School of Advanced Studies of the Romanian Academy
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Summary of Ph.D. Thesis

Stochastic models for nonlinear equations

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Bucharest, 2022

CONTENTS

Keywords. Nonlinear evolution equation, superprocess, non-local branching process, measure-valued process, branching semigroup, continuous additive functional, killed process, excessive function, compact Lyapunov function, right continuous flow, semi-dynamical system, absorbing set, weak generator, log-potential.

INTRODUCTION

This thesis is the result of a Ph. D. Program started in 2017 under the affiliation of the Simion Stoilow Institute of Mathematics of the Romanian Academy (IMAR) in Bucharest, Romania. It is concerned with measure-valued branching processes and the nonlinear equations that describe their underlying dynamics. Thus, a defining characteristic of the exposition is the two-folded approach, both from the probability field perspective of proving such branching processes exist and studying their properties, and from the PDE point of view, to obtain probabilistic representations for the solutions of the above mentioned nonlinear equations.

In the first part, we investigate discrete branching processes arising from the time evolution of systems of particles, in the case when the descendants do not need to start from the place where the parent died, a so-called non-local branching process. The main motivation is to generalise existing results to the case when the killing measure has no density w.r.t. the Lebesgue measure. This generalisation translates itself into considering an abstract continuous additive functional of the base process associated to the perturbing measure by the Revuz correspondence. We first study the properties of a convenient class of such continuous additive functional, rich enough to accept applications of interest. Further, our approach consists in obtaining the mild solutions of the associated nonlinear PDE, which involve the aforementioned continuous additive functional. Our exposition afterwards concentrates in proving the existence of a sufficiently regular branching process having the obtained semigroup as its transition function. The strategy of the proof involves first building an auxiliary process based on a linearised version of the above mentioned nonlinear PDE. The completion of the proof requires several probabilistic and analytical potential theory methods such as the enlargement of the base space and tightness of capacities. Lastly, having obtained the branching process, we return to the PDE aspect of the problem, another main objective, and we obtain a probabilistic representation of the solutions of the nonlinear equation we started with.

In the second part, we study branching processes driven by continuous flows. We show that if the branching mechanism of a superprocess is independent of the spatial variable, then the superprocess is obtained by introducing the branching in the time evolution of the right continuous flow on measures, canonically induced by a right continuous flow as spatial motion. A corresponding result holds for non-local branching processes on the set of all finite configurations of the state space of the spatial motion, provided that the branching procedure is compatible with the right continuous flow. As in the previous part, we give probabilistic representations to the associated nonlinear equations. We also deduce aspects regarding the extended weak generators for these processes.

The thesis presents applications to both parts and highlights their relevance to the current research in this field.

The original results from this thesis are included in

- L. Beznea, O. Lupaşcu-Stamate, and C. I. Vrabie, Stochastic solutions to evolution equations of non-local branching processes, *Nonlinear Analysis* **200** (2020), 112021. https://doi.org/10.1016/j.na.2020.112021;

- L. Beznea and C. I. Vrabie, Continuous flows driving branching processes and their nonlinear evolution equations, *Adv. in Nonlinear Anal.* **11** (2022), 921–936, https://doi.org/10.1515/anona-2021-0229.

1. PRELIMINARIES

The purpose of this chapter is to introduce the preliminary notions that appear in the sequel. We deal with the fundamental notions regarding transition functions and resolvents of kernels, that are omnipresent throughout the thesis, and we define our framework of study for stochastic processes, the right processes, mentioning the regularity properties we are interested in.

2. EVOLUTION EQUATIONS RELATED TO NON-LOCAL BRANCHING PROCESSES

Non-local branching processes arise in modeling the time evolution of a system of particles, which we can describe as follows. A particle starts at a point of a set E and moves according to a Markov process with state space E (called the *base movement*) until a random terminal time when it is destroyed. The particle is then replaced by a finite number of new particles which move further independently, according to the same base movement, until their own terminal times when they are destroyed and replaced by second generation particles, and the process continues in this manner; cf., e.g., [20] and [12].

