HEAT KERNEL ASYMPTOTICS FOR REAL POWERS OF LAPLACIANS

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CONTEXT

- Let (M, g) be a compact, oriented, Riemannian manifold of dimension n.
- Consider $\Delta = d^*d : \mathcal{C}^\infty(M, \mathbb{C}) \longrightarrow \mathcal{C}^\infty(M, \mathbb{C})$.
- We are interested in p_t, the heat kernel of Δ.

THEOREM

Since M is compact and Δ is self-adjoint, the spectrum of Δ is discrete and increases towards ∞. Moreover, there exists a basis of $\mathcal{C}^\infty$ functions $\{\phi_j\}_{j\in\mathbb{N}}$ such that $\Delta\phi_j = \lambda_j\phi_j$.

$$\Delta = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix}$$

The Heat operator

- We define the *heat operator* $e^{-t\Delta}$ on the basis:

$$e^{-t\Delta}\phi_j = e^{-t\lambda_j}\phi_j.$$

- $e^{-t\Delta}$ is an integral operator: $(e^{-t\Delta}f)(x) = \int_M p_t(x,y)f(y)dy$.
- $p \in \mathcal{C}^\infty((0,\infty)\times M \times M, \mathbb{C})$.
- It verifies the heat equation:

$$\begin{cases} (\partial_t + \Delta_x)\, p_t(x,y) = 0. \\ \lim_{t\to 0} e^{-t\Delta} f = f, \ \text{for any } f \in \mathcal{C}^\infty(M), \text{ in } \|\cdot\|_0. \end{cases}$$

$$\text{"}\; e^{-t\Delta} = \begin{bmatrix} e^{-t\lambda_1} & & \\ & e^{-t\lambda_2} & \\ & & \ddots \end{bmatrix} \;\text{"}$$

THE SMALL-TIME ASYMPTOTIC EXPANSION OF p_t

THEOREM (MINAKSHISUNDARAM-PLEIJEL)

The heat kernel p_t has the following small-time asymptotic expansion near the diagonal:

$$p_t(x,y) \overset{t\searrow 0}{\sim} (4\pi t)^{-n/2} e^{\frac{-d(x,y)^2}{4t}} \sum_{j=0}^{\infty} t^j \Psi_j(x,y),$$

where $d(x,y)$ is the geodesic distance between x and y, and the Ψ_j's are recursively defined as solutions of certain ODE's along geodesics.

Theorem (Minakshisundaram-Pleijel)

The heat kernel p_t has the following small-time asymptotic expansion along the diagonal:

$$p_t(x,x) \overset{t \searrow 0}{\sim} t^{-n/2} \sum_{j=0}^{\infty} t^j a_j(x,x),$$

where $a_j : M \longrightarrow \mathbb{C}$ are $\mathcal{C}^\infty$ functions. Moreover, $a_0(x,x) = 1$, and the a_j's are related by a recurrence formula.

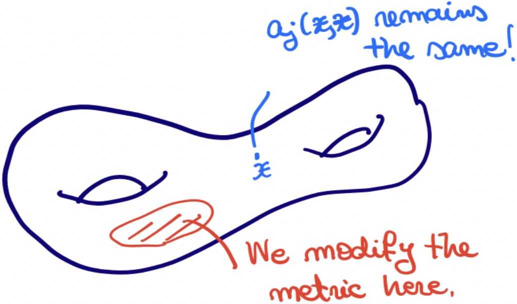
- This means that if we truncate the sum at the N-th term, the difference is of order $\mathcal{O}\left(t^{N+1-n/2}\right)$:

$$\left\| p_t(x,x) - t^{-n/2} \sum_{j=0}^{N} t^j a_j(x,x) \right\|_0 \leq C \cdot t^{N+1-n/2}.$$

- Also true when we take derivatives in t.

THE COEFFICIENTS IN THE ASYMPTOTIC OF p_t

- The a_j's depend only on g and its derivatives $\Rightarrow$ they are LOCAL.



- $a_0 = 1$, $a_1 = c \cdot r_M$.

