \ ,
$$

for any $j \in \mathbb{N} \setminus \{0\}$.

Also, the second and the fifth relations of (3.2) give

$$
-\frac{1}{2}\, z_0 u^{-1} |\mathrm{d}u| A(Z) + aA(Z) + Z(f_0) + u^2 z_1 = 0 \ . \tag{3.3}
$$

Thus, z_j and f_j are determined by z_0 and f_0, for any $j \in \mathbb{N}$.

To complete the proof, just note that the third and fourth relations of (3.2) imply that x_j and y_j are determined by x_0, y_0, z_0 and f_0, for any $j \in \mathbb{N}$. □

We, also, obtain that if E is a real analytic soliton flow on a Riemannian manifold, given by the Gibbons–Hawking construction, then its vertical part is determined by its horizontal part, up to the choice of f_0 satisfying (3.3).

4. A particular case

In this section, we continue the study of Section 3 by tackling a particular case.

Corollary 4.1. *Let (M,g) be given by the Gibbons–Hawking construction, with (N,h) real-analytic. Let $E = fV + t\widetilde{F_1}$ be a real-analytic vector field on (M,g), where $\widetilde{F_1}$ is the horizontal lift of the vector field F_1 on N.*

Then E is a soliton flow on (M,g), with the corresponding constant a, if and only if $f = f_0 - at + \frac{1}{4}\,b\,t^2$, where b is a constant, f_0 is a function on N, and the following relations hold:

$$\begin{aligned} & b = u^{-1}F_1(u)\,, \\ & bA = 2\,\mathrm{d}u \times F_1^\flat\,, \\ & aA + \mathrm{d}f_0 + u^2F_1^\flat = 0\,, \\ & \mathcal{L}_{F_1}h = -\,b\,h\,, \\ & {}^N\mathrm{Ric} + a\,u\,h - uA \odot F_1^\flat = 0\,. \end{aligned} \tag{4.1}$$

Proof. From the first two relations of (3.2) we obtain that $f_j = 0$, for any $j \geq 3$, $f_2 = \frac{1}{4}\,z_1u^{-1}|\mathrm{d}u|$, and $f_1 = -a$.

Note that the third, fourth and fifth relations of (3.2) are equivalent to

$$-\,(j+1)f_{j+1}A + \mathrm{d}f_j + \mathrm{d}u \times F_j^\flat + (j+1)u^2F_{j+1}^\flat = 0\,, \tag{4.2}$$

for any $j \in \mathbb{N}$.

Now, if $j \geq 3$ then (4.2) is trivially satisfied, whilst for $j = 2$ it gives that $\mathrm{d}f_2 = 0$. Thus, $b = z_1u^{-1}|\mathrm{d}u|$ is constant; equivalently, $b = u^{-1}F_1(u)$ is constant.

The second and third relations of (4.1) are equivalent to (4.2) with $j = 1$ and $j = 0$, respectively.

Finally, the last two relations of (4.1) are equivalent to the last two relations of (3.2). □

With the same notations as in Corollary 3.3, it is easy to see that we can weaken the hypotheses of Corollary 4.1 to (N,h) real-analytic and $F_0 = F_2 = 0$.

Also, the second relations of (4.1) imply $A(\mathrm{grad}\,u) = 0$ (condition which can always be satisfied, locally). Moreover, the first two relations of (4.1) determine F_1, whilst the third requires $\mathrm{d}\big(aA + u^2F_1^\flat\big) = 0$ to determine f_0, locally, up to a constant. Consequently, $b = 0$ if and only if $F_1 = 0$ which implies $a = 0$, f_0 is constant (otherwise, $\mathrm{d}A = 0$), and (M,g) is Ricci flat self-dual.

Therefore from now on we shall assume $b \neq 0$.