Consider a right (Markov) process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}^x)$ on E with infinite lifetime, transition function $(T_t)_{t\geqslant 0}$, $T_t f(x) = \mathbb{E}^x(f(X_t))$, $f \in \mathcal{B}_+(E)$, $x \in E$, $t \geqslant 0$, and infinitesimal generator L, which we call the *base movement* or the *spatial motion*; it characterizes the movement of particles between the branching moments.

We fix m to be a $\mathcal{U}$-excessive measure, where $\mathcal{U}$ is the resolvent of kernels associated to X.

The killing rate is given through a perturbing positive smooth measure μ on E. By the Revuz correspondence (see, e.g., [4], [17]), the killed process is obtained from X by killing with the multiplicative functional $(e^{-A_t})_{t\geqslant 0}$, where $A = (A_t)_{t\geqslant 0}$ is a continuous additive functional of X having the Revuz measure μ (w.r.t the measure m). We define the semigroup of sub-Markovian kernels $(T_t^\mu)_{t\geqslant 0}$ by the Feynman-Kac formula

$$T_t^\mu f(x) := \mathbb{E}^x(e^{-A_t} f(X_t)) \ \text{ for all } f \in \mathcal{B}_+(E).$$

It turns out that $(T_t^\mu)_{t\geqslant 0}$ also induces a C_0-semigroup on $L^p(E,m)$ and its L^p-generator may be identified with $L-\mu$, using the Revuz correspondence, in an L^p-weak sense as in [9].

Finally, the birth of new particles is given by a sequence of Markovian kernels B_k from $E^{(k)}$ (the symmetric k-th power of E) to E, for each $k \geqslant 1$, $B_k : \mathrm{b}\mathcal{B}_+(E^{(k)}) \longrightarrow \mathrm{b}\mathcal{B}_+(E)$, and $(b_k)_{k\geqslant 1}$, a sequence of positive Borelian functions on E such that $\sum_{k\geqslant 1} b_k \leqslant 1$. The interpretation is the following: for every $k \geqslant 1$, $b_k(x)$ represents the probability that a particle dies at $x \in E$ and has precisely k descendants; $B_{k,x}$ (the probability on $E^{(k)}$ induced by the Markovian kernel B_k at x) is the distribution of the k descendants in $E^{(k)}$, conditioned that the parent died at $x \in E$.

We can therefore formulate our goal as to study the branching process on $\widehat{E}$, having the underlying (nonlinear) generator on E of the form

$$\mathcal{L}u \ = \ Lu - \mu u + \mu(\sum_{k\geqslant 1} b_k B_k \widehat{u}).$$

We emphasize that another main objective is to study the parabolic PDE associated to this nonlinear operator and obtain a probabilistic representation for its solutions.

The case when

$$\mathcal{L}u \ = \ Lu - \mu u + \mu(\sum_{k\geqslant 1} b_k u^k)$$

was considered by E.B. Dynkin in [14] and corresponds to the particular situation when $B_{k,x} = \delta_{\mathbf{x}}$ for every $x \in E$ and $k \geqslant 1$, where $\mathbf{x} := (x, \ldots, x) \in E^{(k)}$, hence, the descendants start from the point where the parent died. The resulting branching process is called a *local branching process*. By contrast, in our situation, we do not have this restriction, i.e. the descendants do not need to start from the point where the parent died. Therefore, the branching process is called a *non-local branching process*.

In the particular case when A has a density w.r.t. the Lebesgue measure, i.e. $A_t = \int_0^t c(X_s)\mathrm{d}s$, $t \geqslant 0$, then the generator becomes

$$\mathcal{L}u \ = \ Lu - cu + c\sum_{k\geqslant 1} b_k B_k \widehat{u}$$

and the associated branching process was constructed in [6].

