CONVERGENCE RATE OF A SCHWARZ MULTILEVEL METHOD FOR THE CONSTRAINED MINIMIZATION OF NON-QUADRATIC FUNCTIONALS

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Abstract. In [5], the convergence of a subspace correction method applied to the constrained minimization of a functional in a general reflexive Banach space has been proved, provided that the convex set verifies a certain assumption. This assumption is weaker than that in which the convex set is decomposed according to the space decomposition as a sum of subsets. In the Sobolev spaces, the proposed method becomes a multiplicative Schwarz method for the solution of the variational inequalities coming from the minimization of non-quadratic functionals. We prove in this paper that this assumption holds for the one-, two- and multi-level multiplicative Schwarz methods in the finite element space, and we explicitly write the constants in the error estimations depending on the overlapping and mesh parameters. Our error estimates are similar with those obtained for the minimization of quadratic functionals in [4], or with those obtained for the one-obstacle problem in [37].

Key words. domain decomposition methods, variational inequalities, non-quadratic minimization, multigrid and multilevel methods, finite element methods, nonlinear obstacle problems

AMS subject classifications. 65N55, 65N30, 65J15

1. Introduction. Domain decomposition methods provide efficient numerical algorithms to solve very large-scale problems. The great interest in these methods comes from the fact that they are parallelizable on multi-processor machines. Schwarz overlapping methods represent a typical example of such parallelizable methods, they traditionally being classified as multiplicative and additive. The main focus of this paper is the convergence of the multiplicative Schwarz method applied to the constrained minimization of non-quadratic convex functionals.

Naturally, most papers dealing with these methods are dedicated to linear problems. The multiplicative and additive Schwarz methods for elliptic linear problems have been studied by Lions [26]–[28], Chan, Hou and Lions [10], P. Le Tallec [25], A. Quarteroni and A. Valli [33], Bramble, Pasciak, Wang and Xu [8], and Badea [1], for the multiplicative methods, and Dryja [12], Dryja and Widlund [13], [14], and Nepomnyaschikh [32], for the additive version.

For the application of the Schwarz method to the solution of the variational inequalities, we can cite the papers written by Hoffman and Zou [19], Kuznetsov and Neittaanmäki [22], Kuznetsov, Neittaanmäki and Tarvainen [23], [24], Lü, Liem and Shih [29], Zeng and Zhou [42], Tai [35]–[37], Tai and Tseng [39], Badea and Wang [3], Badea, Tai and Wang [4], and Badea [2], [6], [7].

Also, the multilevel and multigrid methods can be viewed as domain decomposition methods and we can cite the results obtained by Kornhuber [21], Mandel [31], and Smith, Bjørstad and Gropp [34].

However, very few papers deal with the application of these methods to nonlinear problems. We can cite in this direction the papers written by Tai and Espedal [38], Tai and Xu [40] for nonlinear equations, Hoffmann and Zhou [20], Lui [30], Zeng and Zhou in [43] for inequalities having nonlinear source terms, and Badea [5] for a general result concerning the convergence of the method for the constrained minimization of

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non-quadratic functionals. Evidently, the above lists of citations are not exhaustive and it can be completed by many other papers.

Almost exclusively, the convergence of the domain decomposition methods for variational inequalities coming from the minimization of a functional is studied in the case when this functional is quadratic. Also, most papers consider the convex set decomposed according to the space decomposition as a sum of convex subsets. The main goal of this paper is to give error estimates for the one-, two- and multi-level Schwarz domain decomposition methods applied to the constrained minimization of the non-quadratic convex functionals over enough general convex sets.

The convergence of a domain decomposition algorithm solving variational inequalities coming from the minimization of quadratic functionals over convex sets is proved in [2]. In that paper, the convex set, defined by constraints on the function values at the points of the domain, is not supposed to be decomposed as a sum of convex subsets. In [40], a subspace correction method applied to the minimization without constraints of a differentiable and convex functional defined in a reflexive Banach space is introduced. Also, in [5], the convergence of an algorithm in a reflexive Banach space for the constrained minimization of convex functionals is proved. There, in order to prove the convergence, a weaker property than that one given in [2] is imposed on the convex set. To the author's knowledge, there are no other papers dealing with the Schwarz method applied to the constrained minimization of non-quadratic functionals. Even if sometimes the conditions on the convex functional are general enough, the authors always consider the space H^1 and implicitly, quadratic functionals. For instance, in [4], using the subspace correction techniques in [8] and [41], and more general conditions in [38] on the convex functional, the convergence rate for the one and two-level algorithms of the method in [2] is given only for the minimization of quadratic functionals. Starting from the general convergence result given in cite [5], we generalize in this paper the results in [4] and [40] to the constrained minimization of non-quadratic functionals. Our error estimates are similar with those obtained for the minimization of quadratic functionals in [4] or [37].

The paper is organized as follows. In Section 2, we state the multiplicative Schwarz method as a subspace correction method in a general reflexive Banach space for the constrained minimization of convex functionals. We also give the convergence theorem of this algorithm which has been proved in [5] provided that a certain assumption holds. In Sections 3, 4 and 5 we prove that the introduced assumption holds and we estimate the error for the one-, two- and multi-level Schwarz methods, respectively, in the finite element spaces. In these cases, we are able to explicitly write the convergence rate depending on the mesh and domain decomposition parameters. The proof for the two- and multi-level methods is based on a lemma which can be viewed as a Friedrichs - Poincaré inequality for the finite element spaces. In Subsection 5.1, we find the convergence rate of the multigrid method from the results obtained for the multi-level method.

Finally, for the writing simplicity, we have considered in the next sections problems in $W^{1,s}$, but all the obtained results hold reading $[W^{1,s}]^d$ in the place of $W^{1,s}$.

2. General convergence result. We enunciate in this section a general algorithm and give an error estimate theorem for it. This general theory, the proof of the theorem included, are given in detail in [5]. We consider that V is a reflexive Banach space and $V_1, \cdots, V_m$, are some closed subspaces of V. Also, let $K \subset V$ be a non empty closed convex set, and we make the following

ASSUMPTION 2.1. *There exists a constant C_0 such that for any $w, v \in K$ and $w_i \in V_i$ with $w + \sum_{j=1}^{i} w_j \in K$, $i = 1, \cdots, m$, there exist $v_i \in V_i$, $i = 1, \cdots, m$, satisfying*

$$w + \sum_{j=1}^{i-1} w_j + v_i \in K \text{ for } i = 1, \cdots, m, \tag{2.1}$$

$$v - w = \sum_{i=1}^{m} v_i, \tag{2.2}$$

and

$$\sum_{i=1}^{m} ||v_i||^p \leq C_0^p \left(||v - w||^p + \sum_{i=1}^{m} ||w_i||^p \right). \tag{2.3}$$

This assumption looks complicated enough, but as we shall see in the following, it is satisfied for a large kind of convex sets in Sobolev spaces. In our proofs, v is the exact solution, w is the solution of the iterative algorithm at a certain iteration, and w_i are its corrections on the subspaces V_i, $i = 1, \cdots, m$. In the case of the convex sets written as a sum of convex subsets, equations (2.1) and (2.2) are always satisfied. We point out that in the case of the problems without constraints or that of the one-obstacle problems, the above assumption can be taken with $w_i = 0$ (see [37], for instance), and, for this reason, equation (2.3) usually is known without the extra terms given by w_i.

We consider a Gâteaux differentiable functional $F : K \to R$, which is supposed to be coercive if K is not bounded, and we assume that for any real number $M > 0$ there exist two functions

$$\alpha_M(\tau) = A_M \tau^p, \quad \beta_M(\tau) = B_M \tau^{q-1}, \tag{2.4}$$

such that

$$< F'(v) - F'(u), v - u > \geq \alpha_M(||v - u||), \text{ for any } u, v \in K, \ ||u||, ||v|| \leq M, \tag{2.5}$$

and

$$\beta_M(||v - u||) \geq ||F'(v) - F'(u)||_{V'}, \text{ for any } u, v \in K, \ ||u||, ||v|| \leq M, \tag{2.6}$$

where F' is the Gâteaux derivative of F, and $A_M > 0$, $B_M > 0$, $p > 1$ and $q > 1$ are some real constants. We have marked here that the constants A_M and B_M depend on M. It is evident that if (2.5) and (2.6) hold, then

$$\begin{aligned} &\alpha_M(||v - u||) \leq < F'(v) - F'(u), v - u > \leq \beta_M(||v - u||)||v - u||, \\ &\text{for any } u, v \in K, \ ||u||, ||v|| \leq M. \end{aligned} \tag{2.7}$$

It follows from (2.7) that we must take $p \geq q$. Following the way in [17] (Lemmas 1.1 and 1.2, pages 61–63), we can prove that

$$\begin{aligned} &< F'(u), v - u > + \lambda_M(||v - u||) \leq F(v) - F(u) \leq \\ &< F'(u), v - u > + \mu_M(||v - u||), \text{ for any } u, v \in K, \ ||u||, ||v|| \leq M, \end{aligned} \tag{2.8}$$

where

$$\lambda(\tau) = \tfrac{A_M}{p}\tau^p, \quad \mu(\tau) = \tfrac{B_M}{q}\tau^q. \tag{2.9}$$

It is well known (see [16]) that if V and F satisfy the above assumptions, then the minimization problem

$$u \in K : F(u) \le F(v), \text{ for any } v \in K \tag{2.10}$$

has a unique solution, and it is also the unique solution of the problem

$$u \in K : \; < F'(u), v - u > \ge 0, \text{ for any } v \in K. \tag{2.11}$$

From (2.8), for a given $M > 0$ such that the solution u of (2.11) satisfies $||u|| \le M$, we have

$$\lambda_M(||v-u||) \le F(v) - F(u), \text{ for any } v \in K, \; ||v|| \le M. \tag{2.12}$$

The proposed algorithm corresponding to the subspaces $V_1, \cdots, V_m$ and the convex set K is written as follows

Algorithm 2.1. *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \ge 0$, we compute sequentially for $i = 1, \cdots, m$, $w_i^{n+1} \in V_i$ satisfying*

$$w_i^{n+1} = \arg \min_{\substack{u^{n+\frac{i-1}{m}} + v_i \in K \\ v_i \in V_i}} G(v_i), \text{ with } G(v_i) = F(u^{n+\frac{i-1}{m}} + v_i), \tag{2.13}$$

and then we update

$$u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}.$$

This algorithm does not assume a decomposition of the convex set K depending on the subspaces V_i, and it has been proposed in [2] in an equivalent form. The above form of this algorithm has been proposed in [4] for the constrained minimization of the quadratic functions. As for problem (2.10), since the subspaces V_i are reflexive Banach spaces, problem (2.13) has a unique solution and it also satisfies the variational inequality

$$\begin{array}{l} w_i^{n+1} \in V_i, \; u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K : \\ < F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1} > \ge 0, \\ \text{ for any } v_i \in V_i, \; u^{n+\frac{i-1}{m}} + v_i \in K. \end{array} \tag{2.14}$$

The introduction of some parameters $\varepsilon_{ij} \ge 0$, $i, j = 1, \cdots, m$, is useful to obtain some sharper error estimations, especially in the case of minimization of the quadratic functionals. Following this way, we assume that for a given $M > 0$, if $v \in K$, $||v|| \le M$, and $v_i \in V_i$, satisfying $v + v_i \in K$, $||v + v_i|| \le M$, $i = 1, \cdots, m$, then we have

$$< F'(v + v_i) - F'(v), w_j > \le \varepsilon_{ij} B_M ||v_i||^{q-1} ||w_j|| \tag{2.15}$$

for any $w_i \in V_i$, $i = 1, \cdots, m$. Evidently, using (2.6), we may always take $\varepsilon_{ij} = 1$, $i, j = 1, \cdots, m$, in (2.15).

The following theorem extends to inequalities the result in [40] concerning the convergence of the method for nonlinear equations.

