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STOCHASTIC INTEGRATION IN RIEMANNIAN MANIFOLDS FROM A FUNCTIONAL-ANALYTIC POINT OF VIEW

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1. MOTIVATION AND CONTEXT

- M a connected Riemannian manifold (therefore separable). Just for simplicity and brevity we shall assume M compact, but the results hold true for arbitrary manifolds.

- **The problem:** can we integrate a real smooth 1-form $\alpha \in \Omega(M)$ along a continuous curve $c : [0, 1] \to M$?

- If c is piece-smooth we have the usual line integral $\int_c \alpha$.

- What happens if c is continuous but not piecewise-smooth? **Several solutions:**

  ◇ the Young integral ('30s),

  ◇ rough paths ('90s),

  ◇ stochastic integration (Itô in the '40s and Stratonovich in the '60s).

1.1 Stochastic integration in $\mathbb{R}^n$

- Let $c : [0, t] \to \mathbb{R}^n$. For every $k \in \mathbb{N}$ consider the system of 2^k points

$$\left(c\left(\frac{t}{2^k}\right), c\left(\frac{2t}{2^k}\right), c\left(\frac{3t}{2^k}\right), \dots, c\left(\frac{2^k t}{2^k}\right)\right) \in (\mathbb{R}^n)^{2^k} .$$

- If c has bounded variation, the Riemann-lie sums

$$\sum_{j=0}^{2^k-1} \alpha\left(c\left(\frac{jt}{2^k}\right)\right)\left[c\left(\frac{(j+1)t}{2^k}\right) - c\left(\frac{jt}{2^k}\right)\right]$$

associated to these systems of points converge to the Stieltjes integral $\int_0^t (\alpha \circ c)\, \mathrm{d}c$.

- What happens if c does not have bounded variation? The limit exists **in measure** and is called the **Itô integral** of α.

- Consider now the "symmetrized" sums

$$\sum_{j=0}^{2^k-1} \frac{1}{2}\left[\alpha\left(c\left(\frac{jt}{2^k}\right)\right)+\alpha\left(c\left(\frac{(j+1)t}{2^k}\right)\right)\right]\left[c\left(\frac{(j+1)t}{2^k}\right)-c\left(\frac{jt}{2^k}\right)\right].$$

- Their limit exists **in measure**, too, and is called the **Stratonovich integral** of α.

- If $x, y \in \mathbb{R}^n$, let $\gamma_{x,y} : [0,1] \to \mathbb{R}^n$ the line sgment $\gamma_{x,y}(\tau) = \tau y + (1-\tau)x$. Let P be a Borel regular probability on $[0,1]$. Consider the Riemann-like sums

$$A_{P,t,k}(\alpha)(c) = \sum_{j=0}^{2^k-1} \int_{[0,1]} \alpha_{\gamma_{c\left(\frac{jt}{2^k}\right),c\left(\frac{(j+1)t}{2^k}\right)}(\tau)} \left(\dot{\gamma}_{c\left(\frac{jt}{2^k}\right),c\left(\frac{(j+1)t}{2^k}\right)}(\tau) \right) \, \mathrm{d}P(\tau) \, .$$

If $P = \delta_0$, we recover the sums that converge to the Itô integral; if $P = \frac{1}{2}(\delta_0 + \delta_1)$, we recover the sums that converge to the Stratonovich integral.

1.2 The statement of the problem

- Not every points x and y of a Riemannian manifold may be joined by a unique minimizing geodesic $\gamma_{x,y} : [0,1] \to M$ with $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$, and therefore we may not define

$$\int_{[0,1]} \alpha_{\gamma_{x,y}(\tau)} \left(\dot{\gamma}_{x,y}(\tau)\right) \, \mathrm{d}P(\tau)$$

for all x and y. We fix this by defining $I_P(\alpha) : M \times M \to \mathbb{R}$ by:

  - ◇ $I_P(\alpha)(x,y) = \int_{[0,1]} \alpha_{\gamma_{x,y}(\tau)}(\dot{\gamma}_{x,y}(\tau)) \, \mathrm{d}P(\tau)$, if there exists a unique minimizing geodesic $\gamma_{x,y}$ as above from x to y;
  - ◇ $I_P(\alpha)(x,y) = 0$, otherwise.