MOTIVATION

- The Atiyah-Singer index theorem:

$$\mathrm{ind(D)} = (2\pi i)^{-n/2} \int_M \hat{A}(M).$$

- Weyl's law:

$$N(\lambda) \overset{\lambda \to \infty}{\sim} \frac{\mathrm{vol}(M)}{(4\pi)^{n/2} \cdot \Gamma\left(\frac{n}{2} + 1\right)} \lambda^{n/2}.$$

DEFINITION

For $\Re s > 0$, the *Gamma function* is defined as:

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt.$$

Remark: Γ extends meromorphically to $\mathbb{C}$ with simple poles at $s = 0, -1, \ldots.$

Part of the bigger picture

The Heat Operator $e^{-t\Delta} \leftarrow \Delta$

$$\updownarrow$$

The heat kernel p_t

$$\updownarrow$$

$$p_t(x,x) \overset{t\to 0}{\sim} t^{-\frac{m}{2}} \cdot \sum_{j=0}^{\infty} t^{j} \cdot \underbrace{a_j(x,x)}_{\text{Locally Computable}}$$

The operator $e^{-t\Delta^r}$

- We study the operator $e^{-t\Delta^r}$, $r \in (0,1)$, defined on the basis as:

$$e^{-t\Delta^r}\phi_j = e^{-t\lambda_j^r}\phi_j.$$

- Important: $\lambda_j \geq 0$!
- Let h_t be the Schwartz kernel of $e^{-t\Delta^r}$.

$$e^{-t\Delta^r} = \begin{bmatrix} e^{-t\lambda_1^r} & & \\ & e^{-t\lambda_2^r} & \\ & & \ddots \end{bmatrix}$$

The asymptotic expansion of h_t along the diagonal

Theorem (-,'22)

If n is even, the small-time asymptotic of h_t along the diagonal is the following:

$$h_t(x,x) \overset{t \searrow 0}{\sim} \sum_{j=0}^{n/2} t^{-\frac{n-2j}{2r}} B_{-\frac{n-2j}{2r}}(x) + \sum_{\substack{j=1 \\ rj \notin \mathbb{N}}}^{\infty} t^j A_j(x).$$

- $B_{-\frac{n-2j}{2r}}(x) = \frac{1}{r} \frac{\Gamma\left(\frac{n-2j}{2r}\right)}{\Gamma\left(\frac{n-2j}{2}\right)} \cdot a_j(x,x) \Rightarrow$ LOCAL!
- What happens with the A_j's? Are they also local?!

THE ASYMPTOTIC EXPANSION OF h_t ALONG THE DIAGONAL

THEOREM

if n is odd, $r = \frac{\alpha}{\beta}$ is rational, and the denominator β is even. In that case,

$$h_{t|_{\mathrm{Diag}}} \overset{t \searrow 0}{\sim} \sum_{j=0}^{(n-1)/2} t^{-\frac{n-2j}{2r}} \cdot A_{-\frac{n-2j}{2r}} + \sum_{\substack{j=1 \\ \alpha \nmid 2j+1}}^{\infty} t^{\frac{2j+1}{2r}} \cdot A_{\frac{2j+1}{2r}} \\ + \sum_{\substack{j=1 \\ rj \notin \mathbb{N}}}^{\infty} t^{j} \cdot A_j + \sum_{\substack{l=1 \\ l\ odd}}^{\infty} t^{l\frac{\beta}{2}} \log t \cdot B_l. \tag{1}$$

Similar expansions are proved in all the other cases.

Motivation

- Duistermaat and Guillemin studied e^{-tP}, where P is a scalar positive elliptic self-adjoint pseudodifferential operator of positive *integer* order.
- Grubb studied e^{-tP} in the context of fiber bundles, when the order of P is positive, not necessary an integer.
- C. Bär and S. Moroianu studied the Schwartz kernel of $e^{-t\Delta^{1/m}}$, $m \in \mathbb{N}$, where Δ is a strictly positive self-adjoint generalised Laplacian.

The bigger picture

Δ

The Heat Operator $e^{-t\Delta}$

The heat kernel p_t

$$p_t(x,x) \overset{t\to 0}{\sim} t^{-\frac{m}{2}}\cdot \sum_{j=0}^{\infty} t^{j}\cdot \underbrace{a_j(x,x)}_{\text{Locally Computable}}$$

The operator $e^{-t\Delta^r}$, $r\in(0,1)$

The Schwartz kernel h_t

LOCAL

$$h_t(x,x) \overset{t\to 0}{\sim} \sum_{j=0}^{m/2} t^{-\frac{m-2j}{2r}} B_{-\frac{m-2j}{2r}}(x) \quad (m=\text{even})$$

$$+ \sum_{\substack{j=1\\ rj\notin\mathbb{N}}}^{\infty} t^{j} A_j(x)$$

NOT LOCAL!