THEOREM 2.1. *We consider that V is a reflexive Banach, $V_1, \cdots, V_m$ are some closed subspaces of V, K is a non empty closed convex subset of V, and F is Gâteaux differentiable functional on K which is supposed to be coercive if K is not bounded. We assume that the functional F satisfies (2.5) and (2.6), and we make Assumption 2.1. On these conditions, if u is the solution of problem (2.10) and u^n, $n \geq 0$, are its approximations obtained from Algorithm 2.1, then we have the following error estimations:*

(i) if $p = q$ we have

$$\begin{aligned} F(u^n) - F(u) &\leq \left(\frac{\hat{C}}{\hat{C}+1}\right)^n \left[F(u^0) - F(u)\right], \\ \|u^n - u\|^p &\leq \frac{\hat{C}+1}{\bar{C}} \left(\frac{\hat{C}}{\hat{C}+1}\right)^n \left[F(u^0) - F(u)\right]. \end{aligned} \tag{2.16}$$

(ii) if $p > q$ we have

$$\begin{aligned} F(u^n) - F(u) &\leq \frac{F(u^0) - F(u)}{\left[1 + n\tilde{C}(F(u^0) - F(u))^{\frac{p-q}{q-1}}\right]^{\frac{q-1}{p-q}}}, \\ \|u - u^n\|^p &\leq \frac{\hat{C}}{\bar{C}} \frac{\left(F(u^0) - F(u)\right)^{\frac{q-1}{p-1}}}{\left[1 + (n-1)\tilde{C}(F(u^0) - F(u))^{\frac{p-q}{q-1}}\right]^{\frac{(q-1)^2}{(p-1)(p-q)}}}. \end{aligned} \tag{2.17}$$

The constants $\hat{C}$, $\bar{C}$ and $\tilde{C}$ are written as

$$\begin{aligned} \hat{C} = \hat{C}(m, C_0, u^0) = B_M\left(\frac{p}{A_M}\right)^{\frac{q}{p}} |\varepsilon_{ij}| \Bigg[(1 + 2C_0)\left(F(u^0) - F(u)\right)^{\frac{p-q}{p(p-1)}} + \\ \left(B_M\left(\frac{p}{A_M}\right)^{\frac{q}{p}} |\varepsilon_{ij}|\right)^{\frac{1}{p-1}} C_0^{\frac{p}{p-1}} / \eta^{\frac{1}{p-1}} \Bigg] / (1 - \eta), \end{aligned} \tag{2.18}$$

$$\bar{C} = \frac{(2 - \eta) A_M}{(1 - \eta) p}, \tag{2.19}$$

$$\tilde{C} = \frac{p - q}{(p-1)\left(F(u^0) - F(u)\right)^{\frac{p-q}{q-1}} + (q-1)\hat{C}^{\frac{p-1}{q-1}}}. \tag{2.20}$$

The value of η in the expressions of $\hat{C}$ and $\bar{C}$ can be arbitrary in $(0, 1)$. On the other hand, we see that the constants in the error estimations of $F(u^n) - F(u)$ in (2.16) and (2.17) are some increasing functions of $\hat{C}$, and there is an $\eta_0 \in (0, 1)$ such that $\hat{C}(\eta_0) \leq \hat{C}(\eta)$ for any $\eta \in (0, 1)$. However, this value η_0 can be found by solving a nonlinear algebraic equation.

We point out that a convergence result can be found (see [5]) under weaker conditions on the functions α_M and β_M than those given in (2.4), and an weaker assumption than Assumption 2.1.

The above algorithm can be viewed as a multiplicative Schwarz method, in a subspace correction variant, if we use the Sobolev spaces. In this way, we consider for a domain Ω in $\mathbf{R}^d$, $d \geq 1$, with Lipschitz continuous boundary $\partial\Omega$, an overlapping decomposition

$$\Omega = \bigcup_{i=1}^{m} \Omega_i \tag{2.21}$$

in which the subdomains Ω_i have Lipschitz continuous boundary, too. We associate with the domain Ω the space $V = W_0^{1,s}(\Omega)$, $1 < s < \infty$, and with the subdomains Ω_i the subspaces $V_i = W_0^{1,s}(\Omega_i)$, $i = 1, \cdots, m$. For convex sets $K \subset V$ satisfying

PROPERTY 2.1. *If $v, w \in K$, and if $\theta \in C^1(\Omega)$ with $0 \leq \theta \leq 1$, then $\theta v + (1-\theta)w \in K$*

it has been proved in [5] the following

PROPOSITION 2.2. *If for the domain decomposition* (2.21) *there exist some continuously differentiable unity partitions $\{\theta_j^i\}_{j=i,\cdots,m}$ associated with $\cup_{j=i}^m \Omega_j$, $i = 1, \cdots, m$, (i.e., for any $i = 1, \cdots, m$, supp$\theta_j^i \subset \Omega_j$, $\theta_j^i \in C^1(\Omega_j)$, and $0 \leq \theta_j^i \leq 1$, for $j = i, \cdots, m$, and $\sum_{j=i}^m \theta_j^i = 1$ on $\cup_{j=i}^m \Omega_j$), then Assumption* 3.1 *holds for any convex set K having Property 2.1.*

Consequently, provided that functional F satisfies (2.5) and (2.6), Algorithm 2.1 converges and we can apply Theorem 2.1 to get the convergence rate. The above Sobolev spaces $W_0^{1,s}$ correspond to Dirichlet boundary conditions. Similar results can be obtained if we consider appropriate subspaces of $W^{1,s}$ for the mixed boundary conditions.

The constant C_0 in Assumption 2.1 depends on the domain decomposition parameters. Consequently, since the constants $\hat{C}$ and $\bar{C}$ in the error estimations in Theorem 2.1 depend on C_0, then these estimations will depend on domain decomposition parameters, too. The goal of the next sections is to prove, for the one-, two-level and multi-level multiplicative Schwarz methods, that Assumption 2.1 also holds for any closed convex K satisfying a similar property to that given in 2.1. In these cases we are able to explicitly write the dependence of C_0 on the domain decomposition and mesh parameters.

3. One-level multiplicative Schwarz method. First, let us consider that the domain $\Omega \subset \mathbf{R}^d$ has an overlapping domain decomposition $\{O_i\}_{1 \leq i \leq M}$ and a simplicial mesh partition $\mathcal{T}_h$ of mesh size h. We assume that $\mathcal{T}_h$ is regular (ie. there exists a constant $C > 0$, independent of h, such that each τ in $\mathcal{T}_h$ contains a ball with the diameter of Ch, and evidently, it is contained in a ball with the diameter of h; see [11], pag. 124, for instance) and it supplies a mesh partition for each subdomain O_i, $i = 1, \cdots, M$, too. In addition, we suppose that there exists a positive constant δ, the overlapping parameter, such that for any $i = 1, \cdots, M$, we have

$$O_i \cap \partial(\bigcup_{j \neq i} O_j) \neq \emptyset \text{ and } \operatorname{dist}(\partial O_i \backslash \partial\Omega, O_i \cap \partial(\bigcup_{j \neq i} O_j) \geq \delta. \tag{3.1}$$

Now, we assume that there exist m colors such that each subdomain O_i can be marked with one color, and the subdomains with the same color do not intersect with each other. For suitable overlaps, one can always choose $m = 2$ if $d = 1$, $m \leq 4$ if $d = 2$, and $m \leq 8$ if $d = 3$. Let Ω_i be the union of the subdomains O_j having the color i. In this way, we have obtained an overlapping decomposition (2.21) with overlaps of size δ. Taking into account (3.1), we can assume that the unity partitions $\{\theta_j^i\}_{j=i,\cdots,m}$ associated with $\cup_{j=i}^m \Omega_j$ in Proposition 2.2 satisfy

$$|\partial_{x_k} \theta_j^i| \leq C/\delta, \text{ for any } i = 1, \cdots, m.j = i, \cdots, m, \text{ and } k = 1, \cdots, d, \tag{3.2}$$

As in (3.2), we denote in the following by C a generic constant which does not depend on either the mesh or the domain decomposition parameters.

In this section we prove for the finite element spaces a similar result to that given in Proposition 2.2 for general Sobolev spaces. The proof is also similar to that given in [4] for the minimization of the quadratic functionals. We consider the piecewise linear finite element space

$$V_h = \{v \in C^0(\bar{\Omega}) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_h,\ v = 0 \text{ on } \partial\Omega\}, \tag{3.3}$$

and also, for $i = 1, \cdots, m$, we take

$$V_h^i = \{v \in V_h : v = 0 \text{ in } \Omega \backslash \Omega_i\} \tag{3.4}$$

as some subspaces of V_h corresponding to the domain decomposition $\Omega_1, \cdots, \Omega_m$. The spaces V_h and V_h^i, $i = 1, \cdots, m$, are considered as subspaces of $W^{1,s}$, for some fixed $1 \le s \le \infty$. We denote by $||\cdot||_{0,s}$ the norm in L^s, and by $||\cdot||_{1,s}$ and $|\cdot|_{1,s}$ the norm and seminorm in $W^{1,s}$, respectively.

In the following, L_h will be the P_1–Lagrangian interpolation operator which uses the function values at the nodes of the mesh $\mathcal{T}_h$. The convex set K_h is defined as a subset of V_h satisfying

Property 3.1. *If $v, w \in K_h$, and if $\theta \in C^1(\Omega)$ with $0 \le \theta \le 1$, then $L_h(\theta v + (1-\theta)w) \in K_h$.*

In order to prove that Assumption 2.1 holds, we follow the same way as in [4] or [5]. Taking into account the additivity of the Lagrangian interpolation L_h, (2.1) and (2.2) in Assumption 2.1 can be recurrently proved. Indeed, first we write

$$v_1 = L_h\left(\theta_1^1(v - w) + (1 - \theta_1^1)w_1\right), \tag{3.5}$$

and prove that

$$\begin{aligned}
&v_1 \in V_h^1 \text{ and } w + v_1 \in K_h,\\
&v - v_1 + w_1 \in K_h,\\
&v - w - v_1 \in W_0^{1,s}(\bigcup_{j=2}^m \Omega_j) \text{ and}\\
&v - w - v_1 = 0 \text{ in } \Omega - \overline{\cup_{j=2}^m \Omega_j}.
\end{aligned}$$

Next, for $i = 2, \cdots, m-1$, we write

$$v_i = L_h\left(\theta_i^i(v - w - \sum_{j=1}^i v_j) + (1 - \theta_i^i)w_i\right), \tag{3.6}$$

and prove

$$\begin{gathered}
v_i \in V_h^i \text{ and } w + \sum_{j=1}^{i-1} w_j + v_i \in K_h,\\
v - \sum_{j=1}^i v_j + \sum_{j=1}^i w_j \in K_h,\\
v - w - \sum_{j=1}^i v_j \in W_0^{1,s}(\bigcup_{j=i+1}^m \Omega_j) \text{ and}\\
v - w - \sum_{j=1}^i v_j = 0 \text{ in } \Omega - \overline{\cup_{j=i+1}^m \Omega_j},
\end{gathered}$$

assuming that these equations hold for $i-1$. Finally, we take

$$v_m = v - w - \sum_{j=1}^{m-1} v_j. \tag{3.7}$$

To prove inequality (2.3) in Assumption 2.1, we first notice that, starting from v_1 given in (3.5), by the recurrent application of (3.6), and then taking v_m given in (3.7), we get that v_i, $i = 1, \cdots, m$, are of the form

$$v_i = L_h\left(\tau_0^i(v-w) + \sum_{j=1}^{i} \tau_j^i w_j\right), \ i = 1, \cdots, m. \tag{3.8}$$

By a simple calculus we get that

$\tau_0^1 = \theta_1^1,\ \tau_1^1 = 1 - \theta_1^1,$
$\tau_0^i = \theta_i^i(1-\theta_{i-1}^{i-1})\cdots(1-\theta_1^1),\ \tau_i^i = 1-\theta_i^i,\ \tau_j^i = -\theta_i^i(1-\theta_{i-1}^{i-1})\cdots(1-\theta_j^j),$
for $i = 2, \cdots, m-1,\ j = 1, \cdots, i-1,$
$\tau_0^m = (1-\theta_{m-1}^{m-1})\cdots(1-\theta_1^1),\ \tau_m^m = 0, \tau_{m-1}^m = -(1-\theta_{m-1}^{m-1}),$
$\tau_j^m = \theta_{m-1}^{m-1}(1-\theta_{m-2}^{m-2})\cdots(1-\theta_j^j),$ for $j = 1, \cdots, m-2.$

Consequently, from (3.2), we have

$$|\tau_j^i| \le 1 \text{ and } |\partial_{x_k}\tau_j^i| \le C(m-1)/\delta,\ i = 1, \cdots, m,\ j = 0, \cdots, i,\ k = 1, \cdots, d. \tag{3.9}$$

For a $v \in V_h$, we can get (see Theorem 3.1.6, in [11], pag. 124, for instance) that

$$||\tau_j^i v - L_h(\tau_j^i v)||_{0,s} \le Ch|\tau_j^i v|_{1,s},\ ||\tau_j^i v - L_h(\tau_j^i v)||_{1,s} \le C|\tau_j^i v|_{1,s},$$

and therefore

$$||L_h(\tau_j^i v)||_{1,s} \le C||\tau_j^i v||_{1,s}, \text{ with } v \in V_h, \tag{3.10}$$

for any $i = 1, \cdots, m$, $j = 0, \cdots, i$. On the other hand, from (3.9) we get

$$||\tau_j^i v||_{0,s} \le ||v||_{0,s}, \quad |\tau_j^i v|_{1,s} \le C(|v|_{1,s} + \frac{m-1}{\delta}||v||_{0,s}), \text{ for any } v \in V_h, \tag{3.11}$$

and therefore, using (3.10), we get

$$||L_h(\tau_j^i v)||_{1,s} \le C(||v||_{1,s} + \frac{m-1}{\delta}||v||_{0,s}), \text{ for any } v \in V_h. \tag{3.12}$$

Now, by a application of (3.12) to (3.8) we get

$$||v_i||_{1,s} \le C(1 + \frac{m-1}{\delta})\left(||v-w||_{1,s} + \sum_{j=1}^{i}||w_j||_{1,s}\right), \quad \text{for any } i = 1, \cdots, m. \tag{3.13}$$

Using this equation we get (2.3) in Assumption 2.1, and we have

Proposition 3.1. *Let $\Omega_1, \cdots, \Omega_m$ be the overlapping decomposition of the domain Ω defined in this section. Then, Assumption 2.1 holds for the piecewise linear finite element spaces, $V = V_h$ and $V_i = V_h^i$, $i = 1, \cdots, m$, and for any convex set*

$K = K_h \subset V_h$ having Property 3.1. The constant in (2.3) of Assumption 2.1 can be taken of the form

$$C_0 = C(m+1)(1 + \frac{m-1}{\delta}), \tag{3.14}$$

where C is independent of the mesh parameter and the domain decomposition.