2.1. Measure-valued branching processes

In the first section, for the convenience of the reader, we introduce the preliminary notions regarding measure-valued branching processes.

Let E be a Lusin topological space. We denote by $M(E)$ the set of all positive finite measures on E. We endow $M(E)$ with the weak topology and we denote by $\mathcal{M}(E)$ the corresponding Borel σ-algebra on $M(E)$.

A second measure space we consider is the set $\widehat{E} \subseteq M(E)$ of all finite sums of Dirac measures on E,

$$\widehat{E} := \left\{ \sum_{i \leqslant i_0} \delta_{x_i} : i_0 \in \mathbb{N}, i_0 \geqslant 1, x_i \in E \text{ for all } 1 \leqslant i \leqslant i_0 \right\} \cup \{\mathbf{0}\},$$

where $\mathbf{0}$ denotes the zero measure. The set $\widehat{E}$ is identified with the union of all symmetric k-th powers $E^{(k)}$ of E,

$$\widehat{E} = \bigcup_{k \geqslant 0} E^{(k)},$$

where $E^{(0)} := \{\mathbf{0}\}$ (see, e.g., [18]). The set $\widehat{E}$ is called the *space of finite configurations of* E and it is endowed with the weak topology on the finite measures on E and the corresponding Borel σ-algebra $\mathcal{B}(\widehat{E})$.

Further $(M, \mathcal{M})$ denotes either $(M(E), \mathcal{M}(E))$ or $(\widehat{E}, \mathcal{B}(\widehat{E}))$.

A right Markov process with state space M is called *branching process* provided that for any two independent copies X and X' of the given process on M, starting respectively from two measures μ and μ', $X + X'$ and the process starting from $\mu + \mu'$ are equal in distribution.

A bounded kernel Q on $(M, \mathcal{M})$ is called *branching kernel* provided that

$$Q_{\mu+\nu} = Q_\mu * Q_\nu \text{ for all } \mu, \nu \in M,$$

where recall that Q_μ denotes the measure on M such that $\int_M h \mathrm{d}Q_\mu = Qh(\mu)$ for all $h \in \mathcal{M}_+$.

It turns out that: *a right Markov process with state space M is a branching process if and only if its transition function is formed by branching kernels.*

For a function $f \in \mathrm{b}\mathcal{B}_+(E)$ we consider the mappings $l_f : M \longrightarrow \mathbb{R}_+$ (the linear functional) and $e_f : M \longrightarrow [0,1]$ (the exponential functional) defined as

$$l_f(\mu) := \langle \mu, f\rangle := \int_E f \mathrm{d}\mu,\ \mu \in M, \quad e_f := e^{-l_f}.$$

For every real-valued, positive, $\mathcal{B}(E)$-measurable function φ define the *multiplicative function* $\widehat{\varphi} : \widehat{E} \longrightarrow \mathbb{R}_+$ as

$$\widehat{\varphi}(\mathbf{x}) = \begin{cases} \prod_k \varphi(x_k), \text{ if } \mathbf{x} = (x_k)_{k\geqslant 1} \in \widehat{E}, \mathbf{x} \neq \mathbf{0}, \\ 1, \quad\quad \text{if } \mathbf{x} = \mathbf{0}, \end{cases}$$

cf. [20]; see also [8].

2.2. Admissible continuous additive functionals

According to [19], page 33, our working assumption is that the CAF $A = (A_t)_{t\geqslant 0}$ is *admissible*, that is,

$$\lim_{t\searrow 0} \sup_{x\in E} \mathbb{E}^x(A_t) = 0.$$

We consider the following property of the continuous additive functional $A = (A_t)_{t\geqslant 0}$:

$$e^{-\alpha t}\mathbb{E}^x e^{-A_t} + \beta\ \mathbb{E}^x \int_0^t e^{-(A_u+\alpha u)}\mathrm{d}A_u \leqslant 1 \ \text{ for all } t, x, \tag{2.1}$$

with $\alpha \geqslant 0$ and $\beta > 0$.