THE IDEA

$$
\begin{tikzcd}
e^{-t\Delta} \arrow[dr, dash, "\text{Mellin Formula}"] \arrow[dd, dash] & & e^{-t\Delta^r},\ r\in(0,1) \arrow[dl, dash, "\text{Inverse Mellin}"'] \arrow[dd, dash] \\
& \Delta^s,\ s\in\mathbb{C} \arrow[d, dash] & \\
P_t \arrow[r, dash] & \zeta_s \arrow[r, dash] & h_t
\end{tikzcd}
$$

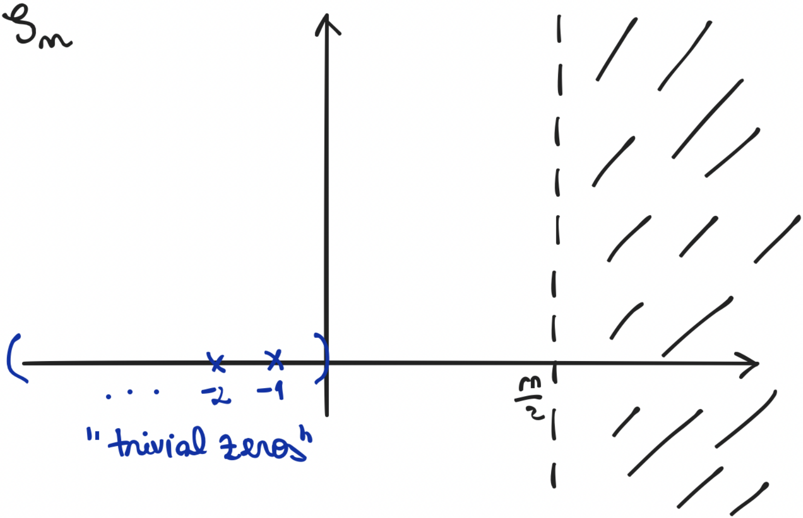
- Consider the Zeta-Epstein function:

$$\zeta_n(s) := \sum_{(k_1,\dots,k_n)\in\mathbb{Z}^n\setminus\{0\}} \left(k_1^2+\dots+k_n^2\right)^{-s} = \sum_{k\in\mathbb{N}^*} k^{-s}R_n(k),$$

  where $R_n(k)$ is the number of representations of k as a sum of n squares.
- Remark that $\zeta_1(s) = 2\zeta(2s)$.
- One can prove that ζ_n is absolutely convergent for $\Re s > \frac{n}{2}$, and it extends meromorphically to $\mathbb{C}$ with "trivial zeros" at $s = -1, -2, \dots$
- Functional equation:

$$\pi^{-s}\zeta_n(s)\Gamma(s) = \pi^{s-n/2}\zeta_n\left(\frac{n}{2}-s\right)\Gamma\left(\frac{n}{2}-s\right).$$

AN UNEXPECTED GUEST



Δ

The Heat Operator $e^{-t\Delta}$ $\qquad$ The operator $e^{-t\Delta^r}$, $r\in(0,1)$

$\Updownarrow$ $\qquad$ $\Updownarrow$

The heat kernel p_t $\qquad$ The Schwartz kernel h_t

$\Updownarrow$ $\qquad$ $\Updownarrow$

$$p_t(x,x) \overset{t\to 0}{\sim} t^{-\frac{m}{2}} \cdot \sum_{j=0}^{\infty} t^{j} \cdot \underbrace{a_j(x,x)}_{\text{Locally Computable}}$$

$$h_t(x,x) \overset{t\to 0}{\sim} \sum_{j=0}^{m/2} t^{-\frac{m-2j}{2r}} B_{-\frac{m-2j}{2r}}(x) + \sum_{\substack{j=1 \\ rj\notin\mathbb{N}}}^{\infty} t^{j} A_j(x)$$

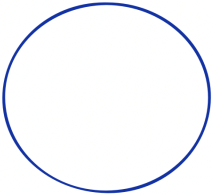
LOCAL

$(m = \text{even})$

NOT LOCAL!

The Non-local coefficients

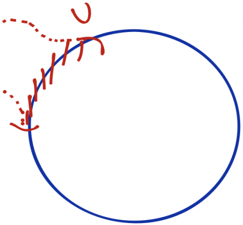
- Claim: $A_j(x,x) = \frac{(-1)^j}{j!} q_{rj}(x,x)$ for $rj \notin \mathbb{N}$ are non-local!
- Strategy: For each dimension n we give a manifold and a Laplacian.



$(S^1, d\theta^2, \Delta)$

$\Delta^s \longleftrightarrow g^{\Delta}_s(\theta,\theta')$

$s \in \mathbb{C}$



$(S^1, \tilde{g}, \tilde{\Delta})$

$\tilde{\Delta}^s \longleftrightarrow g^{\tilde{\Delta}}_s(\theta,\theta')$.