REMARK 3.1. *We notice that the number m of the subdomains Ω_i in the decomposition of Ω is in fact the number of the colors of the overlapping domain decomposition $\{O_i\}_{1\le i\le M}$, and it depends only on the dimension d of the space $\mathbf{R}^d$. Consequently, error estimations (2.16) and (2.17) in Theorem 2.1 depend only on the size δ of the overlaps through the intermediary of the constant C_0 given in (3.14).*

4. Two-level multiplicative Schwarz method. We consider a simplicial mesh partition $\mathcal{T}_h$ of the domain $\Omega \subset \mathbf{R}^d$ of a mesh size h, and a simplicial coarser mesh $\mathcal{T}_H$ with a mesh size H, $\mathcal{T}_h$ being a refinement of $\mathcal{T}_H$. The mesh size h is supposed to approach zero and we shall consider a family of mesh pairs (h, H). We assume that both the families, of fine and coarse meshes, are regular.

As in the previous section, we consider an overlapping decomposition $\Omega = \cup_{i=1}^M O_i$, the mesh partition $\mathcal{T}_h$ of Ω supplying a mesh partition for each O_i, $1 \le i \le M$. Also, we assume that the overlapping size is δ, ie. (3.1) is satisfied. In addition, we suppose that there exists a constant C such that

$$\operatorname{diam}(O_i) \le CH,\ i = 1, \cdots, M. \tag{4.1}$$

Now, we color the subdomains O_i, $i = 1, \cdots, M$, and obtain the subdomains Ω_i, $i = 1, \cdots, m$ as in the previous section. We point out that the domain Ω may be different from

$$\Omega_0 = \bigcup_{\tau \in \mathcal{T}_H} \tau, \tag{4.2}$$

but we assume that if a node of $\mathcal{T}_H$ lies on $\partial\Omega_0$ then it lies on $\partial\Omega$, too, and

$$\Omega \backslash \Omega_0 \subset \bigcup_{x^i \text{ node of } \mathcal{T}_H,\, x^i \in \partial\Omega} S_{x^i}, \tag{4.3}$$

where the sets S_{x^i} are defined as it follows. We first denote by ω_i the union of all $\tau \in \mathcal{T}_H$ having x^i as a vertex. Then, S_{x^i} is the union of ω_i with all $\tau \in \mathcal{T}_h$, $\tau \not\subset \Omega_0$, and which are contained in the smallest sphere centered at x^i and containing ω_i.

Now, we introduce the continuous, piecewise linear finite element space corresponding to the H-level,

$$V_H^0 = \left\{ v \in C^0(\bar{\Omega}_0) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_H,\ v = 0 \text{ on } \partial\Omega_0 \right\}, \tag{4.4}$$

and extending the functions of V_H^0 with zero in $\Omega\backslash\Omega_0$, it becomes a subspace of V_h. The convex set $K_h \subset V_h$ is defined as a subset of V_h having Property 3.1.

The two-level Schwarz method is also obtained from Algorithm 2.1 in which we take $V = V_h$, $K = K_h$ and the subspaces $V_0 = V_H^0$, $V_1 = V_h^1$, $V_2 = V_h^2$, $\cdots$, $V_m = V_h^m$. As in the previous section, the spaces V_h, V_H^0, V_h^1, V_h^2, $\cdots$, V_h^m, are considered as subspaces of $W^{1,s}$ for $1 \le s \le \infty$. We notice that this time, the decomposition of the domain Ω contains m overlapping subdomains, but we use $m+1$ subspaces of V, V_0, V_1, $\cdots$, V_m, in Algorithm 2.1. Naturally, this algorithm will converge if Assumption

2.1 written for $m+1$ subspaces, will be satisfied for the above choice of the convex set K and the subspaces V_0, V_1, $\cdots$, V_m, of V. As in the previous section, we prove that Assumption 2.1 holds and find the constant C_0 depending on the mesh and domain decomposition parameters. First, we have the following lemma in which inequality (4.5) can be viewed as one of Friedrichs-Poincaré type for the finite element spaces.

LEMMA 4.1. *Let $\omega \subset \mathbf{R}^d$ be a domain of diameter H, and ω_i, $i = 0, 1, \cdots, N$, be an overlapping decomposition of it, $\omega = \cup_{i=0}^{N} \omega_i$. We consider a simplicial regular mesh partition $\mathcal{T}_h$ of ω and assume that it supplies a mesh partition for each ω_i, $i = 0, 1, \cdots, N$, too. Let $x^0 \in \bar{\omega}_0$ be a node of $\mathcal{T}_h$. We assume that the overlapping partition of ω satisfies:*

(i) for any $x \in \bar{\omega}_0$, the line segment $[x^0, x]$ lies in $\bar{\omega}_0$,

(ii) for $N > 0$, if $\omega_i \cap \omega_j \neq \emptyset$, $0 \le i \neq j \le N$, then for any $x \in \bar{\omega}_i$, $y \in \bar{\omega}_j$ and $z \in \bar{\omega}_i \cap \bar{\omega}_j$, the line segments $[x, z]$ and $[y, z]$ lie in $\bar{\omega}_i$ and $\bar{\omega}_j$, respectively.
On these conditions, if v is a continuous function which is linear on each $\tau \in \mathcal{T}_h$, and $v(x^0) = 0$, then

$$||v||_{0,s,\omega} \le C(N, s) C(d, s) H C_{d,s}(H, h) |v|_{1,s,\omega}, \tag{4.5}$$

where

$$C_{d,s}(H,h) = \begin{cases} 1 & \text{if } d = s = 1 \text{ or } 1 \le d < s \le \infty \\ \left(\ln \frac{H}{h} + 1\right)^{\frac{d-1}{d}} & \text{if } 1 < d = s < \infty \\ \left(\frac{H}{h}\right)^{\frac{d-s}{s}} & \text{if } 1 \le s < d < \infty, \end{cases} \tag{4.6}$$

$$C(d,s) = \begin{cases} C & \text{if } d = s = 1 \text{ or } 1 = s < d < \infty \\ C\left(d\frac{s-1}{s-d}\right)^{\frac{s-1}{s}} & \text{if } 1 \le d < s \le \infty \\ C d^{\frac{d-1}{d}} & \text{if } 1 < d = s < \infty \\ C\left(d\frac{s-1}{d-s}\right)^{\frac{s-1}{s}} & \text{if } 1 < s < d < \infty. \end{cases} \tag{4.7}$$

and

$$C(N,s) = \begin{cases} 1 & \text{if } N = 0 \\ \text{if } (N+1)\frac{C_\omega^{(N+1)/s} - 1}{C_\omega^{1/s} - 1} & \text{if } N \neq 0 \end{cases} \tag{4.8}$$

with

$$C_\omega = \max_{\omega_i \cap \omega_j \neq \emptyset} \frac{|\omega_i|}{|\omega_i \cap \omega_j|} \tag{4.9}$$

In (4.9) we have denoted by $|\quad|$ the measure of a set, and we have marked in (4.5) that the norm in L^s and the semi-norm in $W^{1,s}$, $1 \le s \le \infty$, refer to the domain ω. The constant C in (4.7) is independent of H, h, d, s and the decomposition of ω.

Proof. In this proof, we use the polar coordinates. The Jacobian determinant of the transformation from the rectangular coordinates to the polar coordinates can be written as

$$J(r, \varphi) = r^{d-1} E(\varphi),$$

where $E(\varphi)$ is an algebraic expression of cosines and sines of the component angles of φ.

We first consider that $N=0$, ie. the decomposition of ω in the statement of the lemma has only one element, $\omega_0=\omega$. Consequently, for any $x\in\bar{\omega}$, the line segment $[x^0,x]$ lies in $\bar{\omega}$. We take the origin of the system of coordinates at the point x^0, and, using the polar coordinates, a point $x=(x_1,\cdots,x_d)$, will be written as $x=(r,\varphi)$, φ being the system of $d-1$ angles giving the direction of the vector x. We denote by r_φ the maximum size of the radius in the direction φ of the points in $\bar{\omega}$, and consequently, the points on $\partial\omega$ will be written as (r_φ,φ). We denote by o the union of the $\tau\in\mathcal{T}_h$ having a vertex at x^0, let r_0 be the distance from x^0 to $\partial o\backslash\partial\omega$. We consider the open ball with the center at x^0 of radius r_0, $B_{r_0}(x^0)$. For two points $x'=(r',\varphi)\in\omega\cap B_{r_0}(x^0)$ and $x=(r,\varphi)\in\omega\backslash\bar{B}_{r_0}(x^0)$, we have

$$\begin{aligned}
&|v(x)|=|v(r,\varphi)|\le|v(r',\varphi)|+|\textstyle\int_{r'}^{r}\frac{\partial v}{\partial r}(\rho,\varphi)d\rho|=\\
&|\tfrac{\partial v}{\partial r}(r',\varphi)|r'+|\textstyle\int_{r'}^{r}\frac{\partial v}{\partial r}(\rho,\varphi)d\rho|\le|\nu_1\frac{\partial v}{\partial x_1}(r',\varphi)+\cdots+\nu_d\frac{\partial v}{\partial x_d}(r',\varphi)|r'+\\
&|\textstyle\int_{r'}^{r_\varphi}\left(\nu_1\frac{\partial v}{\partial x_1}(\rho,\varphi)+\cdots+\nu_d\frac{\partial v}{\partial x_d}(\rho,\varphi)\right)d\rho|\le\\
&(|\tfrac{\partial v}{\partial x_1}(r',\varphi)|+\cdots+|\tfrac{\partial v}{\partial x_d}(r',\varphi)|)r'+\\
&\textstyle\int_{r'}^{r_\varphi}\left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|+\cdots+|\frac{\partial v}{\partial x_d}(\rho,\varphi)|\right)d\rho,
\end{aligned}\tag{4.10}$$

where $(\nu_1,\cdots,\nu_d)$ is the unity vector giving the direction of $x=(r,\varphi)$ in the rectangular system of coordinates $(x_1,\cdots,x_d)$. In the following, we find (4.5) for the various values of d and s starting from (4.10).
For $d=s=1$ or $1\le d<s\le\infty$, we take $r'=0$ in (4.10). If $d=s=1$ we get

$$|v(x)|=|v(r,\varphi)|\le\int_0^{r_\varphi}|\frac{\partial v}{\partial x_1}(\rho,\varphi)|d\rho.$$

Here, we may have $\varphi=0$ and $\varphi=\pi$ if x^0 is a inner point in ω, and only $\varphi=0$ or only $\varphi=\pi$ if $x^0\in\partial\omega$. Integrating again from 0 to $r_\varphi\le H$, we get (4.5) for $N=0$ and $d=s=1$. If $1\le d<s=\infty$, we have

$$|v(x)|\le r_\varphi d\max_{1\le j\le d}\sup_{0\le\rho\le r_\varphi}|\frac{\partial v}{\partial x_j}(\rho,\varphi)|\le CdH|v|_{1,\infty,\omega}.$$

If $1\le d<s<\infty$ we have

$$|v(x)|^s\le d^{s-1}\left[\int_0^{r_\varphi}\rho^{\frac{1-d}{s-1}}d\rho\right]^{s-1}\int_0^{r_\varphi}\left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s+\cdots+|\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right)\rho^{d-1}d\rho.$$

Multiplying the above inequality by r^{d-1} and integrating from 0 to $r_\varphi\le H$ we get

$$\begin{aligned}
&\textstyle\int_0^{r_\varphi}|v(r,\varphi)|^s r^{d-1}dr\le\\
&\left(d\tfrac{s-1}{s-d}\right)^{s-1}(CH)^s\textstyle\int_0^{r_\varphi}\left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s+\cdots+|\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right)\rho^{d-1}d\rho.
\end{aligned}$$