Lemma 4.2. *Let u be a harmonic function on a three-dimensional Riemannian manifold (N,h). Let B be a local solution of the monopole equation $\mathrm{d}u = *\,\mathrm{d}B$ and let w be a function satisfying $(\mathrm{grad}\,w)(u) = bu^3 + aB^\sharp(u)$.*

Then there exists functions v and f_0 such that the first three relations of (4.1) *are satisfied, with $A = B + \mathrm{d}v$ and $F_1 = -au^{-2}B^\sharp + u^{-2}\operatorname{grad} w$, if and only if*

$$b\,\mathrm{d}v - 2u^{-2}\,\mathrm{d}u \times \mathrm{d}w + bB + 2au^{-2}\,\mathrm{d}u \times B = 0\,. \tag{4.3}$$

Consequently, the first three equations of (4.1) *can be, locally, solved, up to a gauge transformation, if and only if there exists a function w such that the following two assertions hold:*

(i) $(\operatorname{grad} u)(w) = bu^3 + aB^\sharp(u)$;

(ii) $-2u^{-2}\,\mathrm{d}u \times \mathrm{d}w + bB + 2au^{-2}\,\mathrm{d}u \times B$ *is closed.*

Proof. As A satisfies $\mathrm{d}u = *\,\mathrm{d}A$, it must (locally) be of the form $A = B + \mathrm{d}v$, for a suitable function v.

Also, the third relation of (4.1) is satisfied, for a suitable f_0, if and only if $a\,\mathrm{d}u = -*\,\mathrm{d}\big(u^2F_1^\flat\big)$. Thus, $F_1 = -au^{-2}B^\sharp + u^{-2}\operatorname{grad} w$ for a suitable function w. Then the first relation of (4.1) is satisfied if and only if $\operatorname{grad} w = bu^3 + aB(\operatorname{grad} u)$.

To complete the proof, just note that, now, the second relation of (4.1) is equivalent to (4.3). □

We do not have a reference for the following simple and easy to prove lemma.

Lemma 4.3. *Let α and β be two one-forms on a Riemannian manifold, and let d^* be the codifferential. Then*

$$\mathrm{d}^*(\alpha\wedge\beta) = (\mathrm{d}^*\alpha)\,\beta - \alpha\,(\mathrm{d}^*\beta) - \big[\alpha^\sharp,\beta^\sharp\big]^\flat\,.$$

Next, we use Lemma 4.3 to characterise the suitable functions w of Lemma 4.2.

Proposition 4.4. *The first three equations of* (4.1) *can be, locally, solved, up to a gauge transformation, if and only if there exists a function w on (N,h) such that the following two equations hold:*

$$\begin{aligned}&(\operatorname{grad} u)(w) = bu^3 + aB^\sharp(u)\,,\\ &\big(2u^{-2}\Delta w - 2au^{-2}\,\mathrm{d}^*B + b\big)\,\mathrm{d}u + 2\big[u^{-2}\operatorname{grad} u, \operatorname{grad} w - aB^\sharp\big]^\flat\\ &\qquad\qquad - 4u^{-3}|\mathrm{d}u|^2(\mathrm{d}w - aB) = 0\,.\end{aligned} \tag{4.4}$$

Proof. We have to show that the second relation of (4.4) holds if and only if the one-form appearing in (ii) of Lemma 4.2 is closed; equivalently,

$$*\,\mathrm{d}\big(-2u^{-2}*(\mathrm{d}u\wedge\mathrm{d}w) + 2au^{-2}*(\mathrm{d}u\wedge B) + bB\big) = 0\,. \tag{4.5}$$

We calculate the left hand side of (4.5) by using that $*\,\mathrm{d}B = \mathrm{d}u$ and Lemma 4.3:

$$\begin{aligned}&-2\,\mathrm{d}^*\big((u^{-2}\,\mathrm{d}u)\wedge(\mathrm{d}w - aB)\big) + b\,\mathrm{d}u\\ &= -2\big(\mathrm{d}^*(u^{-2}\,\mathrm{d}u)\big)(\mathrm{d}w - aB) + 2(\mathrm{d}^*\mathrm{d}w - a\,\mathrm{d}^*B)\,u^{-2}\,\mathrm{d}u\\ &\qquad +2\big[u^{-2}\operatorname{grad} u, \operatorname{grad} w - aB^\sharp\big] + b\,\mathrm{d}u\,.\end{aligned} \tag{4.6}$$

Now, just note that, as u is harmonic, we have $\mathrm{d}^*(u^{-2}\,\mathrm{d}u) = 2u^{-3}|\mathrm{d}u|^2$. □

Note that, the first two relations of (4.7), below, are tensorial in $\operatorname{grad} w$.