- We want to show that the Riemann-like sums

$$A_{P,t,k}(\alpha)(c) = \sum_{j=0}^{2^k-1} I_P(\alpha) \left(c\left(\frac{jt}{2^k}\right), c\left(\frac{(j+1)t}{2^k}\right) \right)$$

converge in measure to some limit denoted $\mathrm{Int}_P(\alpha)$. When $P = \delta_0$ the limit will turn out to be the Itô integral, and when $P = \frac{1}{2}(\delta_0 + \delta_1)$ the limit will turn out to be the Stratonovich integral.

2. A COMPLEX MEASURE DENSITY WITH RESPECT TO THE WIENER MEASURE

2.1 The space of continuous curves

- Fix a point $x_0 \in M$ once and for all and define the space

$$\mathcal{C}_t = \{c : [0,t] \to M \mid \text{continuous, with } c(0) = x_0\}.$$

Endow $\mathcal{C}_t$ with the distance $D(c_1, c_2) = \max_{s\in[0,t]} d(c_1(s), c_2(s))$, where d is the natural distance on M. This makes $\mathcal{C}_t$ separable!

- Define the (continuous) projections $\pi_k : \mathcal{C}_t \to M^{2^k}$ by

$$\pi_k(c) = \left(c\left(\frac{t}{2^k}\right), \ldots, c\left(\frac{2^k t}{2^k}\right)\right) \; .$$

We say that $f : \mathcal{C}_t \to \mathbb{C}$ is **cylindrical** if and only if there exists $f_k : M^{2^k} \to \mathbb{C}$ such that $f = f_k \circ \pi_k$.

Endow $\mathcal{C}_t$ with the Wiener measure w_t, characterized by the property that

$$\int_{\mathcal{C}_t} f_k \circ \pi_k \, \mathrm{d}w_t = \int_M \mathrm{d}x_1 \, h\left(\frac{t}{2^k}, x_0, x_1\right) \ldots \int_M \mathrm{d}x_{2^k} \, h\left(\frac{t}{2^k}, x_{2^k-1}, x_{2^k}\right) f_k(x_1, \ldots, x_{2^k}) \, .$$

Let Cyl_t be the space of continuous (and therefore bounded, since M is compact) cylindrical functions.

Theorem. Cyl_t *is dense in* $L^1(\mathcal{C}_t)$.

2.2 The heat kernel associated to α

- Consider the differential operator $\nabla = \mathrm{d} + \mathrm{i}\alpha$.

- The operator $\nabla^* \nabla$ defined on $C_0^\infty(M) \subset L^2(M)$ is positive-definite and symmetric, therefore it admits the self-adjoint Friedrichs extension H_α, which in turn generates the semigroup $(\mathrm{e}^{-sH_\alpha})_{s \geq 0}$ în $L^2(M)$.

- This semigroup admits a unique smooth integral kernel h_α, that we shall call **the heat kernel associated to** α: for all $f \in L^2(M)$

$$[\mathrm{e}^{-sH_\alpha} f](x) = \int_M h_\alpha(s, x, y) \, f(y) \, \mathrm{d}y \ .$$

- If $\alpha = 0$, the associated heat kernel is called **the heat kernel** on M, denoted h.

- In general, $|h_\alpha| \leq h$ (**the diamagnetic inequality**).