Lemma 2.1. *([7]) Assume that there exists a sequence $A^n = (A^n_t)_{t\geqslant 0}$, $n \geqslant 1$, of continuous additive functionals such that each A^n satisfies* (2.1) *with constants α_n and β_n. Suppose that A^n_t converges a.s. to A_t for each $t > 0$, $\lim_{n\to\infty} \alpha_n = \alpha \in \mathbb{R}$ and $\lim_{n\to\infty} \beta_n = \beta \in \mathbb{R}_+$, $\beta \neq 0$. Then A satisfies* (2.1) *with these constants α and β.*

Lemma 2.2. *([7]) The following assertions hold.*

(i) For an arbitrary CAF, $A = (A_t)_{t\geqslant 0}$, (2.1) *holds with any $\beta \leqslant 1$ and $\alpha \geqslant 0$.*

(ii) If the CAF A is admissible then for each $\beta > 0$ there exists $\alpha > 0$ such that (2.1) *holds.*

(iii) If the CAF A has a density $c \in \mathrm{b}\mathcal{B}_+(E)$ w.r.t. the Lebesgue measure, i.e., for all $t > 0$ we have $A_t = \int_0^t c(X_s)\mathrm{d}s$, then it is admissible. In addition, $(A_t)_{t\geqslant 0}$ satisfies (2.1) *for any α and β such that $\alpha \geqslant \|c\|_\infty(\beta - 1)$.*

2.3. The integral evolution equation

In the third section, we construct a nonlinear integral evolution equation involving the base process X and the continuous additive functional A. A particular case of it leads us to the mild solutions of the parabolic PDE written as

$$\frac{\mathrm{d}}{\mathrm{d}t} H_t\varphi = \mathcal{L} H_t\varphi,\ t \geqslant 0, \tag{2.2}$$

with the initial condition $H_0\varphi = \varphi$, $0 \leqslant \varphi \leqslant 1$, where recall that $\mathcal{L}$ is the nonlinear operator on E, $\mathcal{L}u = Lu - \mu u + \mu(\sum_{k\geqslant 1} b_k B_k \widehat{u})$.

The pair of sequences $((B_k)_{k\geqslant 1}, (b_k)_{k\geqslant 1})$ induces a sub-Markovian kernel B from $\widehat{E}$ to E,

$$Bg := \sum_{k\geqslant 1} b_k B_k(g|_{E^{(k)}}) \text{ for all } g \in \mathcal{B}_+(\widehat{E}).$$

For simplicity we write $B_k g$ instead of $B_k(g|_{E^{(k)}})$ and with this convention we have

$$B = \sum_{k\geqslant 1} b_k B_k.$$

Consider $m_1 := \| \sum_{k\geqslant 1} k b_k \|_\infty$ and assume that $m_1 < \infty$. Denote by $\mathcal{B}_{\mathsf{u}}$ the set of all $\varphi \in \mathcal{B}_+(E)$ such that $\varphi \leqslant 1$.

A map $H : \mathcal{B}_{\mathsf{u}} \longrightarrow \mathcal{B}_{\mathsf{u}}$ is called *absolutely monotonic* provided that there exists a sub-Markovian kernel $\mathbf{H} : \mathrm{b}\mathcal{B}_+(\widehat{E}) \longrightarrow \mathrm{b}\mathcal{B}_+(E)$ such that $H\varphi = \mathbf{H}\widehat{\varphi}$ for all $\varphi \in \mathcal{B}_{\mathsf{u}}$.