Theorem 4.5. *Let u be a harmonic function on a three-dimensional real-analytic Riemannian manifold (N,h). Let B be a (local) solution of the monopole equation $\mathrm{d}u = *\,\mathrm{d}B$ and let w be a function on N satisfying*

(4.7)
$$(\operatorname{grad} w)(u) = bu^3 + aB^\sharp(u)\ ,$$
$$\nabla_{\operatorname{grad} w}\,\mathrm{d}u = -u^{-1}|\mathrm{d}u|^2\big(\mathrm{d}w - aB\big) + \tfrac{5}{2}\,bu^2\,\mathrm{d}u - a\big[\operatorname{grad} u, B^\sharp\big]^\flat + \tfrac{1}{2}\,a\big(\mathcal{L}_{B^\sharp}h\big)(\operatorname{grad} u, \cdot)\ ,$$
$$\nabla\mathrm{d}w = \tfrac{1}{2}\,a\,\mathcal{L}_{B^\sharp}h - \tfrac{1}{2}\,bu^2h + 2u^{-1}\,\mathrm{d}u \odot \big(\mathrm{d}w - aB\big)\ ,$$

where $a, b \in \mathbb{R}$, $b \neq 0$.

Then, locally on N, there exist functions v and f_0, unique up to constants, such that if (M,g) is given by (N,h), u and $A = B + \mathrm{d}v$, through the Gibbons–Hawking construction, and $E = \big(f_0 - at + \frac{1}{4}\,b\,t^2\big)\frac{\partial}{\partial t} + tF$, where F is the horizontal lift of $u^{-2}\big(\operatorname{grad} w - aB^\sharp\big)$, then the following assertions are equivalent:

(i) *E is a soliton flow on (M,g), with the corresponding constant a;*

(ii) $^N\mathrm{Ric} + a\,u\,h - u^{-1}A \odot (\mathrm{d}w - aB) = 0$.

Moreover, any soliton flow on a real-analytic Riemannian manifold, given by the Gibbons–Hawking construction (with u nonconstant), is obtained this way, if its horizontal part is t times a nonzero basic vector field.

Proof. If we denote $G = \operatorname{grad} w - aB^\sharp$ then $a\,\mathrm{d}u = - * \mathrm{d}G^\flat$, and (4.4) is equivalent to

$$\begin{aligned} G(u) &= bu^3 \\ 2\big[u^{-2}\operatorname{grad} u, G\big] &= -\big(2u^{-2}\,\mathrm{d}^*G + b\big)\operatorname{grad} u + 4u^{-3}|\mathrm{d}u|^2G\ . \end{aligned} \tag{4.8}$$

Further, as $F_1 = F = u^{-2}G$, the fourth relation of (4.1) holds if and only if

$$\mathcal{L}_G h = -bu^2h + 4u^{-1}\,\mathrm{d}u \odot G^\flat\ . \tag{4.9}$$

Then (4.9) and the first relation of (4.8) imply $\mathrm{d}^*G = -\frac{bu^2}{2}$ (which, together with the second relation of (4.8), gives $G\big(|\mathrm{d}u|^2\big) = 3\,b\,u^2|\mathrm{d}u|^2$).

Thus, if (4.9) holds then (4.8) is equivalent to the first relation of (4.7) and the following $\big[\operatorname{grad} u, G\big] = 2u^{-1}|\mathrm{d}u|^2G - 2bu^2\operatorname{grad} u$; further, the latter is equivalent to

(4.10)
$$\nabla_{\operatorname{grad} u}(\operatorname{grad} w) = \nabla_{\operatorname{grad} w}(\operatorname{grad} u) + a\big[\operatorname{grad} u, B^\sharp\big] + 2u^{-1}|\mathrm{d}u|^2\big(\operatorname{grad} w - aB^\sharp\big) - 2bu^2\operatorname{grad} u\ .$$