2.3 A complex measure density

- Define the linear functional $W_{\alpha,t}$ on $\mathrm{Cyl}_t \subset L^1(\mathcal{C}_t)$ by

$$W_{\alpha,t}(f_k \circ \pi_k) = \int_M \mathrm{d}x_1\, h_\alpha\left(\frac{t}{2^k}, x_0, x_1\right) \dots \int_M \mathrm{d}x_{2^k}\, h_\alpha\left(\frac{t}{2^k}, x_{2^k-1}, x_{2^k}\right) f_k(x_1, \dots, x_{2^k}) \ .$$

- From the diamagnetic inequality $|h_\alpha| \leq h$ we get that

$$|W_{\alpha,t}(f_k \circ \pi_k)| \leq \int_M \mathrm{d}x_1\, h\left(\frac{t}{2^k}, x_0, x_1\right) \dots \int_M \mathrm{d}x_{2^k}\, h\left(\frac{t}{2^k}, x_{2^k-1}, x_{2^k}\right) |f_k|(x_1, \dots, x_{2^k}) =$$
$$= \int_{\mathcal{C}_t} |f_k \circ \pi_k|\, \mathrm{d}w_t = \|f_k \circ \pi_k\|_{L^1(\mathcal{C}_t)} \ ,$$

so $W_{\alpha,t}$ is continuous in the norm $\|\cdot\|_{L^1(\mathcal{C}_t)}$.

- Since Cyl_t is dense in $L^1(\mathcal{C}_t)$, we may extend $W_{\alpha,t}$ to a continuous linear functional on $L^1(\mathcal{C}_t)$, so there exists a unique $\rho_{\alpha,t} \in L^\infty(\mathcal{C}_t) = L^1(\mathcal{C}_t)^*$ such that

$$W_{\alpha,t}(f) = \int_{\mathcal{C}_t} f\, \rho_{\alpha,t}\, \mathrm{d}w_t \ .$$

2.4 The explicit formula of $\rho_{\alpha,t}$

- **We have a problem:** $\rho_{\alpha,t}$ is too abstract, therfore difficult to use!
- Consider the sequence of measurable functions on $\mathcal{C}_t$ define by

$$S_{P,t,k}(\alpha)(c) = A_{P,t,k}(\alpha)(c) + \sum_{j=0}^{2^k-1} \frac{t}{2^k} (\mathrm{d}^*\alpha) \left(c\left(\frac{jt}{2^k}\right)\right) \int_{[0,1]} (2\tau - 1)\,\mathrm{d}P(\tau) \ .$$

Theorem. $\mathrm{e}^{\mathrm{i}\, S_{P,t,k}(\alpha)} \to \rho_{\alpha,t}$ *în* $L^2(\mathcal{C}_t)$*, uniformly with respect to* t *from bounded subsets of* $(0,\infty)$*.*

- The core of the proof is the evaluation when $k \to \infty$ of the scalar product

$$\left\langle \rho_{\alpha,t}, \mathrm{e}^{\mathrm{i} S_{P,t,k}(\alpha)} \right\rangle_{L^2(\mathcal{C}_t)} = \int_{\mathcal{C}_t} \mathrm{e}^{-\mathrm{i} S_{P,t,k}(\alpha)}\, \rho_{\alpha,t}\, \mathrm{d}w_t = W_{\alpha,t}(\mathrm{e}^{-\mathrm{i} S_{P,t,k}(\alpha)}) =$$

$$= \int_M \mathrm{d}x_1\, h_\alpha\left(\frac{t}{2^k}, x_0, x_1\right) \mathrm{e}^{-\mathrm{i}\, I_P(\alpha)(x_0,x_1) - \frac{t}{2^k}(\mathrm{d}^*\alpha)(x_0) \int_{[0,1]}(2\tau-1)\,\mathrm{d}P(\tau)} \dots$$

$$\dots \int_M \mathrm{d}x_{2^k}\, h_\alpha\left(\frac{t}{2^k}, x_{2^k-1}, x_{2^k}\right) \mathrm{e}^{-\mathrm{i}\, I_P(\alpha)(x_{2^k-1},x_{2^k}) - \frac{t}{2^k}(\mathrm{d}^*\alpha)(x_{2^k-1}) \int_{[0,1]}(2\tau-1)\,\mathrm{d}P(\tau)} =$$