Theorem 2.3. *([7]) Let $\alpha \geqslant 0$ and $\beta > 0$ be such that* (2.1) *holds. Then for any $\varphi \in \mathcal{B}_{\mathsf{u}}$ the evolution equation*

$$h_t(x) \;=\; e^{-\alpha t}\mathbb{E}^x(e^{-A_t}\varphi(X_t)) + \beta\mathbb{E}^x\int_0^t e^{-A_u-\alpha u} B\widehat{h_{t-u}}(X_u)\mathrm{d}A_u, \tag{2.3}$$

has a unique locally bounded solution $[0,\infty)\times E \ni (t,x) \longmapsto H_t\varphi(x)$ such that $H_t\varphi \in \mathcal{B}_{\mathsf{u}}$ and the following assertions hold.

(i) The map $\varphi \longmapsto H_t\varphi$ is an absolutely monotonic operator and there exists a positive increasing function $t \longmapsto C(t)$ such that

$$\sup_{0\leqslant s\leqslant t} \|H_s(\varphi) - H_s(\psi)\|_\infty \leqslant C(t)\|\varphi - \psi\|_\infty \;\; \textit{for all} \;\; \varphi,\psi \in \mathcal{B}_{\mathsf{u}}.$$

In particular, for any $t \geqslant 0$ the map $\varphi \longmapsto H_t\varphi$ is Lipschitz.

(ii) The family $(H_t)_{t\geqslant 0}$ *is a nonlinear sub-Markovian semigroup of operators on* $\mathcal{B}_{\mathsf{u}}$*. Moreover, if* $\sum_{k\geqslant 1} b_k = 1$ *and* α*,* β *satisfy* (2.1) *with equality, then* $H_t 1 = 1$ *for all* $t \geqslant 0$*.*

(iii) If the function $t \longmapsto \mathbb{E}^x(e^{-A_t}\varphi(X_t))$ *is right continuous for all* $x \in E$*, then* $t \longmapsto H_t\varphi(x)$ *is also right continuous on* $[0,\infty)$*.*

(iv) If $T_t\varphi$ *converges to* φ *(i.e.* $\mathbb{E}^{\cdot}\varphi(X_t) \longrightarrow \varphi$ *) uniformly as* $t \searrow 0$*, then* $H_t\varphi$ *also converges uniformly to* φ *as* $t \searrow 0$*.*

In addition, it is required the following lemma, adapted from the monograph [19], Proposition 2.12, which leads to a main instrument in the proof of Theorem 2.3, as a substitute for Gronwall's Lemma and for proving the semigroup property from assertion *(ii)*.

Lemma 2.4. *([7]) Let* $r > 0$ *and* $\varphi \in \mathcal{B}_{\mathsf{u}}$*. Then the function* $(t,x) \longmapsto h_t(x)$ *is a solution of* (2.3) *for all* $t \geqslant 0$ *if and only if it satisfies the equation* (2.3) *for* $0 \leqslant t \leqslant r$ *and* $(t,x) \longmapsto h_{t+r}(x)$ *solves the equation*

$$h_{t+r}(x) \;=\; e^{-\alpha t}\mathbb{E}^x(e^{-A_t}h_r(X_t)) \;+\; \beta\mathbb{E}^x\int_0^t e^{-A_u-\alpha u}B\widehat{h_{t+r-u}}(X_u)\mathrm{d}A_u,\; t \geqslant 0. \quad (2.4)$$

From now on we denote by $(H_t)_{t\geqslant 0}$ *the nonlinear semigroup given by equation (2.3) when* $\alpha = 0$ *and* $\beta = 1$

Corollary 2.5. *([7]) For* $t \geqslant 0$ *consider the branching kernel* $\widehat{\mathbf{H}}_t$ *on* $\widehat{E}$ *such that* $H_t\varphi = \widehat{\mathbf{H}}_t\widehat{\varphi}|_E$ *for all* $\varphi \in \mathcal{B}_{\mathsf{u}}$*. Then the following assertions hold.*

(i) $(\widehat{\mathbf{H}}_t)_{t\geqslant 0}$ *is a sub-Markovian semigroup of branching kernels on* $(\widehat{E}, \mathcal{B}(\widehat{E}))$*.*