By multiplication of this equation with the Jacobian part depending on φ, $E(\varphi)$, and integrating over the $d-1$ dimensional domain of the angles φ, we get (4.5) for $N=0$ and $1\le d<s<\infty$.
Now, from (4.10) for an arbitrary $0<r'<r_0$, we get

$$\begin{aligned}
&|v(x)|=|v(r,\varphi)|\le(|\tfrac{\partial v}{\partial x_1}(r',\varphi)|+\cdots+|\tfrac{\partial v}{\partial x_d}(r',\varphi)|)r_0+\\
&\textstyle\int_{r_0}^{r_\varphi}\left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|+\cdots+|\frac{\partial v}{\partial x_d}(\rho,\varphi)|\right)d\rho.
\end{aligned}\tag{4.11}$$

Also, since for a fixed φ, $\frac{\partial v}{\partial r}(r',\varphi)$ is constant for $r'\in(0,r_0)$, we have

$$|v(x')|^s = |v(r',\varphi)|^s \le \frac{(r')^{s-d}}{d}\int_0^{r_0}|\frac{\partial v}{\partial\rho}(\rho,\varphi)|^s\rho^{d-1}d\rho \le \\ d^{s-2}(r')^{s-d}\int_0^{r_0}\left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s+\cdots+|\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right)\rho^{d-1}d\rho.$$

Multiplying the above inequality by $(r')^{d-1}$, and integrating from 0 to r_0, we get

$$\int_0^{r_0}|v(\rho,\varphi)|^s\rho^{d-1}d\rho \le \\ \frac{d^{s-2}}{s}r_0^s\int_0^{r_0}\left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s+\cdots+|\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right)\rho^{d-1}d\rho. \tag{4.12}$$

Now, if $1=s<d<\infty$ we get from (4.11),

$$|v(x)| \le \frac{1}{d}r_0^{1-d}\int_0^{r_0}(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|+\cdots+|\frac{\partial v}{\partial x_d}(\rho,\varphi)|)\rho^{d-1}d\rho+ \\ r_0^{1-d}\int_{r_0}^{r_\varphi}\left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|+\cdots+|\frac{\partial v}{\partial x_d}(\rho,\varphi)|\right)\rho^{d-1}d\rho \le \\ r_0^{1-d}\int_0^{r_\varphi}\left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|+\cdots+|\frac{\partial v}{\partial x_d}(\rho,\varphi)|\right)\rho^{d-1}d\rho.$$

Using the regularity of the mesh $\mathcal{T}_h$, we have $\frac{r_\varphi}{r}\le C\frac{H}{h}$, and therefore,

$$\int_{r_0}^{r_\varphi}|v(\rho,\varphi)|\rho^{d-1}d\rho \le CH\left(\frac{H}{h}\right)^{d-1}\int_0^{r_\varphi}\left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|+\cdots+|\frac{\partial v}{\partial x_d}(\rho,\varphi)|\right)\rho^{d-1}d\rho.$$

From this last inequality and (4.12) we get (4.5) for $N=0$ and $1=s<d<\infty$ by a multiplication with $E(\varphi)$ and integrating over the domain of the angles φ.
Starting again from (4.11), for $1<d=s<\infty$ or $1<s<d<\infty$, we get

$$|v(x)|^s \le (2d)^{s-1}(|\frac{\partial v}{\partial x_1}(r',\varphi)|^s+\cdots+|\frac{\partial v}{\partial x_d}(r',\varphi)|^s)r_0^s+ \\ (2d)^{s-1}\left[\int_{r_0}^{r_\varphi}\rho^{\frac{1-d}{s-1}}d\rho\right]^{s-1}\int_{r_0}^{r_\varphi}\left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s+\cdots+|\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right)\rho^{d-1}d\rho = \\ 2^{s-1}d^s r_0^{s-d}\int_0^{r_0}\left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s+\cdots+|\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right)\rho^{d-1}d\rho+ \\ (2d)^{s-1}\left[\int_{r_0}^{r_\varphi}\rho^{\frac{1-d}{s-1}}d\rho\right]^{s-1}\int_{r_0}^{r_\varphi}\left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s+\cdots+|\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right)\rho^{d-1}d\rho.$$

Consequently,

$$\int_{r_0}^{r_\varphi}|v(\rho,\varphi)|^s\rho^{d-1}d\rho \le \\ (2d)^{s-1}r_\varphi^d r_0^{s-d}\int_0^{r_0}\left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s+\cdots+|\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right)\rho^{d-1}d\rho+ \\ 2^{s-1}d^{s-2}r_\varphi^d\left[\int_{r_0}^{r_\varphi}\rho^{\frac{1-d}{s-1}}d\rho\right]^{s-1}\cdot \\ \int_{r_0}^{r_\varphi}\left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s+\cdots+|\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right)\rho^{d-1}d\rho. \tag{4.13}$$

Now, from (4.13), if $1<d=s<\infty$ we get

$$\int_{r_0}^{r_\varphi}|v(\rho,\varphi)|^d\rho^{d-1}d\rho \le \\ 2^{d-1}r_\varphi^d\max\left\{d^{d-1},d^{d-2}\left(\ln\frac{r_\varphi}{r_0}\right)^{d-1}\right\} \\ \int_0^{r_\varphi}\left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^d+\cdots+|\frac{\partial v}{\partial x_d}(\rho,\varphi)|^d\right)\rho^{d-1}d\rho.$$

Using regularity of the mesh $\mathcal{T}_h$, we get

$$\int_{r_0}^{r_\varphi}|v(\rho,\varphi)|^d\rho^{d-1}d\rho \le \\ d^{d-1}(CH)^d\left(\ln\frac{H}{h}+1\right)^{d-1}\int_0^{r_\varphi}\left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^d+\cdots+|\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right)\rho^{d-1}d\rho.$$

This inequality together with (4.12) prove (4.5) for $N = 0$ and $1 < d = s < \infty$. Finally, if $1 < s < d < \infty$, we get from (4.13),

$$\int_{r_0}^{r_\varphi} |v(\rho,\varphi)|^s \rho^{d-1} d\rho \le 2^{s-1} \max \left\{ d^{s-1} r_\varphi^d r_0^{s-d}, d^{s-2} r_\varphi^s (\tfrac{s-1}{d-s})^{s-1} \left[\left(\tfrac{r_\varphi}{r_0}\right)^{\frac{d-s}{s-1}} - 1 \right]^{s-1} \right\}$$
$$\int_0^{r_\varphi} \left(|\tfrac{\partial v}{\partial x_1}(\rho,\varphi)|^s + \cdots + |\tfrac{\partial v}{\partial x_d}(\rho,\varphi)|^s \right) \rho^{d-1} d\rho,$$

and consequently,

$$\int_{r_0}^{r_\varphi} |v(\rho,\varphi)|^s \rho^{d-1} d\rho \le$$
$$(d\tfrac{s-1}{d-s})^{s-1} (CH)^s \left(\tfrac{H}{h}\right)^{d-s} \int_0^{r_\varphi} \left(|\tfrac{\partial v}{\partial x_1}(\rho,\varphi)|^s + \cdots + |\tfrac{\partial v}{\partial x_d}(\rho,\varphi)|^s \right) \rho^{d-1} d\rho.$$

Using again (4.12) and the last inequality, we get (4.5) for $N = 0$ and $1 < s < d < \infty$.

Assume now that $N > 0$, ie. we have more than one subdomain ω_i, $i = 0, 1, \cdots, N$ in the overlapping decomposition of ω. Such a decomposition is considered when there exist points $x \in \bar{\omega}$ for which the line segment $[x^0, x]$ do not wholly lie in $\bar{\omega}$. Let ω_i and ω_j, $i \neq j$, be two fixed subdomains such that $\omega_i \cap \omega_j \neq \emptyset$. We consider a fixed point $z \in \omega_i \cap \omega_j$, and denoting by z^k and ϕ_k the nodes of $\mathcal{T}_h$ in $\bar{\omega}_i \cap \bar{\omega}_j$ and the corresponding functions in the nodal basis, respectively, for a given $1 \le s < \infty$, we have

$$||v||_{0,s,\omega_j} - |\sum_k v(z^k)\phi_k(z)||\omega_j|^{1/s} \le ||v - \sum_k v(z^k)\phi_k(z)||_{0,s,\omega_j} =$$
$$||\sum_k \left(v - v(z^k)\right) \phi_k(z)||_{0,s,\omega_j} \le \sum_k ||v - v(z^k)||_{0,s,\omega_j} \phi_k(z).$$

Since $v - v(z^k)$ vanishes at z^k, we get from the first part of the proof and the last equation that

$$||v||_{0,s,\omega_j} - |\sum_k v(z^k)\phi_k(z)||\omega_j|^{1/s} \le \sum_k C(d,s) H C_{d,s}(H,h) |v|_{1,s,\omega_j} \phi_k(z) =$$
$$C(d,s) H C_{d,s}(H,h) |v|_{1,s,\omega_j},$$

and integrating over $\omega_i \cap \omega_j$, we get

$$|\omega_i \cap \omega_j| ||v||_{0,s,\omega_j} \le |\omega_j|^{1/s} \int_{\omega_i \cap \omega_j} |v| + |\omega_i \cap \omega_j| C(d,s) H C_{d,s}(H,h) |v|_{1,s,\omega_j} \le$$
$$|\omega_j|^{1/s} |\omega_i \cap \omega_j|^{(s-1)/s} ||v||_{0,s,\omega_i \cap \omega_j} + |\omega_i \cap \omega_j| C(d,s) H C_{d,s}(H,h) |v|_{1,s,\omega_j}.$$

Consequently, we have

$$||v||_{0,s,\omega_j} \le \left(\frac{|\omega_j|}{|\omega_i \cap \omega_j|} \right)^{1/s} ||v||_{0,s,\omega_i} + C(d,s) H C_{d,s}(H,h) |v|_{1,s,\omega_j}. \tag{4.14}$$

It is easy to see that equation (4.14) holds for $s = \infty$, too. Taking into account that

$$||v||_{0,s,\omega_0} \le C(d,s) H C_{d,s}(H,h) |v|_{1,s,\omega_0}, \tag{4.15}$$

from (4.14) and (4.15), we get (4.5) for $N > 0$. □

Remark 4.1. *As we have already said at the beginning of this section, we are interested in the error estimation for a family of pairs (H, h). In general, since the* *mesh $\mathcal{T}_h$ is regular, the overlapping decomposition of ω in Lemma 4.1 can be taken such that the number N and the constant C_ω in (4.9) are bounded and independent of (H,h). In this point of view, the constants $C(d,s)$, $C(N,s)$ and C_ω, written in (4.7)–(4.9), can be considered as independent of H and h, and assimilated to the generic constant C. In the following we write (4.5) as*

$$||v||_{0,s,\omega} \le CHC_{d,s}(H,h)|v|_{1,s,\omega}, \tag{4.16}$$

where $C = C(N,s)C(d,s)$ and $C_{d,s}(H,h)$ is given in (4.6).

The above lemma can be very useful in various error estimations. The following result, for instance, extends to $W^{1,s}$ that in Lemma 2.3 in [9].

Corollary 4.2. *Let ω be a domain of diameter H and having a simplicial regular mesh partition $\mathcal{T}_h$. If v is a continuous function which is linear on each $\tau \in \mathcal{T}_h$, and $v = 0$ on $\partial\omega$, then for any $1 \le s \le \infty$ we have*

$$||v||_{0,\infty,\omega} \le CH^{\frac{s-d}{s}} C_{d,s}(H,h)|v|_{1,s,\omega}, \tag{4.17}$$

where $C_{d,s}(H,h)$ is given in (4.6), and C is independent of H and h.