$$= \left[\left(T_{\frac{t}{2^k}} \right)^{2^k} 1 \right](x_0) = \int_M h(t, x_0, x)\, 1\, \mathrm{d}x = w_t(\mathcal{C}_t) \ ,$$

where

$$(T_s f)(x) = \int_M h_\alpha(s, x, y)\, \mathrm{e}^{-\mathrm{i}\, I_P(\alpha)(x,y) - s\, (\mathrm{d}^*\alpha)(x) \int_{[0,1]} (2\tau - 1)\, \mathrm{d}P(\tau)}\, f(y)\, \mathrm{d}y\ .$$

• Using in $C(M)$ Chernoff's theorem about the approximation of contraction semigroups in Banach spaces, we get that

$$\lim_{k\to\infty} \left(T_{\frac{t}{2^k}} \right)^{2^k} 1 = \mathrm{e}^{-tL}\, 1\ ,$$

where L is the generator of the heat semigroup acting in $C(M)$.

• The term containing $\mathrm{d}^*\alpha$ stems naturally from the need to satisfy one of the hypotheses of this theorem. The uniformity with respect to t comes from Chernoff's theorem, too.

• It is interesting to notice that P is nowhere to be found in the limit! This will soon turn out to have major consequences.

• Under these conditions we shall show that $\rho_{\alpha,t}$ is the exponential of a measurable function.

Theorem. *There exists a unique real function* $\operatorname{Strat}_t(\alpha) \in L^0(\mathcal{C}_t)$ *such that* $\rho_{\alpha,t} = \mathrm{e}^{\mathrm{i}\,\operatorname{Strat}_t(\alpha)}$.

Proof. Define $U_s : L^2(\mathcal{C}_t) \to L^2(\mathcal{C}_t)$ by $U_s f = \rho_{s\alpha,t} f$.

We shall show that $\mathbb{R} \ni s \mapsto U_s \in \mathcal{B}(L^2(\mathcal{C}_t))$ satisfies the hypotheses in Stone's theorem.

- Since $\mathrm{e}^{\mathrm{i}\,S_{P,t,k}(\alpha)} \to \rho_{\alpha,t}$ in $L^2(\mathcal{C}_t)$, it is clear that U_s is unitary.
- To show that $s \mapsto U_s$ is a group,

$$U_{s+s'} f = \rho_{(s+s')\alpha,t} f = \lim_k \mathrm{e}^{\mathrm{i}\,S_{P,t,k}((s+s')\alpha)} = \lim_k \mathrm{e}^{\mathrm{i}\,S_{P,t,k}(s\alpha)} \mathrm{e}^{\mathrm{i}\,S_{P,t,k}(s\alpha)} = \rho_{s\alpha,t} \rho_{s'\alpha,t} f =$$
$$= U_s U_{s'} f \ .$$

- To show the strong continuity, we use that $\mathcal{C}_t$ is separable, so $L^2(\mathcal{C}_t)$ is separable. It follows that strong continuity with respect to s is equivalent to weak measurability, and

$$\langle U_s f, g \rangle_{L^2(\mathcal{C}_t)} = \langle \rho_{s\alpha,t} f, g \rangle_{L^2(\mathcal{C}_t)} = \lim_{k\to\infty} \langle \mathrm{e}^{\mathrm{i} S_{P,t,k}(s\alpha)} f, g \rangle_{L^2(\mathcal{C}_t)} \ ,$$

which is measurable being the pointwise limit of a sequence of measurable functions.