On the other hand, (4.9) is equivalent to

$$\nabla\mathrm{d}w = \tfrac{1}{2}\,a\,\mathcal{L}_{B^\sharp}h - \tfrac{1}{2}\,bu^2h + 2u^{-1}\,\mathrm{d}u \odot \big(\mathrm{d}w - aB\big)\ ; \tag{4.11}$$

which, together with $G(u) = bu^3$, implies

$$\nabla_{\operatorname{grad} u}(\mathrm{d}w) = \tfrac{1}{2}\,a\big(\mathcal{L}_{B^\sharp}h\big)\big(\operatorname{grad} u, \cdot\big) + \tfrac{1}{2}\,bu^2\,\mathrm{d}u + u^{-1}|\mathrm{d}u|^2\big(\mathrm{d}w - aB\big)\ . \tag{4.12}$$

Thus, if (4.11) holds then (4.10) is equivalent to the second relation of (4.7).

We have, thus, shown that the first four relations of (4.1) can be locally solved, with

a suitable $A = B + \mathrm{d}v$ and f_0, if and only if (4.7) holds. Together with Corollary 4.1, this completes the proof. □

We end this section with the following application of Theorem 4.5. We omit the proof.

Corollary 4.6. *Let (M, g) be given by the Gibbons–Hawking construction, with (N, h) real-analytic, and the fibres of u are flat and geodesic.*

Let $E = fV + t\widetilde{F}$ be a real-analytic vector field on (M, g), where $\widetilde{F}$ is the horizontal lift of the vector field F on N.

Then E is a soliton flow on (M, g), with the corresponding constant a, if and only if $F = 0$, $a = 0$, f_0 is constant, and (M, g) is Ricci flat self-dual.

5. Ricci solitons and the Beltrami fields construction

Let $\varphi : (M, g) \to (N, h)$ be a twistorial harmonic morphism with $V(\lambda^{-2}) \neq 0$. Then, up to homotheties, we may suppose that $V(\lambda^{-2}) = 2$. It follows that, locally, there exists a one-form A on N satisfiying the Beltrami fields equation $\mathrm{d}A + 2 * A = 0$ and such that

$$g = \rho^2 h + \rho^{-2}(\rho\,\mathrm{d}\rho + A)^2 \,,$$

with $\varphi : M = N \times (0, \infty) \to N$ the projection; in particular, $\lambda = \rho^{-1}$, $V = \rho^{-1}\frac{\partial}{\partial\rho}$, and $\theta = \rho\,\mathrm{d}\rho + A$.

Note that, if (N, h) is real-analytic then, as A satisfies $\Delta A = 4A$, we have that, also, g is real-analytic. Furthermore, the complexification of g is defined on $N^{\mathbb{C}} \times (\mathbb{C} \setminus \{0\})$.

Theorem 5.1. *Let E be a real-analytic soliton flow on a Riemannian manifold (M, g) given by the Beltrami fields construction.*

Suppose that E admits a complexification on a set containing $U \times \gamma$, where U is an open subset of N and γ is a circle on $\mathbb{C}$, centred at 0.

Then the following assertions are equivalent:

(i) *(M, g) is Einstein;*

(ii) *$\int_\gamma \rho\,(\mathcal{L}_E h)^{\mathbb{C}}\,\mathrm{d}\rho = 0$, on $U^{\mathbb{C}}$.*

Furthermore, a sufficient condition for (i) *and* (ii) *to hold is that there exists $j \in \mathbb{N} \setminus \{0\}$ such that the trace-free part of $\int_\gamma \rho^{2j+1}\,(\mathcal{L}_E h)^{\mathbb{C}}\,\mathrm{d}\rho$ is zero, on $U^{\mathbb{C}}$.*

The proof of Theorem 5.1 will be given below.

From now on, in this section, *we shall denote by X, Y and Z the projections onto N of the corresponding vector fields appearing in* (2.3). Then, as $\Omega = \mathrm{d}A$, we have $Z = \frac{1}{|A|}A^\sharp$, $A(X) = A(Y) = 0$. Hence, the horizontal lifts of X, Y and Z are X, Y and $-|A|\rho^{-1}\frac{\partial}{\partial\rho} + Z$, respectively. Consequently, $\Omega(X, Y) = \big(-|A|\rho^{-1}\frac{\partial}{\partial\rho} + Z\big)(\rho^2) = -2|A|$.