Example

Take $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$, $g = d\theta^2$, and Δ. Then for $\Re s > \frac{1}{2}$ (actually for all $s \in \mathbb{C}$),

$$q^{\Delta}_{-s}(\theta,\theta) = \left(\frac{1}{2\pi}\sum_{k\in\mathbb{Z}^*}(k^2)^{-s}\right) = \frac{1}{\pi}\zeta(2s).$$

We are interested in $s = -rj \notin -\mathbb{N}$.
We modify the metric locally in an open set $U \subset S^1 \rightsquigarrow \tilde{g}, \tilde{\Delta}$ and

$$q^{\tilde{\Delta}}_{-s}(\theta,\theta) = \frac{p^{2s-1}}{\pi}\zeta(2s).$$

Remark: $\zeta(2s) = 0 \Leftrightarrow 2s \in \{-2,-4,...\} \Leftrightarrow s \in \{-1,-2,...\}$, which is not our case since $s = -rj \notin -\mathbb{N}$!

In n-dimensions, we take $M = S^1 \times S^1 \times ... \times S^1$, ζ_n.

THE ASYMPTOTIC EXPANSION OF h_t AWAY FROM THE DIAGONAL

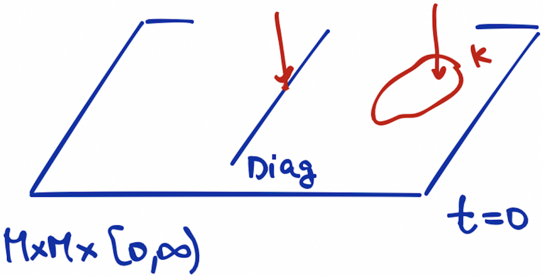
THEOREM (-,'22)

The Schwartz kernel h_t of the operator $e^{-t\Delta^r}$ is $\mathcal{C}^\infty$ on $[0,\infty) \times (M \times M \setminus \mathrm{Diag})$. Furthermore, let $K \subset M \times M \setminus \mathrm{Diag}$ be a compact set. Then the Taylor series of $h_{t|_K}$ as $t \searrow 0$ is the following:

$$h_{t|_K} \overset{t \searrow 0}{\sim} \sum_{j=1}^{\infty} t^j q_{rj|_K} \frac{(-1)^j}{j!}.$$

Moreover, if $r = \frac{\alpha}{\beta}$ is rational with α, β coprime, then the coefficient of t^j vanishes for $j \in \beta\mathbb{N}^*$.

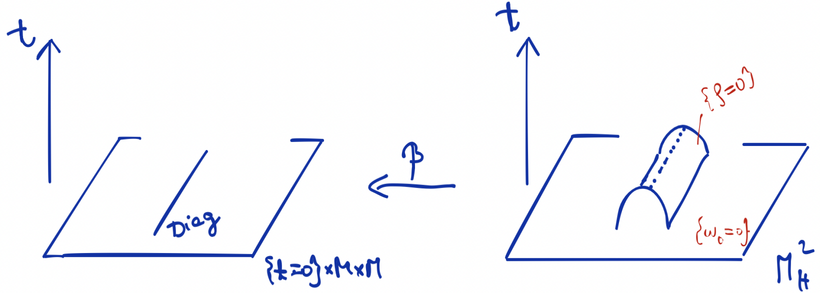
Simultaneous formula?



- Question: Can we find a simultaneous formula for both cases?
- For $r = 1/2$, the answer is yes!

THE STANDARD HEAT KERNEL p_t ON THE HEAT SPACE

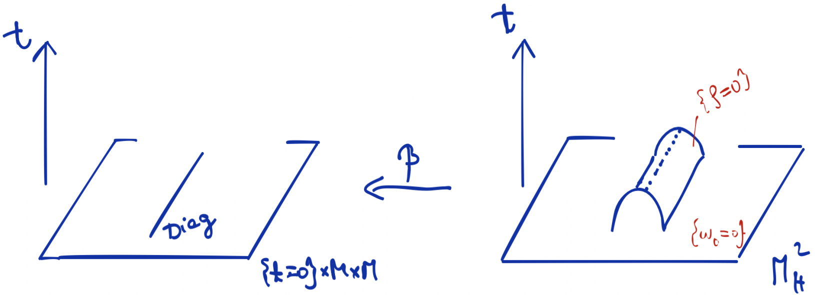
- Melrose used his blow-up techniques to give a conceptual interpretation for the asymptotic of the usual heat kernel p_t.
- The *heat space* M^2_H is obtained by performing a *parabolic* blow-up of $\{t = 0\} \times \text{Diag}$ in $[0, \infty) \times M \times M$.
- M^2_H is a manifold with corners with boundary hypersurfaces given by the boundary defining function ρ and ω_0.