From Stone's theorem, there exists a unique self-adjoint generator $\mathrm{Strat}_t(\alpha)$ such that

$$\rho_{s\alpha,t} f = U_s f = \mathrm{e}^{\mathrm{i}\ \mathrm{Strat}_t(\alpha)} f \ .$$

One shows that $\mathrm{Strat}_t(\alpha)$ commutes with all the functions in $L^\infty(\mathcal{C}_t)$, so it is itself the multiplication by some function from $L^0(\mathcal{C}_t)$, which we shall denote by $\mathrm{Strat}_t(\alpha)$, too. It is real-valued, because $\mathrm{Strat}_t(\alpha)$ is self-adjoint. □

- Notice that $\mathrm{Strat}_t(\alpha)$ does not depend on P, even though its construction does depend on it!

- We have seen that $\mathrm{e}^{\mathrm{i}\, S_{P,t,k}(\alpha)} \to \mathrm{e}^{\mathrm{i}\,\mathrm{Strat}_t(\alpha)}$ in $L^2(\mathcal{C}_t)$. What can one say about the convergence of the exponents?

Theorem. *$S_{P,t,k} \to \mathrm{Strat}_t(\alpha)$ in $L^0(\mathcal{C}_t)$ (i.e. in measure), uniformly with respect to t from bounded subsets of $(0,\infty)$.*

Proof. Using the resolvents of $S_{P,t,k}$ and $\mathrm{Strat}_t(\alpha)$ we have that

$$\left\| \frac{1}{\mathrm{i} + S_{P,t,k}(\alpha)} - \frac{1}{\mathrm{i} + \mathrm{Strat}_t(\alpha)} \right\|_{L^2(\mathcal{C}_t)} = \| R_{-\mathrm{i}}(S_{P,t,k}(\alpha)) - R_{-\mathrm{i}}(\mathrm{Strat}_t(\alpha)) \|_{L^2(\mathcal{C}_t)} =$$
$$= \| R_{-\mathrm{i}}(S_{P,t,k}(\alpha))\, 1 - R_{-\mathrm{i}}(\mathrm{Strat}_t(\alpha))\, 1 \|_{L^2(\mathcal{C}_t)} \leq$$
$$\leq \int_0^\infty \mathrm{e}^{-s}\, \| \mathrm{e}^{\mathrm{i} S_{P,t,k}(s\alpha)} - \rho_{s\alpha,t} \|_{L^2(\mathcal{C}_t)}\, \mathrm{d}s \to 0$$

and now we finish with the dominated convergence theorem. □

3. A GENERAL CONCEPT OF STOCHASTIC INTEGRAL

• If c is a twice continuously differentiable curve then $A_{P,t,k} \to \int_c \alpha$. If we want the stochastic integrals to be probabilistic analogues of the line integral, then...

Definition. *We shall say that* $\mathrm{Int}_{P,t} : \Omega^1(M) \times L^0(\mathcal{C}_t) \to \mathbb{R}$ *is a* ***stochastic integral*** *if and only if* $A_{P,t,k}(\alpha) \to \mathrm{Int}_{P,t}(\alpha)$ *în* $L^0(\mathcal{C}_t)$.

• In the above definition we have given up the uniformity with respect to t because it is not clear whether this is an essential property or a merely accidental one.

• **Important**: the stochastic integral $\mathrm{Int}_{P,t}$ is not a function, but an element from $L^0(\mathcal{C}_t)$, so it is not defined pointwise!

- Is there any relationship between $\mathrm{Int}_{P,t}$ and Strat_t? Yes, there is!

If we pass to the limit in

$$S_{P,t,k}(\alpha)(c) = A_{P,t,k}(\alpha)(c) + \sum_{j=0}^{2^k-1} \frac{t}{2^k}(\mathrm{d}^*\alpha)\left(c\left(\frac{jt}{2^k}\right)\right)\int_{[0,1]}(2\tau - 1)\,\mathrm{d}P(\tau)\ ,$$

obținem că

$$\mathrm{Strat}_t(\alpha) = \mathrm{Int}_{P,t}(\alpha) + \int_{[0,1]}(2\tau - 1)\,\mathrm{d}P(\tau)\int_0^t (\mathrm{d}^*\alpha)(c(s))\,\mathrm{d}s\ .$$