Thus, Proposition 2.1 gives the following result.

Corollary 5.2. *Let (M, g) be given by the Beltrami fields construction. Then a vector field $E = fV + F$, with F horizontal, is a soliton flow on (M, g), with the corresponding*

constant a, if and only if the following five relations hold:

$$
\begin{aligned}
&\rho\,\frac{\partial f}{\partial \rho} + a\rho^2 - f + z|A| = 0\,,\\
&X(f) + 2y|A| + \rho^3\,\frac{\partial x}{\partial \rho} = 0\,,\\
&Y(f) - 2x|A| + \rho^3\,\frac{\partial y}{\partial \rho} = 0\,,\\
&-|A|\rho^{-1}\frac{\partial f}{\partial \rho} + Z(f) + \rho^3\,\frac{\partial z}{\partial \rho} = 0\,,\\
&\rho^2\,\mathcal{L}_F h + 2\,{}^N\mathrm{Ric} + 2\big(f - z|A| + a\rho^2 - 2\big)h = 0 \text{ on } \mathscr{H}\,.
\end{aligned}
\tag{5.1}
$$

We shall, now, rewrite (5.1) under the hypothesis of Theorem 5.1 .

Corollary 5.3. *Let (M,g) be given by the Beltrami fields construction, with (N,h) real-analytic. Let $E = fV + F$ be a real-analytic vector field on (M,g), where F is horizontal.*

Suppose that E admits a complexification on a set containg $U \times \gamma$, where U is an open subset of N and γ is a circle on $\mathbb{C}$, centred at 0.

Then f and F admit Laurent series expansions $f = \sum_{j=-\infty}^{\infty} \rho^j f_j$ and $F = \sum_{j=-\infty}^{\infty} \rho^j \widetilde{F_j}$, where f_j are functions on N, and $\widetilde{F_j}$ are the horizontal lifts of vector fields $F_j = x_j X + y_j Y + z_j Z$ on N.

Furthermore, E is a soliton flow on (M,g), with the corresponding constant a, if and only if the following relations hold:

$$
\begin{aligned}
&(j-1)f_j + z_j|A| = 0\,,\ \big(j \in \mathbb{Z} \setminus \{2\}\big)\,,\\
&f_2 + z_2|A| + a = 0\,,\\
&X(f_j) + 2y_j|A| + (j-2)x_{j-2} = 0\,,\ \big(j \in \mathbb{Z}\big)\,,\\
&Y(f_j) - 2x_j|A| + (j-2)y_{j-2} = 0\,,\ \big(j \in \mathbb{Z}\big)\,,\\
&-(j+2)|A|f_{j+2} + Z(f_j) + (j-2)z_{j-2} = 0\,,\ \big(j \in \mathbb{Z}\big)\,,\\
&\mathcal{L}_{F_{j-2}}\, h - 2jA \odot F_j^\flat + 2\big(f_j - z_j|A|\big)h = 0\,,\ \big(j \in \mathbb{Z} \setminus \{0\,,\,2\}\big)\,,\\
&\mathcal{L}_{F_{-2}}\, h + 2\,{}^N\mathrm{Ric} + 2\big(f_0 - z_0|A| - 2\big)h = 0\,,\\
&\mathcal{L}_{F_0}\, h - 4A \odot F_2^\flat + 2\big(f_2 - z_2|A| + a\big)h = 0\,.
\end{aligned}
\tag{5.2}
$$

Proof. We shall denote by the same symbol an object and its complexification. Also, all the objects are assumed complex-analytic.

As the domain (of the complexification) of E contains U times a circle, with $U \subseteq N$ open, it, also, contains U times an open annulus. Then the existence of the Laurent series expansions, for f, x, y, z, and, consequently, F, follows by applying a standard argument.

It is easy to see that the first four relations of (5.1) are equivalent to the first five relations of (5.2) .