Proof. Let $x^0 \in \bar{\omega}$ be the point where $|v(x^0)| = ||v||_{0,\infty,\omega}$, and $x \in \omega$ a current point. We note that x^0 is a node of $\mathcal{T}_h$. For $1 \le s < \infty$, we have

$$|v(x^0)|^s \le 2^{s-1}|v(x^0) - v(x)|^s + 2^{s-1}|v(x)|^s,$$

and integrating it over ω, using (4.16), we get

$$\begin{aligned}|\omega|||v||^s_{0,\infty,\omega} &\le 2^{s-1}||v(x^0) - v(x)||^s_{0,s,\omega} + 2^{s-1}||v(x)||^s_{0,s,\omega} \le \\ &2^{s-1}\left(CHC_{d,s}(H,h)\right)^s |v(x)|^s_{1,s,\omega} + 2^{s-1}||v(x)||^s_{0,s,\omega}.\end{aligned}$$

Now, since $v = 0$ on $\partial\omega$, we can apply the classical Friedrichs-Poincaré inequality and obtain (4.17). If $s = \infty$, the proof is similar. □

Coming back to the two-level method, let us denote by x^i a node of $\mathcal{T}_H$, by ϕ_i the linear nodal basis function associated with x^i and $\mathcal{T}_H$, and by ω_i the support of ϕ_i. We point out that we consider all the nodal basis functions, including those corresponding to the nodes on $\partial\Omega_0$. Given a $v \in V_h$, let us write

$$I_i^- v = \min_{x\in\omega_i} v(x)^- \text{ and } I_i^+ v = \min_{x\in\omega_i} v(x)^+, \tag{4.18}$$

where $v(x)^- = \max(0, -v(x))$ and $v(x)^+ = \max(0, v(x))$. Since v is piecewise linear, $I_i^- v$ or $I_i^+ v$ are attained at a node of $\mathcal{T}_h$ if they are not zero. For a $v \in V_h$, we define

$$I_H^- v := \sum_{x^i \text{ node of } \mathcal{T}_H} (I_i^- v)\phi_i(x) \text{ and } I_H^+ v := \sum_{x^i \text{ node of } \mathcal{T}_H} (I_i^+ v)\phi_i(x), \tag{4.19}$$

and we write

$$I_H v = I_H^+ v - I_H^- v. \tag{4.20}$$

The following result extends that given in [37], where similar operators to I_i^+ have been introduced.

Lemma 4.3. *For any $v \in V_h$ we have*

$$||I_H v - v||_{0,s,\Omega_0} \le CHC_{d,s}(H,h)|v|_{1,s,\Omega_0} \tag{4.21}$$

and

$$||I_H v||_{0,s,\Omega_0} \le C||v||_{0,s,\Omega_0} \text{ and } |I_H v|_{1,s,\Omega_0} \le CC_{d,s}(H,h)|v|_{1,s,\Omega_0} \tag{4.22}$$

where Ω_0 *is the union of the simplexes in* $\mathcal{T}_H$ *written in (4.2),* $C_{d,s}(H,h)$ *is defined in (4.6), and* C *is independent of* H*,* h *and* δ*. Equations (4.21) and (4.22) also hold if* Ω_0 *is replaced by* Ω*. Moreover, if* $\mathcal{K}$ *is a convex and closed set in* V_h *having Property 3.1, with* $0 \in \mathcal{K}$*, then for any* $v \in \mathcal{K}$ *we have* $I_H v \in \mathcal{K} \cap V_H^0$*.*

Proof. Let us take an ω_i, the support of the linear basis function ϕ_i corresponding to the node x^i of $\mathcal{T}_H$, and a $v \in V_h$. If v vanishes at a point in ω_i, then $I_i^+ v = I_i^- v = 0$ and v^+ and v^- vanish at some nodes of $\mathcal{T}_h$ in ω_i. Applying Lemma 4.1, we get

$$\begin{aligned} &||v||^s_{0,s,\omega_i} = ||v^+ - v^-||^s_{0,s,\omega_i} = ||v^+||^s_{0,s,\omega_i} + ||v^-||^s_{0,s,\omega_i} \le \\ &[CHC_{d,s}(H,h)]^s \left[|v^+|^s_{1,s,\omega_i} + |v^-|^s_{1,s,\omega_i}\right] = [CHC_{d,s}(H,h)]^s |v|^s_{1,s,\omega_i}. \end{aligned} \tag{4.23}$$

Consequently,

$$||v - I_i^+ v + I_i^- v||_{0,s,\omega_i} \le CHC_{d,s}(H,h)|v|_{1,s,\omega_i}. \tag{4.24}$$

If $v \neq 0$ at any point of ω_i then either $v^+ = I_i^+ v = 0$ or $v^- = I_i^- v = 0$. Consequently, there exits at least a node of $\mathcal{T}_h$ in ω_i at which $v - I_i^+ v + I_i^- v = v^+ - v^- - I_i^+ v + I_i^- v$ vanishes. From Lemma 4.1, since $I_i^+ v - I_i^- v$ is a constant, we get again (4.24). We notice that, since for any $x \in \omega_i$ the line segment $[x^i, x]$ lies in ω_i, we can take a decomposition of ω_i as in Lemma 4.1 having $N \le 1$. Assuming that $N = 1$, let ω_{i0} and $\omega_{i1} = \omega$ be this decomposition. Since ω_{i0} contains at least one $\tau \in \mathcal{T}_H$ and the mesh $\mathcal{T}_H$ is regular, then, according to (4.9), C_{ω_i} can be taken independent of H and h. Consequently, $C(N,s)$ in (4.8) is independent of H and h. Now, using (4.24), we get

$$\begin{aligned} &||I_H v - v||^s_{0,s,\omega_i} = ||\sum_{x^j \in \omega_i} \left[I_j^+ v - I_j^- v - v\right] \phi_j||^s_{0,s,\omega_i} \le \\ &C \sum_{x^j \in \omega_i} ||I_j^+ v - I_j^- v - v||^s_{0,s,\omega_i \cap \omega_j} \le [CHC_{d,s}(H,h)]^s \sum_{x^j \in \omega_i} |v|^s_{1,s,\omega_j}. \end{aligned}$$

Above, x^j are nodes of $\mathcal{T}_H$, and we used the fact that, since the mesh is regular, the maximum number of ω_j which non-emptily intersect a given ω_i is bounded and independent of H. Now, we use again this property to obtain

$$\begin{aligned} &||I_H v - v||^s_{0,s,\Omega_0} \le \sum_{x^i \in \Omega_0} ||I_H v - v||^s_{0,s,\omega_i} \le \\ &[CHC_{d,s}(H,h)]^s \sum_{x^i \in \Omega_0} \sum_{x^j \in \omega_i} |v|^s_{1,s,\omega_j} \le [CHC_{d,s}(H,h)]^s \sum_{x^i \in \Omega_0} |v|^s_{1,s,\omega_i}. \end{aligned}$$

Also, from the regularity of the mesh, it follows that each ω_i contains a bounded number of simplexes of $\mathcal{T}_H$ which is independent of H. Consequently, we have

$$||I_H v - v||^s_{0,s,\Omega_0} \le [CHC_{d,s}(H,h)]^s \sum_{\tau \in \mathcal{T}_H} |v|^s_{1,s,\tau},$$

and in this way, we get (4.21).

In order to prove (4.22), we notice first that, from the definition of $I_i^+ v$ and $I_i^- v$, we have for any $x \in \omega_i$,

$$\begin{aligned} &0 \le I_i^+ v - I_i^- v \le v(x) \text{ if } v(x) \ge 0, \text{ and} \\ &0 \ge I_i^+ v - I_i^- v \ge v(x) \text{ if } v(x) \le 0, \end{aligned} \tag{4.25}$$

and therefore,

$$|I_i^+ v - I_i^- v| \le |v(x)| \text{ for any } x \in \omega_i. \tag{4.26}$$

Using this inequality, we obtain

$$\begin{aligned} &||I_H v||_{0,s,\omega_i}^s = || \sum_{x^j \in \omega_i} (I_j^+ v - I_j^- v)\phi_j||_{0,s,\omega_i}^s \le \\ &\int_{\omega_i} (\sum_{x^j \in \omega_i} |I_j^+ - I_j^-|\phi_j)^s = \int_{\omega_i} (\sum_{x^j \in \omega_i} |v(x)|\phi_j)^s = \int_{\omega_i} |v(x)|^s = ||v||_{0,s,\omega_i}^s. \end{aligned}$$

Taking again into account the regularity of the mesh, we get

$$||I_H v||_{0,s,\Omega_0}^s \le \sum_{x^i \in \Omega_0} ||I_H v||_{0,s,\omega_i}^s \le \sum_{x^i \in \Omega_0} ||v||_{0,s,\omega_i}^s \le C \sum_{\tau \in \mathcal{T}_H} ||v||_{0,s,\tau}^s,$$

and therefore, the first equation in (4.22) holds. To prove the second equation in (4.22), first we write

$$\begin{aligned} &|I_H v|_{1,s,\omega_i}^s = | \sum_{x^j \in \omega_i} (I_j^+ v - I_j^- v)\phi_j|_{1,s,\omega_i}^s \le \\ &CH^{d-s} \max_{x^k, x^l \in \omega_i, \omega_k \cap \omega_l \ne \emptyset} |(I_k^+ v - I_k^- v) - (I_l^+ v - I_l^- v)|^s. \end{aligned}$$

Since $\omega_k \cap \omega_l \ne \emptyset$, taking into account the definition of $I_i^+ v$ and $I_i^- v$ in (4.18), we get that $I_k^+ v - I_k^- v$ and $I_l^+ v - I_l^- v$ can not be both different from zero and they have different signs. Therefore, if we write

$$|I_p^+ v - I_p^- v| - |I_q^+ v - I_q^- v| = \max_{x^k, x^l \in \omega_i, \omega_k \cap \omega_l \ne \emptyset} |(I_k^+ v - I_k^- v) - (I_l^+ v - I_l^- v)|,$$

using (4.26), we get

$$|I_H v|_{1,s,\omega_i}^s \le CH^{d-s}(|I_p^+ v - I_p^- v| - |I_q^+ v - I_q^- v|)^s \le CH^{d-s}(|v(x)| - |I_q^+ v - I_q^- v|)^s$$

for any $x \in \omega_p \cap \omega_q$. Since the mesh $\mathcal{T}_h$ is regular, we have $H^d \le C|\omega_p \cap \omega_q|$, and integrating the above equation over $\omega_p \cap \omega_q$ we get

$$\begin{aligned} &|I_H v|_{1,s,\omega_i}^s \le CH^{-s} \textstyle\int_{\omega_p \cap \omega_q} (|v(x)| - |I_q^+ v - I_q^- v|)^s = \\ &CH^{-s} \textstyle\int_{\omega_p \cap \omega_q} |v(x) - (I_q^+ v - I_q^- v)|^s \le CH^{-s} \int_{\omega_q} |v(x) - (I_q^+ v - I_q^- v)|^s. \end{aligned}$$

If there exists a point in ω_q at which v vanishes, then $I_q^+ v = I_q^- v = 0$, and, as in (4.23), we get

$$|I_H v|_{1,s,\omega_i} \le CC_{d,s}(H,h)|v|_{1,s,\omega_q}.$$

Also, if $v > 0$ or $v < 0$ in ω_q, then there exists $x^q \in \omega_q$, node of $\mathcal{T}_h$, such that $v(x^q) = I_q^+ v - I_q^- v$, and we get again the above inequality applying Lemma 4.1. Finally, using again the fact that the mesh $\mathcal{T}_H$ is regular, we get the second equation in (4.22).

To prove that (4.21) and (4.22) hold on Ω, we see that $I_H v = 0$ on all the sets S_{x^i} introduced in (4.3). Therefore, (4.22) holds on all sets S_{x^i}. Also, since $v(x^i) = 0$,

from Lemma 4.1, we get that (4.21) holds on S_{x^i}. Consequently, the above reasoning we made for Ω_0 can be done for Ω, too.
From (4.25), (4.19) and (4.20), we get that for any $x \in \Omega$, we have

$$0 \le I_H v(x) \le v(x) \text{ if } v(x) \ge 0, \text{ and } 0 \ge I_H v(x) \ge v(x) \text{ if } v(x) \le 0 \tag{4.27}$$

Therefore, we can find a $\theta(x) \in C^1(\Omega)$, $0 \le \theta(x) \le 1$, such that $\theta(x^i) = I_H v(x^i)/v(x^i)$ if $I_H v(x^i) \ne 0$, and $\theta(x^i) = 0$ if $I_H v(x^i) = 0$, at any node x^i of $\mathcal{T}_h$. Consequently, we can write $I_H v = L_h(\theta v + (1-\theta)0)$. Finally, if $0, v \in \mathcal{K}$, and $\mathcal{K}$ has Property 3.1, we get that $I_H v \in \mathcal{K}$. □

Now, we can prove the following proposition which shows that the constant C_0 in Assumption 2.1 is independent of the mesh and domain decomposition parameters if H/δ and H/h are constant. This result is similar to that given in [4] for the inequalities coming from minimization of the quadratic functionals. In the first part of the proof, the construction of v_i, $i = 1, \cdots, m$, is similar to that given for the one-level method. In the second part we define an appropriate v_0 using the previous lemma.

PROPOSITION 4.4. *Let $\Omega_1, \cdots, \Omega_m$ be the overlapping decomposition of the domain Ω defined in this section. Then Assumption 2.1 is verified for the piecewise linear finite element spaces, $V = V_h$ and $V_0 = V_H^0$, $V_i = V_h^i$, $i = 1, \cdots, m$, defined in (3.3), (3.4) and (4.4), respectively, and any convex set $K = K_h$ satisfying Property 3.1. The constant in (2.3) of Assumption 2.1 can be taken of the form*

$$C_0 = C(m+2)^{1-\frac{1}{p}} \left(1 + (m-1)\frac{H}{\delta}\right) C_{d,s}(H,h), \tag{4.28}$$

where C is independent of the mesh and domain decomposition parameters, and $C_{d,s}(H,h)$ is given in (4.6).