3.1 Consequeces

This has several important consequences:

- The only thing that the operator $\mathrm{Int}_{P,t}$ retains from P is its first order moment: $M_1(P) = \int_{[0,1]} \tau \, \mathrm{d}P(\tau)$.
- Any two stochastic integrals $\mathrm{Int}_{P,t}$ and $\mathrm{Int}_{Q,t}$ are connected by the simple formula

$$\mathrm{Int}_{P,t}(\alpha)(c) = \mathrm{Int}_{Q,t}(\alpha)(c) - 2 \int_{[0,1]} \tau \, \mathrm{d}(P - Q)(\tau) \int_0^t (\mathrm{d}^* \alpha)(c(s)) \, \mathrm{d}s \ ,$$

so there exists essentially a single stochastic integral (any one of them), all the others being mere translations of this one by a simple term.

- If P is the Lebesgue measure $\mathrm{Leb}_{[0,1]}$, or $\delta_{\frac{1}{2}}$, or $\frac{1}{2}(\delta_0 + \delta_1)$, then $M_1(P) = \frac{1}{2}$, hence $\mathrm{Strat}_t = \mathrm{Int}_{P,t}$. Comparing this with the probabilistic literature, we discover that Strat_t is the Stratonovich integral!
- If $P = \delta_0$, then comparing this with the probabilistic literature we discover that $\mathrm{Int}_{\delta_0,t}$ is the Itô integral (that we shall denote Ito_t)!

3.2 Properties of the stochastic integrals

Theorem. *The Stratonovich and the Itô integral of* α *are equal if and only if* $\mathrm{d}^*\alpha = 0$.

Proof.

$$\mathrm{Strat}_t(\alpha)(c) = \mathrm{Ito}_t(\alpha)(c) - \int_0^t (\mathrm{d}^*\alpha)(c(s))\,\mathrm{d}s \ .$$

□

The proof of this fact is surprisingly complicated in the probabilistic approach, while it is elementary in the one presented here!

Theorem. $\mathrm{Strat}_t(\mathrm{d}f)(c) = f(c(1)) - f(x_0)$ *for every real-valued continuously differentiable function* f *and for almost all* $c \in \mathcal{C}_t$.

Proof. (Just the intuition, without the technical details.)

Choosing $P = \mathrm{Leb}_{[0,1]}$ we get

$$\mathrm{Strat}_t(\mathrm{d}f)(c) = \lim_k A_{\mathrm{Leb}_{[0,1]},t,k}(\mathrm{d}f)(c) = \lim_k \sum_{j=0}^{2^k-1} I_{\mathrm{Leb}_{[0,1]}}(\mathrm{d}f)\left(c\left(\frac{jt}{2^k}\right), c\left(\frac{(j+1)t}{2^k}\right)\right) \text{"} = \text{"}$$

$$\text{"} = \text{"} \lim_k \sum_{j=0}^{2^k-1} \int_0^1 (\mathrm{d}f)_{\gamma_{c\left(\frac{jt}{2^k}\right),c\left(\frac{(j+1)t}{2^k}\right)}(\tau)} \left(\dot{\gamma}_{c\left(\frac{jt}{2^k}\right),c\left(\frac{(j+1)t}{2^k}\right)}(\tau)\right) \mathrm{d}\tau =$$

$$= \lim_k \sum_{j=0}^{2^k-1} \left[f\left(c\left(\frac{(j+1)t}{2^k}\right)\right) - f\left(c\left(\frac{jt}{2^k}\right)\right)\right] = f(c(1)) - f(c(0)) \ .$$

$\square$

As an immediate consequence we have "Itô's lemma".