(ii) For all $t \geqslant 0$ *and* $f \in \mathrm{b}\mathcal{B}_+(E)$ *define* $V_t f \in \mathrm{b}\mathcal{B}_+(E)$ *by* $V_t f := -\ln H_t(e^{-f})$*. Then* $(V_t)_{t\geqslant 0}$ *is a nonlinear semigroup on* $\mathrm{b}\mathcal{B}_+(E)$ *(called the* cumulant semigroup*) and*

$$\widehat{\mathbf{H}}_t\widehat{\varphi} = \widehat{e^{-V_t f}},$$

where $\varphi := e^{-f}$*.*

2.4. Construction of the auxiliary process

In the fourth and last section of this chapter, we construct an auxiliary right Markov process on E, based on a linearised version of the integral evolution equation from the previous section.

Theorem 2.6. *([7]) Let $\alpha \geqslant 0$ and $\beta > 0$ be such that* (2.1) *holds and K be a sub-Markovian kernel on E. Then for any $f \in \mathrm{b}\mathcal{B}_+(E)$ the linear evolution equation*

$$r_t(x) = e^{-\alpha t} T_t^\mu f(x) + \beta \mathbb{E}^x \int_0^t e^{-A_u - \alpha u} K r_{t-u}(X_u) \mathrm{d}A_u, \ t \geqslant 0, \tag{2.5}$$

has a unique solution $Q_t f \in \mathrm{b}\mathcal{B}_+(E)$, the function $[0, \infty) \times E \ni (t, x) \longmapsto Q_t f(x)$ is measurable, and the following assertions hold.

(i) The family $(Q_t)_{t \geqslant 0}$ is a semigroup of sub-Markovian kernels on $(E, \mathcal{B}(E))$ and it is the transition function of a Borel right process with state space E.

(ii) The function $t \longmapsto Q_t f(x)$ is right continuous on $[0, \infty)$ for all $x \in E$ if and only if the transition function $t \longmapsto T_t^\mu f(x)$ has the same property.

(iii) The resolvent of kernels $\mathcal{U}^o = (U_q^o)_{q>0}$ on $(E, \mathcal{B}(E))$ induced by $(Q_t)_{t \geqslant 0}$ satisfies

$$U_q^o \ = \ U_{\alpha+q}^\mu + J_q K U_q^o \ = \ U_{\alpha+q}^\mu + G_q U_{\alpha+q}^\mu,$$

where $\mathcal{U}^\mu = (U_q^\mu)_{q>0}$ is the resolvent of kernels induced by $(T_t^\mu)_{t \geqslant 0}$, and J_q, G_q are the bounded kernels on E defined as

$$J_q f(x) := \beta \mathbb{E}^x \int_0^\infty e^{-A_u - \alpha u} e^{-qu} f(X_u) \mathrm{d}A_u \ \text{ and } \ G_q := \sum_{k=0}^\infty (J_q K)^k.$$

In addition we have $\mathcal{E}(\mathcal{U}_q^o) \subset \mathcal{E}(\mathcal{U}_{\alpha+q}^\mu)$, $G_q(\mathcal{E}(\mathcal{U}_{\alpha+q}^\mu)) \subset \mathcal{E}(\mathcal{U}_q^o)$, and $[\mathrm{b}\mathcal{E}(\mathcal{U}_q^o)] = [\mathrm{b}\mathcal{E}(\mathcal{U}_q^\mu)]$.

Proposition 2.7. *([7]) There exists a sub-Markovian kernel K on $(E, \mathcal{B}(E))$ and a semigroup $(Q_t)_{t \geqslant 0}$ which solves the evolution equation (2.5) and such that*

$$e^{-\alpha t} \widehat{\mathbf{H}}_t(l_f) = l_{Q_t f} \ \text{ for all } f \in \mathrm{b}\mathcal{B}_+(E) \text{ and } t > 0,$$

where $\alpha \geqslant 0$ is such that (2.1) is satisfied for $\beta = m_1$.