Proof. Let us consider $w \in K_h$, $w_0 \in V_H^0$ and $w_i \in V_h^i$ such that $w + \sum_{j=0}^{i} w_i \in K_h$, $i = 0, \cdots, m$, and let v be another element in K_h. In the following, we use unity partitions $(\theta_j^i)_{j=i,\cdots,m}$, of the domains $\cup_{j=i,m}^{m} \Omega_j$, $i = 1, \cdots, m$, having property (3.2).

Step 1. We assume that we have got a $v_0 \in V_H^0$ satisfying

$$w + v_0,\ v + w_0 - v_0 \in K_h, \tag{4.29}$$

and we recursively construct $v_i \in V_h^i$, $i = 1, \cdots, m$, which satisfy (2.1) and (2.2) in Assumption 2.1. To this end, we define

$$v_1 = L_h\left(\theta_1^1(v - w - v_0) + (1 - \theta_1^1) w_1\right), \tag{4.30}$$

and, as in the previous section, we get

$$\begin{aligned}
& v_1 \in V_h^1 \text{ and } w + w_0 + v_1 \in K_h, \\
& v - v_0 - v_1 + w_0 + w_1 \in K_h, \\
& v - w - v_0 - v_1 \in W_0^{1,s}(\bigcup_{j=2}^{m} \Omega_j) \text{ and} \\
& v - w - v_0 - v_1 = 0 \text{ in } \Omega - \overline{\cup_{j=2}^{m} \Omega_j}.
\end{aligned}$$

Also, for $i = 2, \cdots, m-1$ we write

$$v_i = L_h\left(\theta_i^i(v - w - \sum_{j=0}^{i-1} v_j) + (1 - \theta_i^i) w_i\right), \tag{4.31}$$

and we prove

$$v_i \in V_h^i \text{ and } w + \sum_{j=0}^{i-1} w_j + v_i \in K_h,$$
$$v - \sum_{j=0}^{i} v_j + \sum_{j=0}^{i} w_j \in K_h,$$
$$v - w - \sum_{j=0}^{i} v_j \in W_0^{1,s}(\bigcup_{j=i+1}^{m} \Omega_j) \text{ and}$$
$$v - w - \sum_{j=0}^{i} v_j = 0 \text{ in } \Omega - \overline{\cup_{j=i+1}^{m} \Omega_j},$$

assuming that these equations hold for $i-1$. Finally, we take

$$v_m = v - w - \sum_{j=0}^{m-1} v_j \tag{4.32}$$

and we get that (2.1) and (2.2) in Assumption 2.1 hold.

Step 2. We define in this step a $v_0 \in V_H^0$ satisfying (4.29) and prove that condition (2.3) in Assumption 2.1 is satisfied with the constant C_0 given in (4.28). It is easy to see that (4.29) is equivalent with

$$v_0 - w_0 \in (K_h - (w + w_0)) \cap (v - K_h)\,, \tag{4.33}$$

and also, since $v, w + w_0 \in K_h$, we get

$$v - w - w_0 \in (K_h - (w + w_0)) \cap (v - K_h)\,. \tag{4.34}$$

We write $\mathcal{K} = (K_h - (w + w_0)) \cap (v - K_h)$, and from the above equation and Lemma 4.3, we get that $I_H(v - w - w_0) \in \mathcal{K}$. From (4.21) and (4.22) we have

$$||v - w - w_0 - I_H(v - w - w_0)||_{0,s} \le CHC_{d,s}(H,h)|v - w - w_0|_{1,s} \tag{4.35}$$

and

$$\begin{aligned} ||I_H(v - w - w_0)||_{0,s} &\le CC_{d,s}(H,h)||v - w - w_0||_{0,s} \\ |I_H(v - w - w_0)|_{1,s} &\le CC_{d,s}(H,h)|v - w - w_0|_{1,s}, \end{aligned} \tag{4.36}$$

where $C_{d,s}(H,h)$ is defined in (4.6). Now, we take

$$v_0 = w_0 + I_H(v - w - w_0), \tag{4.37}$$

and, from 4.34, the second part of Lemma 4.3, and (4.33), we get that it satisfies condition (4.29). To prove condition (2.3) in Assumption 2.1, we first notice that, starting from v_1 given in (4.30), by the recurrent application of (4.31), as in the proof of Proposition 3.1, we get v_i, $i = 1, \cdots, m$, of the form

$$v_i = L_h(\tau_0^i(v - w - v_0) + \sum_{j=1}^{i} \tau_j^i w_j),\ i = 1, \cdots, m, \tag{4.38}$$

where τ_j^i, $i=1,\cdots,m$, $j=0,\cdots,i$, satisfy (3.9). Using (3.10) and (3.11), we get

$$||L_h(\tau_j^i w_j)||_{1,s} \le C||\tau_j^i w_j||_{1,s} \le C(||w_j||_{1,s} + \frac{m-1}{\delta}||w_j||_{0,s}).$$

It follows from (4.1) that the diameters of the connected component of Ω_i are less than CH, and since $w_i \in V_h^i$, using the classical Friedrichs-Poincaré inequality, we get

$$||L_h(\tau_j^i w_j)||_{1,s} \le C[1+(m-1)\frac{H}{\delta}]|w_j|_{1,s},\ i=1,\cdots,m,\ j=1,\cdots,i. \tag{4.39}$$

On the other hand, taking into account (3.10), (3.11), (4.37) and (4.35), we get

$$\begin{array}{l}||L_h(\tau_0^i(v-w-v_0))||_{1,s} \le C[|v-w-v_0|_{1,s} + (1+\frac{m-1}{\delta})||v-w-v_0||_{0,s}] = \\ C[|v-w-v_0|_{1,s} + (1+\frac{m-1}{\delta})||v-w-w_0-I_H(v-w-w_0)||_{0,s}] \le \\ C\left[|v-w-v_0|_{1,s} + (m-1)C_{d,s}(H,h)\frac{H}{\delta}|v-w-w_0|_{1,s}\right] \le \\ C(|v-w|_{1,s}+|v_0|_{1,s}) + C(m-1)C_{d,s}(H,h)\frac{H}{\delta}(|v-w|_{1,s}+|w_0|_{1,s}).\end{array}$$

Consequently, we have

$$\begin{array}{l}||L_h(\tau_0^i(v-w-v_0))||_{1,s} \le C\left[1+(m-1)\frac{H}{\delta}\right]C_{d,s}(H,h)\cdot \\ (|v-w|_{1,s}+|w_0|_{1,s}) + C|v_0|_{1,s},\ i=1,\cdots,m.\end{array} \tag{4.40}$$

Also, from (4.37) and (4.36), we get

$$\begin{array}{l}|v_0|_{1,s} = |w_0+I_H(v-w-w_0)|_{1,s} \le |w_0|_{1,s} + |I_H(v-w-w_0)|_{1,s} \le \\ |w_0|_{1,s} + CC_{d,s}(H,h)|v-w-w_0|_{1,s},\end{array}$$

and therefore,

$$|v_0|_{1,s} \le CC_{d,s}(H,h)(|v-w|_{1,s}+|w_0|_{1,s}). \tag{4.41}$$

Now, from (4.40) and (4.41), we get

$$\begin{array}{l}||L_h(\tau_0^i(v-w-v_0))||_{1,s} \le \\ C\left[1+(m-1)\frac{H}{\delta}\right]C_{d,s}(H,h)(|v-w|_{1,s}+|w_0|_{1,s}),\ i=1,\cdots,m.\end{array} \tag{4.42}$$

Finally, from (4.38), (4.39), (4.41), and (4.42) we obtain that condition (2.3) in Assumption 2.1 holds with C_0 given in (4.28). ◻

REMARK 4.2. *As in Remark 3.1, we notice that, since the number m of the subdomains Ω_i is the number of colors of the overlapping domain decomposition $\{O_i\}_{1\le i\le M}$, the error estimates in Theorem 2.1 depends only on C_0 given in (4.28). Therefore, if the overlapping size δ and the mesh sizes H and h are chosen such that H/h and H/δ are constant, then the convergence rate of the two-level multiplicative Schwarz method is independent of the mesh and domain decomposition parameters.*

5. Multi-level multiplicative Schwarz method. We consider over the domain $\Omega \subset \mathbf{R}^d$ a family of regular meshes $\mathcal{T}_{h_j}$ of mesh sizes h_j, $j=1,\cdots,L$, such that $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$, $j=1,\cdots,L-1$. We write

$$\Omega_j = \bigcup_{\tau\in\mathcal{T}_{h_j}} \tau \tag{5.1}$$

and we assume that $\Omega=\Omega_L$. As in the previous section, we assume that, if a node of $\mathcal{T}_{h_j}$ lies on $\partial\Omega_j$ then it lies on $\partial\Omega_{j+1}$, too, that is, it lies on $\partial\Omega$. Also, for the nodes

$x^j \in \partial\Omega$ of $\mathcal{T}_{h_j}$, $j = 1, \cdots, L-1$, we consider the union of the all $\tau \in \mathcal{T}_{h_j}$ having x^j as a vertex, ω_j, and we define the set S_{x^j} as the union of ω_j with all $\tau \in \mathcal{T}_{h_{j+1}}$, $\tau \not\subset \Omega_j$, which are contained in the smallest sphere which is centered at x^j and contains ω_j. We assume that

$$\Omega_{j+1} \backslash \Omega_j \subset \bigcup_{x^j \text{ node of } \mathcal{T}_{h_j},\, x^j \in \partial\Omega} S_{x^j} \text{ for } j = 1, \cdots, L-1. \tag{5.2}$$

Since the mesh $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$, we have $h_{j+1} \leq h_j$, and we assume that there exists a constant γ, independent of the number of meshes, L, such that

$$1 < \gamma \leq \frac{h_j}{h_{j+1}}, \quad j = 1, \cdots, L-1. \tag{5.3}$$

At each level $j = 1, \cdots, L$, we consider an overlapping decomposition $\{O_j^i\}_{1 \leq i \leq M_j}$ of Ω_j, and we assume that the mesh partition $\mathcal{T}_{h_j}$ of Ω_j supplies a mesh partition for each O_j^i, $1 \leq i \leq M_j$. Also, we assume that the overlapping size for the domain decomposition at the level $1 \leq j \leq L$ is δ_j, i.e.,

$$O_j^i \cap \partial(\bigcup_{l \neq i} O_j^l) \neq \emptyset \text{ and } \mathrm{dist}(\partial O_j^i \backslash \partial\Omega_j, O_j^i \cap \partial(\bigcup_{l \neq i} O_j^l) \geq \delta_j \tag{5.4}$$

is satisfied. In addition, we suppose that there exists a constant C such that

$$\mathrm{diam}(O_{j+1}^i) \leq C h_j,\ j = 1, \cdots, L-1,\ i = 1, \cdots, M_j. \tag{5.5}$$

Now, at each level $j = 1, \cdots, L$, we color the subdomains O_j^i, $i = 1, \cdots, M_j$, and obtain the overlapping subdomains Ω_j^i, $i = 1, \cdots, m_j$, as in the previous section. Finally, we assume that $m_1 = 1$, and let us write

$$m = \max_{j=1,\cdots,L} m_j. \tag{5.6}$$

At each level $j = 1, \cdots, L$, we introduce the linear finite element spaces,

$$V_{h_j} = \{v \in C^0(\bar{\Omega}_j) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_{h_j},\ v = 0 \text{ on } \partial\Omega_j\}, \tag{5.7}$$

and, for $i = 1, \cdots, m_j$, we write

$$V_{h_j}^i = \{v \in V_{h_j} : v = 0 \text{ in } \Omega_j \backslash \Omega_j^i\} \tag{5.8}$$

The convex set will be a subset K_{h_L} of V_{h_L} having Property 3.1.

In order to prove that Assumption 2.1 holds for the convex set $K = K_{h_L}$ and the spaces $V = V_{h_L}$, $V_j^i = V_{h_j}^i$, $j = 1, \cdots, L$, $i = 1, \cdots, m_j$, and to find the constant C_0 in (2.3) as a function of the domain decomposition and mesh parameters, we need the following lemma. This result generalizes to more than two levels the second inequality (4.22) in Lemma 4.3. To this end, we introduce operators $I_{h_k} : V_{h_{k+1}} \to V_{h_k}$, $k = 1, \cdots, L-1$, which are similar to the operator $I_H : V_h \to V_H$ defined in (4.20).