Let S be a vector field on N and let $\widetilde{S}$ be its horizontal lift. As $\widetilde{S} = -A(S)\rho^{-1}\frac{\partial}{\partial\rho} + S$, we have $\widetilde{S}(\rho^j) = -jA(S)\rho^{j-2}$. Consequently,

$$(\mathcal{L}_{\rho^j\widetilde{F_j}}\, h)(\widetilde{S},\widetilde{S}) = \rho^j(\mathcal{L}_{F_j}\, h)(S,S) - 2j\rho^{j-2}\big(A\odot F_j^\flat\big)(S,S)\ ,$$

from which the last three relations follow quickly. □

We can, now, give the proof of Theorem 5.1 .

Proof of Theorem 5.1. The equivalence of assertions (i) and (ii) is a consequence of the seventh formula of (5.2) .

On working with homogeneous quadratic polynomials, instead of symmetric bilinear forms, and identifying forms and vector fields on N, through h , the sixth formula of (5.2) is equivalent to the following:

$$\begin{aligned} -\tfrac{1}{2}\,\mathcal{L}_{F_{j-2}}\, h = {} & (f_j - z_j|A|)X^2 + (f_j - z_j|A|)Y^2 + (f_j - 2z_j|A|)Z^2 \\ & - jx_j|A|XZ - jy_j|A|YZ\ . \end{aligned} \tag{5.3}$$

The last assertion follows quickly from the first formula of (5.2) and (5.3) . □

Remark 5.4. Under the same hypotheses as in Theorem 5.1 , and with similar proofs, the following three statements follow quickly:

(a) The vertical part of E is determined by the horizontal part (in fact, by $z = A(E)$).

(b) (M,g) is self-dual if and only if the trace-free part of $\int_\gamma \rho\,(\mathcal{L}_E h)^{\mathbb{C}}\,\mathrm{d}\rho$ is zero, on $U^{\mathbb{C}}$.

(c) F_0 and F_1 uniquely determine all F_j , with $j \in \mathbb{N}$.

We end with a particular case.

Corollary 5.5. *Let (M,g) be a Riemannian manifold given by the Beltrami fields construction, with A the corresponding one-form on N.*

Suppose that E admits a complexification on a set containg $U \times \gamma$, where U is an open subset of N and γ is a circle on $\mathbb{C}$, centred at 0 .

Then $E = fV + F$, with $F = xX + yY + zZ$ and z a function on N, is a soliton flow, with the corresponding constant a , if and only if the following assertions hold:

(i) $a = 0$, $f = z|A|$ *and* $Z(f) = 0$;

(ii) $x + \mathrm{i}y = u\,e^{-\mathrm{i}\rho^{-2}|A|} + \frac{1}{2\mathrm{i}|A|}(X + \mathrm{i}\,Y)(f)$, *where u is a function on N;*

(iii) $\int_\gamma \rho\,(\mathcal{L}_E h)^{\mathbb{C}}\,\mathrm{d}\rho + 2\,{}^N\mathrm{Ric} - 4h = 0$, *on* $U^{\mathbb{C}}$;

(iv) $\int_\gamma \rho^{-j+1}(\mathcal{L}_E h)^{\mathbb{C}}\,\mathrm{d}\rho = A \odot \int_\gamma \rho^{-j-1}(F^\flat)^{\mathbb{C}}\,\mathrm{d}\rho$, *on* $U^{\mathbb{C}}$, *for any* $j \in \mathbb{Z}\setminus\{0\}$.

Proof. As $\frac{\partial z}{\partial\rho} = 0$, the first assertion of (5.1) is equivalent to

$$\frac{\partial}{\partial\rho}(\rho^{-1}f + a\rho - \rho^{-1}z|A|) = 0\ ;$$

equivalently, $f = -a\rho^2 + v\rho + z|A|$, where v is a function on N.

By using this, we obtain that the fourth relation of (5.1) is equivalent to

$$-|A|^{-1}\rho^{-1}(v - 2a\rho) + \rho\, Z(v) + Z(z|A|) = 0\,;$$

that is, $a = v = Z(v) = 0$.

Then the second and the third relations of (5.1) are equivalent to assertion (ii). Also, by using Corollary 5.3, we obtain that the fifth relation of (5.1) is equivalent to (iii) and (iv). □

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