THE STANDARD HEAT KERNEL ON THE HEAT SPACE

THEOREM (MELROSE)

The heat kernel p_t belongs to $\rho^{-n} \cdot \mathcal{C}^\infty \left(M_H^2\right)$, and vanishes rapidly at the boundary hypersurface $\{\omega_0 = 0\}$.



h_t ON THE STANDARD BLOW-UP SPACE

- We study h_t, the Schwartz kernel of $e^{-t\Delta^{1/2}}$.
- Let $\mathrm{M_{heat}} = [[0,\infty) \times M \times M, \{t=0\} \times \mathrm{Diag}]$ be the *standard* blow-up.
- The blow down map is given locally by:

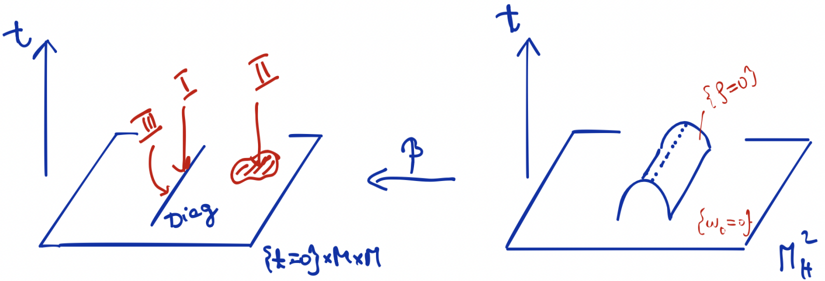
$$\beta_H : \mathrm{M_{heat}} \longrightarrow [0,\infty) \times M \times M$$
$$\beta_H(\rho, \omega, x') = (\rho\omega_0, \rho\omega' + x', x'),$$

where

$$\omega \in \mathbb{S}^n_H = \{\omega = (\omega_0, \omega') \in \mathbb{R}^{n+1} : \ \omega_0 \geq 0, \ \omega_0^2 + |\omega'|^2 = 1\}.$$

THEOREM (-,'22)

If n is even, then the Schwartz kernel h_t of the operator $e^{-t\Delta^{1/2}}$ belongs to $\rho^{-n}\omega_0 \cdot \mathcal{C}^\infty(\mathrm{M}_{\mathrm{heat}})$. Furthermore, if n is odd, $h_t \in \rho^{-n}\omega_0 \cdot \mathcal{C}^\infty(\mathrm{M}_{\mathrm{heat}}) + \rho \log \rho \cdot \omega_0 \cdot \mathcal{C}^\infty(\mathrm{M}_{\mathrm{heat}})$.



THE KERNEL h_t AS A POLYHOMOGENEOUS FUNCTION

THEOREM (-,'22)

For $r = \frac{1}{2}$, the heat kernel h_t of the operator $e^{-t\Delta^{1/2}}$ is a polyhomogeneous conormal section on the linear heat space $\mathrm{M}_{\mathrm{heat}}$ with values in $\mathcal{E} \boxtimes \mathcal{E}^*$. The index set for the lateral boundary is:

$$F_{\mathrm{lb}} = \{(k, 0) : k \in \mathbb{N}^*\}.$$

If n is even, the index set of the front face is:

$$F_{\mathrm{ff}} = \{(-n + k, 0) : k \in \mathbb{N}\},$$

while for n odd the index set towards ff is given by:

$$F_{\mathrm{ff}} = \{(-n + k, 0) : k \in \mathbb{N}\} \cup \{(k, 1) : k \in \mathbb{N}^*\}.$$

WHY $r = 1/2$?

- Legendre duplication formula:

$$\frac{\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)} = \frac{1}{\sqrt{2\pi}} 2^{s-\frac{1}{2}} \Gamma\left(\frac{s+1}{2}\right).$$

- Gauss multiplication formula, $m \in \mathbb{N}^*$:

$$\Gamma(s)\Gamma\left(s + \frac{1}{m}\right) \ldots \Gamma\left(s + \frac{m-1}{m}\right) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}-ms} \Gamma(ms).$$

- Already in the case $r = 1/3$ our method leads to complicated computations involving Bessel modified functions.
- However, it seems reasonable to expect that the Schwartz kernel h_t of the operator $e^{-t\Delta^r}$ for $r \in (0, 1)$ can be lifted to a polyhomogeneous conormal section in a certain "transcendental" heat space M^r_{Heat} depending on r.

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Thank you!