LEMMA 5.1. *For a given $1 \leq j < L-1$, let $v_k,\ w_k \in V_{h_k}$, $k = j+1, \cdots, L-1$, such that*

$$v_k = w_k + I_{h_k}(v_{k+1}). \tag{5.9}$$

Then,

$$
\begin{aligned}
(5.10)\qquad & |I_{h_j} v_{j+1}|_{1,s,\Omega_j} \le C(L-j)^{\frac{s-1}{s}} \{ \sum_{k=j+1}^{L-1} C_{d,s}(h_j,h_k)^s |w_k|^s_{1,s,\Omega_j} + \\
& C_{d,s}(h_j,h_L)^s |v_L|^s_{1,s,\Omega_j} \}^{\frac{1}{s}}.
\end{aligned}
$$

Moreover, (5.10) also holds if its seminorms over Ω_j are replaced with seminorms over Ω_k, for any $k = j+1, \cdots, L$.

Proof. Let ω_j be the support of the nodal basis function in V_{h_j} corresponding to the node x^j of $\mathcal{T}_{h_j}$. Then there exists two nodes of $\mathcal{T}_{h_j}$, $x_1^j, x_2^j \in \omega_j$, such that

$$
(5.11)\qquad |I_{h_j} v_{j+1}|^s_{1,s,\omega_j} \le C h_j^{d-s} |(I_{h_j} v_{j+1})(x_1^j) - (I_{h_j} v_{j+1})(x_2^j)|^s
$$

Starting from (5.11), we prove that, for each $k = j, \cdots, L-1$, there exist two nodes of $\mathcal{T}_{h_{k+1}}$, $x_1^{k+1} \in \omega_k^1$ and $x_2^{k+1} \in \omega_k^2$, ω_k^1 and ω_k^2 being the supports of the nodal basis function in V_{h_k} corresponding to the nodes x_1^k and x_2^k of $\mathcal{T}_{h_k}$, respectively, such that

$$
\begin{aligned}
(5.12)\qquad & |I_{h_j} v_{j+1}|^s_{1,s,\omega_j} \le \\
& C h_j^{d-s} \left[\sum_{k=j+1}^{L-1} |w_k(x_1^k) - w_k(x_2^k)| + |v_L(x_1^L) - v_L(x_2^L)| \right]^s.
\end{aligned}
$$

First, we assume that, starting with x_1^j and x_2^j in (5.11), at each level $k = j, \cdots, L-1$, the values of $(I_{h_k} v_{k+1})(x_1^k)$ and $(I_{h_k} v_{k+1})(x_2^k)$ are obtained as values of v_{k+1} at two nodes $x_1^{k+1} \in \omega_k^1$ and $x_2^{k+1} \in \omega_k^2$, respectively, that is, we have

$$
(5.13)\qquad (I_{h_k} v_{k+1})(x_1^k) = v_{k+1}(x_1^{k+1}) \text{ and } (I_{h_k} v_{k+1})(x_2^k) = v_{k+1}(x_2^{k+1}).
$$

Therefore, starting from (5.11), using (5.9), for any $j+1 \le N \le L-1$, we get

$$
\begin{aligned}
(5.14)\qquad & |I_{h_j} v_{j+1}|^s_{1,s,\omega_j} \le C h_j^{d-s} \\
& \left[\sum_{k=j+1}^{N} |w_k(x_1^k) - w_k(x_2^k)| + |(I_{h_N} v_{N+1})(x_1^N) - (I_{h_N} v_{N+1})(x_2^N)| \right]^s,
\end{aligned}
$$

and consequently, we have get (5.12). Now, we prove that there exist some nodes x_1^k and x_2^k of $\mathcal{T}_{h_k}$ such that (5.12) holds, even if (5.13) does not hold for all $k = j, \cdots, L-1$. Let us assume that (5.13) holds for $k = j, \cdots, N$, $j+1 \le N \le L-1$. Therefore we can get (5.14). From the definition the operators I_i^+ and I_i^- in (4.18), it follows that if, for instance, $(I_{h_N} v_{N+1})(x_1^N) \ne v_{N+1}(x)$ for any node $x \in \omega_N^1$ of $\mathcal{T}_{h_{N+1}}$, then $(I_{h_N} v_{N+1})(x_1^N) = 0$ and v_{N+1} takes both positive and negative values at the nodes of $\mathcal{T}_{h_{N+1}}$ in ω_N^1. Consequently, if both $(I_{h_N} v_{N+1})(x_1^N) \ne v_{N+1}(x)$ for any node $x \in \omega_N^1$ of $\mathcal{T}_{h_{N+1}}$, and $(I_{h_N} v_{N+1})(x_2^N) \ne v_{N+1}(x)$ for any node $x \in \omega_N^2$ of $\mathcal{T}_{h_{N+1}}$, then $(I_{h_N} v_{N+1})(x_1^N) = (I_{h_N} v_{N+1})(x_2^N) = 0$, and we get that (5.12) holds for some arbitrary nodes of $\mathcal{T}_{h_k}$, $x_1^k \in \omega_{k-1}^1$, $x_2^k \in \omega_{k-1}^2$, $N+1 \le k \le L$. Also, if $(I_{h_N} v_{N+1})(x_1^N) \ne v_{N+1}(x)$ for any node $x \in \omega_N^1$ of $\mathcal{T}_{h_{N+1}}$, but there exists $x_2^{N+1} \in \omega_N^2$, node of $\mathcal{T}_{h_{N+1}}$, such that $(I_{h_N} v_{N+1})(x_2^N) = v_{N+1}(x_2^{N+1})$, then

$$
|(I_{h_N} v_{N+1})(x_1^N) - (I_{h_N} v_{N+1})(x_2^N)| = |v_{N+1}(x_2^{N+1})| \le |v_{N+1}(x_1^{N+1}) - v_{N+1}(x_2^{N+1})|
$$

where $x_1^{N+1} \in \omega_N^1$ is an arbitrary node of $\mathcal{T}_{h_{N+1}}$ for which $v_{N+1}(x_1^{N+1})$ and $v_{N+1}(x_2^{N+1})$ have different signs. In this way we get that (5.14) holds for $N+1$, and we can continue the same reasoning for $N+2 \le k \le L-1$.

If we write $\omega_{j-1}^1 = \omega_{j-1}^2 = \omega_j$, since, for $k = j, \cdots, L$, the above nodes x_1^k and x_2^k of $\mathcal{T}_{h_k}$ belong to ω_{k-1}^1 and ω_{k-1}^2, respectively, and $\operatorname{diam}(\omega_{k-1}^1)$, $\operatorname{diam}(\omega_{k-1}^2) \le 2h_k$, then x_1^k and x_2^k, $k = j, \cdots, L$, belong to the sphere centered at x^j and having the radius of $2h_j + h_{j+1} + h_{j+2} + \cdots + h_{L-1}$. Using (5.3), we get that they belongs to the sphere centered at x^j with the radius of $\frac{2\gamma-1}{\gamma-1}h_j$. Consequently, if we write

$$\tilde{\omega}_j = \bigcup_{\tau \in \mathcal{T}_{h_j},\ \mathrm{dist}(\mathrm{x^j},\tau) \le \frac{\gamma}{\gamma-1}\mathrm{h_j}} \tau, \tag{5.15}$$

then $x_1^k, x_2^k \in \tilde{\omega}_j$, $k = j, \cdots, L$. For any $x \in \tilde{\omega}_j$, we get from (5.12),

$$|I_{h_j} v_{j+1}|_{1,s,\omega_j}^s \le C h_j^{d-s} [2(L-j)]^{s-1}$$
$$\{\sum_{k=j+1}^{L-1} [|w_k(x_1^k) - w_k(x)|^s + |w_k(x_2^k) - w_k(x)|^s] +$$
$$|v_L(x_1^L) - v_L(x)|^s + |v_L(x_2^L) - v_L(x)|^s\},$$

and integrating over $\tilde{\omega}_j$ we have,

$$(\tfrac{2\gamma-1}{\gamma-1}h_j)^d |I_{h_j} v_{j+1}|_{1,s,\omega_j}^s \le C h_j^{d-s} [2(L-j)]^{s-1}$$
$$\{\sum_{k=j+1}^{L-1} [||w_k(x_1^k) - w_k||_{0,s,\tilde{\omega}_j}^s + ||w_k(x_2^k) - w_k||_{0,s,\tilde{\omega}_j}^s] +$$
$$||v_L(x_1^L) - v_L||_{0,s,\tilde{\omega}_j}^s + ||v_L(x_2^L) - v_L||_{0,s,\tilde{\omega}_j}^s\}.$$

From this inequality and (4.16), we get

$$|I_{h_j} v_{j+1}|_{1,s,\omega_j}^s \le C(L-j)^{s-1} (\tfrac{\gamma-1}{2\gamma-1})^d$$
$$\{\sum_{k=j+1}^{L-1} C_{d,s}(2h_j \frac{2\gamma-1}{\gamma-1}, h_k)^s |w_k|_{1,s,\tilde{\omega}_j}^s + C_{d,s}(2h_j \frac{2\gamma-1}{\gamma-1}, h_L)^s |v_L|_{1,s,\tilde{\omega}_j}^s\},$$

and, taking into account the definition of $C_{d,s}$ in (4.6), we have

$$|I_{h_j} v_{j+1}|_{1,s,\omega_j}^s \le C(L-j)^{s-1} \{\sum_{k=j+1}^{L-1} C_{d,s}(h_j, h_k)^s |w_k|_{1,s,\tilde{\omega}_j}^s +$$
$$C_{d,s}(h_j, h_L)^s |v_L|_{1,s,\tilde{\omega}_j}^s\}.$$

Finally, since the mesh $\mathcal{T}_{h_j}$ is regular and γ is independent of L and of the mesh parameters, then ω_j and $\tilde{\omega}_j$ contain a bounded number of simplexes in $\mathcal{T}_{h_j}$, which is also independent of L and of the mesh parameters. Consequently, we get (5.10). Since the nodes of $\mathcal{T}_{h_j}$ belonging to $\partial\Omega_j$ lie also on $\partial\Omega_{j+1}$, and $v_{j+1} = 0$ on $\partial\Omega_{j+1}$, it follows that $I_{h_j} v_{j+1} = 0$ on $\partial\Omega_j$. Consequently, they are extended with zero to Ω_k, $j+1 \le k \le L$, and (5.10) holds for these domains, too. □

The following proposition shows that Assumption 2.1 holds for the multi-level method and writes the constant C_0 as a function of the domain decomposition and mesh parameters.

PROPOSITION 5.2. *Let, for each level $j = 1, \cdots, L$, $\Omega_j^1, \cdots, \Omega_j^{m_j}$ be the overlapping decomposition of the domain Ω_j defined in this section with $\Omega_L = \Omega$ and $m_1 = 1$. Then Assumption 2.1 is verified for the piecewise linear finite element spaces, $V = V_{h_L}$ and $V_j^i = V_{h_j}^i$, $j = 1, \cdots, L$, $i = 1, \cdots, m_j$ defined in (5.7) and (5.8), respectively, and any convex set $K = K_{h_L} \subset V_{h_L}$ with Property 3.1. The constant in (2.3) of Assumption 2.1 can be taken of the form*

$$C_0 = Cm^2(L+1)^{2-\frac{1}{p}-\frac{1}{s}} \sum_{j=1}^{L} [1 + (m-1)\frac{h_{j-1}}{\delta_j}] C_{d,s}(h_{j-1}, h_L) \tag{5.16}$$

in which we take $h_0 = h_1$, C is independent of the mesh and domain decomposition parameters, and $C_{d,s}(H, h)$ is given in (4.6).