Theorem. *If $f : M \to \mathbb{R}$ is twice continuously differentiable, and if Δ is the Laplace-Beltrami on M, then*

$$f(c(1)) = f(x_0) + \mathrm{Ito}_t(\mathrm{d}f)(c) + \int_0^t (\Delta f)(c(s))\,\mathrm{d}s$$

for almost all $c \in \mathcal{C}_t$.

Proof. Using the previous theorem, the proof is short and elementary:

$$f(c(1)) = f(x_0) + \mathrm{Strat}_t(\mathrm{d}f)(c) = f(x_0) + \mathrm{Ito}_t(\mathrm{d}f)(c) - \int_0^t \mathrm{d}^*(\mathrm{d}f)(c(s))\,\mathrm{d}s =$$
$$= f(x_0) + \mathrm{Ito}_t(\mathrm{d}f)(c) + \int_0^t (\Delta f)(c(s))\,\mathrm{d}s\ ,$$

where we have used the basic Hodge-theoretic formula $\Delta = -\mathrm{d}^*\mathrm{d}$. □

The Stratonovich integral is special among the stochastic integrals, it being the one that has emerged naturally as the generator of a one-parameter unitary group. In the following we shall see one more argument in favour of its special nature.

In the following theorem $\mathcal{C}_{t,x}$ is the space of the continuous curves that begin at x; $w_{t,x}$ is the Wiener measure on it, and $\mathrm{Strat}_{t,x}$ is the Stratonovich integral on it.

Theorem (The Feynman-Kac-Itô formula)**.** *Let* $V : M \to \mathbb{R}$ *be continuous, with* $\inf V > -\infty$*. If* $f \in L^2(M)$*, then*

$$(\mathrm{e}^{-tH_{\mathrm{d}+\mathrm{i}\alpha,V}} f)(x) = \int_{\mathcal{C}_{t,x}} \mathrm{e}^{\mathrm{i}\,\mathrm{Strat}_{t,x}(\alpha) - \int_0^t V(c(s))\,\mathrm{d}s} f(c(t))\,\mathrm{d}w_{t,x}(c)$$

for all $t > 0$ *and almost all* $x \in M$*.*

4. FINAL COMMENTS

Many of the properties of the stochastic integrals have easy proofs in the theoretical framework constructed above because the technical difficulties are taken care of by classical functional analysis theorems:

- the existence of the Friedrichs extension of $(\mathrm{d}+\mathrm{i}\alpha)^*(\mathrm{d}+\mathrm{i}\alpha)$, which generates a contraction seimgroup in $L^2(M)$;
- the fact that this semigroup admits an integral kernel h_α;
- the fact that h_α is smooth (which follows from the hypoellipticity of $\partial_t + H_\alpha$);
- the diamagnetic inequality;
- Chernoff's theorem about the approximation of contraction semigroups in Banach spaces;
- Stone's theorem.

4.1 Improving the results

- If α has compact support then $\mathrm{Int}_{P,t}(\alpha) \in L^2(\mathcal{C}_t)$.
- Moreover, if M is a Riemannian symmetric space, then $A_{\delta_0,t,k}(\alpha) \to \mathrm{Ito}_t(\alpha)$ in $L^2(\mathcal{C}_t)$.

4.2 The case of arbitrary manifolds

- The convergence $e^{i\,S_{P,t,k}(\alpha)} \to \rho_{\alpha,t}$ in $L^2(\mathcal{C}_t)$ has been proved using Chernoff's theorem in an essential way. Its use forces us to work with compact manifolds (possibly with boundary) for two technical reasons:

  ◇ First, it is necessary to come up with an essential domain for the generator of the heat semigroup on $C_b(M)$, made of compactly-supported smooth functions; such a domain is not known so far on arbitrary manifolds.

  ◇ Second, it is necessary to show that certain integrals are finite, the integrands of which are very general, their only regularity property being their continuity.

- In order to treat the case of arbitrary manifolds, the solution is to obtain the desired conclusions on relatively-compact connected domains with smooth boundary, followed by the exhaustion of the manifold with such domains.