Proof. Let us consider $w \in K_{h_L}$, $w_j^i \in V_{h_j}^i$, $j = 1, \cdots, L$, $i = 1, \cdots, m_j$, such that $w + \sum_{j=1}^{k-1} \sum_{i=1}^{m_j} w_j^i + \sum_{i=1}^{l} w_k^i \in K_{h_L}$, $k = 1, \cdots, L$, $l = 1, \cdots, m_k$, and let v be another element in K_{h_L}. For $j = 1, \cdots, L$, we write

$$w_j^0 = \sum_{i=1}^{m_j} w_j^i \text{ and } w_j = \sum_{k=1}^{j} w_j^0 = \sum_{k=1}^{j} \sum_{i=1}^{m_j} w_j^i.$$

Since $v, w + w_{L-2} \in K_{h_L}$, and also, $w + w_{L-2} + w_{L-1}^0 \in K_{h_L}$ and $w + w_{L-2} + w_{L-1}^0 + \sum_{i=1}^{l} w_L^i \in K_{h_L}$, $l = 1, \cdots, m_L$, as in the proof of Proposition 4.4, we get that there exist $v_{L-1}^0 \in V_{h_{L-1}}$ and $v_L^i \in V_{h_L}^i$, $i = 1, \cdots, m_L$ such that

$$w + w_{L-2} + v_{L-1}^0 \in K_{h_L}, \tag{5.17}$$

$$w + w_{L-2} + w_{L-1}^0 + \sum_{i=1}^{l-1} w_L^i + v_L^l \in K_{h_L},\ l = 1, \cdots, m_L, \tag{5.18}$$

$$v - w - w_{L-2} = v_{L-1}^0 + \sum_{i=1}^{m_L} v_L^i, \tag{5.19}$$

and

$$\begin{aligned} &v_{L-1}^0 = w_{L-1}^0 + I_{h_{L-1}}(v - w - w_{L-2} - w_{L-1}^0) \\ &v_L^i = L_{h_L}(\tau_0^i(v - w - w_{L-2} - v_{L-1}^0) + \sum_{l=1}^{i} \tau_l^i w_L^l),\ i = 1, \cdots, m_L, \end{aligned} \tag{5.20}$$

where τ_j^i, $i = 1, \cdots, m$, $j = 0, \cdots, i$, satisfy (3.9). In this way, using (5.17), we get that $w + w_{L-3} + w_{L-2}^0 + v_{L-1}^0, w + w_{L-3} \in K_{h_L}$, and also, $w + w_{L-3} + w_{L-2}^0 \in K_{h_L}$ and $w + w_{L-3} + w_{L-2}^0 + \sum_{i=1}^{l} w_{L-1}^i \in K_{h_L}$, $l = 1, \cdots, m_{L-1}$. Consequently, there exist $v_{L-2}^0 \in V_{h_{L-2}}$ and $v_{L-1}^i \in V_{h_{L-1}}^i$, $i = 1, \cdots, m_{L-1}$ such that

$$w + w_{L-3} + v_{L-2}^0 \in K_{h_L}, \tag{5.21}$$

$$w + w_{L-3} + w_{L-2}^0 + \sum_{i=1}^{l-1} w_{L-1}^i + v_{L-1}^l \in K_{h_L},\ l = 1, \cdots, m_{L-1}, \tag{5.22}$$

$$w_{L-2}^0 + v_{L-1}^0 = v_{L-2}^0 + \sum_{i=1}^{m_{L-1}} v_{L-1}^i, \tag{5.23}$$

and

$$\begin{aligned} & v_{L-2}^0 = w_{L-2}^0 + I_{h_{L-2}}(v_{L-1}^0) \\ & v_{L-1}^i = L_{h_{L-1}}(\tau_0^i(w_{L-2}^0 + v_{L-1}^0 - v_{L-2}^0) + \\ & \sum_{l=1}^{i} \tau_l^i w_{L-1}^l), \ i = 1, \cdots, m_{L-1}. \end{aligned} \tag{5.24}$$

Starting from (5.21) we successively get for $j = 3, \cdots, L-1$ that $w + w_{L-j-1} + w_{L-j}^0 + v_{L-j+1}^0, w + w_{L-j-1} \in K_{h_L}$, and also, $w + w_{L-j-1} + w_{L-j}^0 \in K_{h_L}$ and $w + w_{L-j-1} + w_{L-j}^0 + \sum_{i=1}^{l} w_{L-j+1}^i \in K_{h_L}$, $l = 1, \cdots, m_{L-j+1}$. Consequently, there exist $v_{L-j}^0 \in V_{h_{L-j}}$ and $v_{L-j+1}^i \in V_{h_{L-j+1}}^i$, $i = 1, \cdots, m_{L-j+1}$ such that

$$w + w_{L-j-1} + v_{L-j}^0 \in K_{h_L}, \tag{5.25}$$

$$\begin{aligned} & w + w_{L-j-1} + w_{L-j}^0 + \sum_{i=1}^{l-1} w_{L-j+1}^i + \\ & v_{L-j+1}^l \in K_{h_L}, \ l = 1, \cdots, m_{L-j+1}, \end{aligned} \tag{5.26}$$

$$w_{L-j}^0 + v_{L-j+1}^0 = v_{L-j}^0 + \sum_{i=1}^{m_{L-j+1}} v_{L-j+1}^i, \tag{5.27}$$

and

$$\begin{aligned} & v_{L-j}^0 = w_{L-j}^0 + I_{h_{L-j}}(v_{L-j+1}^0) \\ & v_{L-j+1}^i = L_{h_{L-j+1}}(\tau_0^i(w_{L-j}^0 + v_{L-j+1}^0 - v_{L-j}^0) + \\ & \sum_{l=1}^{i} \tau_l^i w_{L-j+1}^l), \ i = 1, \cdots, m_{L-j+1}. \end{aligned} \tag{5.28}$$

If we write $v_1^1 = v_1^0$, since $m_1 = 1$, then equations (5.18), (5.22) and (5.26) prove that equation (2.1) of Assumption 2.1 holds. Also, we get (2.2) of Assumption 2.1 from (5.19), (5.23) and (5.27). Now, if we write

$$v_L^0 = v - w - w_{L-1}, \tag{5.29}$$

we get from (5.20), (5.24) and (5.28) that

$$\begin{aligned} & v_{j-1}^0 = w_{j-1}^0 + I_{h_{j-1}}(v_j^0), \\ & v_j^i = L_{h_j}(\tau_0^i(w_{j-1}^0 + v_j^0 - v_{j-1}^0) + \sum_{l=1}^{i} \tau_l^i w_j^l), \\ & \text{for } j = 2, \cdots, L, \ i = 1, \cdots, m_j. \end{aligned} \tag{5.30}$$

Similarly to (4.39), we get that

$$\begin{aligned} & ||L_{h_j}(\tau_l^i w_j^l)||_{1,s} \le C[1 + (m_j - 1)\tfrac{h_{j-1}}{\delta_j}]|w_j^l|_{1,s}, \\ & j = 2, \cdots, L, \ i = 1, \cdots, m_j, \ l = 1, \cdots, i. \end{aligned} \tag{5.31}$$

Replacing v_{j-1}^0 given in the first equation (5.30) in the second equation of (5.30), and using (4.21) and (4.22), we get

$$\begin{aligned}&||L_{h_j}(\tau_0^i(w_{j-1}^0+v_j^0-v_{j-1}^0))||_{1,s}=||L_{h_j}(\tau_0^i(v_j^0-I_{h_{j-1}}v_j^0))||_{1,s}\le\\&C[|v_j^0-I_{h_{j-1}}v_j^0|_{1,s}+(1+\tfrac{m_{j-1}}{\delta_j})||v_j^0-I_{h_{j-1}}v_j^0||_{0,s}]\le\\&C\{[1+C_{d,s}(h_{j-1},h_j)]|v_j^0|_{1,s}+(1+\tfrac{m_j-1}{\delta_j})h_{j-1}C_{d,s}(h_{j-1},h_j)]|v_j^0|_{1,s}\}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&||L_{h_j}(\tau_0^i(w_{j-1}^0+v_j^0-v_{j-1}^0))||_{1,s}\le\\&C[1+(m_j-1)\tfrac{h_{j-1}}{\delta_j}]C_{d,s}(h_{j-1},h_j)|v_j^0|_{1,s}\text{ for } j=2,\cdots,L,\ i=1,\cdots,m_j.\end{aligned}\tag{5.32}$$

From the second equation in (5.30), (5.31) and (5.32), for $j=2,\cdots,L$ and $i=1,\cdots,m_j$, we get

$$\begin{aligned}&||v_j^i||_{1,s}\le C[1+(m_j-1)\tfrac{h_{j-1}}{\delta_j}]C_{d,s}(h_{j-1},h_j)|v_j^0|_{1,s}+\\&C[1+(m_j-1)\frac{h_{j-1}}{\delta_j}]\sum_{l=1}^{i}|w_j^l|_{1,s},\end{aligned}$$

and using (5.6), we have

$$\begin{aligned}&||v_j^i||_{1,s}\le C[1+(m-1)\tfrac{h_{j-1}}{\delta_j}]C_{d,s}(h_{j-1},h_j)|v_j^0|_{1,s}+\\&C[1+(m-1)\frac{h_{j-1}}{\delta_j}]\sum_{l=1}^{m_j}|w_j^l|_{1,s}\text{ for } j=2,\cdots,L.\end{aligned}\tag{5.33}$$

The first equation in (5.30) shows that the conditions of Lemma 5.1 are satisfied, and we get from (5.10) that for $j=1,\cdots,L-1$,

$$\begin{aligned}&|v_j^0|_{1,s}\le C(L-j)^{\frac{s-1}{s}}[\sum_{k=j}^{L-1}C_{d,s}(h_j,h_k)^s|w_k^0|_{1,s}^s+\\&C_{d,s}(h_j,h_L)^s|v_L^0|_{1,s}^s]^{\frac{1}{s}}.\end{aligned}$$

Since $C_{d,s}(h_j,h_k)\le C_{d,s}(h_j,h_L)$, $j=1,\cdots L-1$, $j\le k\le L-1$, using (5.29), we get

$$\begin{aligned}&|v_j^0|_{1,s}\le C(L-1)^{\frac{s-1}{s}}C_{d,s}(h_j,h_L)[\sum_{k=1}^{L-1}|w_k^0|_{1,s}+|v-w|_{1,s}]\\&\text{for } j=1,\cdots,L-1.\end{aligned}\tag{5.34}$$

From (4.6), we have $C_{d,s}(h_{j-1},h_j)C_{d,s}(h_j,h_L)\le C_{d,s}(h_{j-1},h_L)$, and using it, we get from (5.33) and (5.34),

$$\begin{aligned}&||v_j^i||_{1,s}\le C(L-1)^{\frac{s-1}{s}}[1+(m-1)\tfrac{h_{j-1}}{\delta_j}]C_{d,s}(h_{j-1},h_L)\\&[\sum_{k=1}^{L}\sum_{l=1}^{m_k}|w_k^l|_{1,s}+|v-w|_{1,s}],\quad\text{for } j=2,\cdots,L.\end{aligned}$$

Since $m_1=1$ and since we have written $v_1^1=v_1^0$ which vanishes on $\partial\Omega$, it follows from (5.34) that the above equation also holds for $j=1$ with $h_0=h_1$. From this equation we get

$$||v_j^i||_{1,s} \le Cm^{\frac{p-1}{p}}(L+1)^{\frac{p-1}{p}}(L-1)^{\frac{s-1}{s}}[1+(m-1)\tfrac{h_{j-1}}{\delta_j}]C_{d,s}(h_{j-1},h_L)$$
$$(5.35)\qquad [\sum_{k=1}^{L}\sum_{l=1}^{m_k}|w_k^l|_{1,s}^p + |v-w|_{1,s}^p]^{\frac{1}{p}},$$

and (5.16) follows from it. □

5.1. Multigrid method. In the above multi-level method a mesh is the refinement of that one on the previous level, but the domain decompositions are almost independent from a level to another one. The multigrid method is obtained from the multi-level method by taking the subsets O_j^i of a particular form: we associate at each node x_j^i of $\mathcal{T}_{h_j}$, $j=1,\cdots,L$, $i=1,\cdots,M_j$, an O_j^i defined as the union of the simplexes in $\mathcal{T}_{h_j}$ having x_j^i as a vertex. Consequently, the subspaces $V_{h_j}^i$ will be direct sums of some one-dimensional spaces generated by the nodal basis functions associated with the nodes of $\mathcal{T}_{h_j}$. Evidently, all the previous assumptions on the domain decompositions are satisfied and we can take $\delta_j = h_j$. In the multigrid methods, the construction of a finer mesh from a coarse one, is made following the same procedure of division of the simplexes at each level. Therefore, we can replace equation (5.3) by

$$(5.36)\qquad 1 < \gamma \le \frac{h_j}{h_{j+1}} \le C\gamma, \quad j=1,\cdots,L-1,$$

where the constant C is independent of the number of meshes. Starting from the expression of the constant C_0 in (5.16), using (5.36), we have

$$\begin{aligned}&Cm^2(L+1)^{2-\frac{1}{p}-\frac{1}{s}}\sum_{j=1}^{L}[1+(m-1)\frac{h_{j-1}}{\delta_j}]C_{d,s}(h_{j-1},h_L) \le\\ &Cm^2(L+1)^{2-\frac{1}{p}-\frac{1}{s}}L[1+(m-1)\gamma]C_{d,s}(h_1,h_L) \le\\ &Cm^3L^{3-\frac{1}{p}-\frac{1}{s}}\gamma C_{d,s}(h_1,h_L)\end{aligned}$$

If we write $h=h_1$ and denote by H the diameter of Ω, then the constant C_0 can be taken as

$$(5.37)\qquad C_0 = CL^{3-\frac{1}{p}-\frac{1}{s}}\gamma C_{d,s}(H,h).$$

We point out that an iteration of Algorithm 2.1 using the one-dimensional spaces generated by the basis functions corresponding to the nodes of the L meshes represents half of a V-cycle multigrid iteration. Since a full V-cycle multigrid iteration uses more than once these one-dimensional spaces, in order to describe it, we should repeat them in the list of the subspaces used by Algorithm 2.1. Consequently, for the multigrid method, only L in the expression of C_0 in (5.37) should be multiplicated by a constant. Therefore, C_0 given in (5.37) is valid for the multigrid method, too.

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