3. CONSTRUCTION OF NON-LOCAL BRANCHING PROCESSES

3.1. The main existence result

In the first section, we focus on an existence result for a sufficiently regular branching process having $(\widehat{\mathbf{H}}_t)_{t \geqslant 0}$ as transition function. Recall that by Kolmogorov's theorem, a transition function together with an initial distribution uniquely determine a Markov process. However, for various applications, the simple existence of a Markov process is not enough and we are interested in additional path regularity properties of the resulting process.

The proof follows the approach from Section 4 in [6], a method developed in [10] (see also [5], Appendix A, for a concise presentation) for proving the existence of right Markov process in infinite dimensional situations, if we start with a given resolvent of kernels or a transition function, namely $(\widehat{\mathbf{H}}_t)_{t\geqslant 0}$ on $\widehat{E}$ in our case.

Let $\mathcal{A} = \overline{[\mathrm{b}\mathcal{E}(\mathcal{U}_q^{\mu})]}$, where the closure is in the supremum norm. We need the following additional hypothesis, which is a similar condition to that used in [6]:
$(*)$ There exists a countable subset $\mathcal{F}_o$ of $\mathrm{b}\mathcal{E}(\mathcal{U}_q^{o})$ which is additive, $0 \in \mathcal{F}_o$, and separates the finite measures on E and a separable vector lattice $\mathcal{C} \subseteq \mathcal{A}$ such that $\{e^{-u} : u \in \mathcal{F}_o\} \subseteq \mathcal{C}$ and $V_t(\mathcal{F}_o) \subseteq \overline{\mathcal{C}}$, where V_t is the nonlinear cumulant semigroup introduced in Corollary 2.5.

Theorem 3.1. *([7]) If the base process X is standard and condition $(*)$ holds then there exists a branching càdlàg process with state space $\widehat{E}$ which has $(\widehat{\mathbf{H}}_t)_{t\geqslant 0}$ from Corollary 2.5 as its transition function. If in addition $B1 = 1$, then the process is standard.*

3.2. Probabilistic representation of the solution of the PDE

Corollary 3.2. *([7]) Let $\widehat{X}$ be the branching process with state space $\widehat{E}$ and transition function $(\widehat{\mathbf{H}}_t)_{t\geqslant 0}$, given by Theorem 3.1. Then the following assertions hold.*

(i) If $\varphi \in \mathcal{B}_+(E)$, $\varphi \leqslant 1$, then the solution $(H_t\varphi)_{t\geqslant 0}$ of equation (2.2) has the representation

$$H_t\varphi(x) = \widehat{\mathbb{E}}^{\delta_x}\widehat{\varphi}(\widehat{X}_t),\ x \in E,\ t \geqslant 0. \tag{3.1}$$

(ii) Let $X' = (X'_t)_{t\geqslant 0}$ be the right process with state space E and transition function $(Q_t)_{t\geqslant 0}$, given by Theorem 2.6 and Proposition 2.7, λ be a σ-finite measure and h a Borel, positive, bounded function on E. Then for all $t \geqslant 0$

$$e^{-\alpha t}\widehat{\mathbb{E}}^{\lambda}\langle h, \widehat{X}_t\rangle = \mathbb{E}^{\lambda} h(X'_t), \tag{3.2}$$

where recall that $\mathbb{E}^{\lambda} := \int_E \mathbb{E}^x \lambda(\mathrm{d}x)$ and on the left hand side λ is considered as a measure on $\widehat{E} \supset E$; if $\mu \in \widehat{E}$ then $\langle h, \mu\rangle := \int h\mathrm{d}\mu$.

3.3. Application. The reflecting Brownian motion as spatial motion

In the last section, we apply the results obtained by taking as base movement the reflecting Brownian motion on the closure of a smooth bounded Euclidean domain, and the killing rate to be given by the local time on the boundary.

Corollary 3.3. *([7]) Consider the Neumann problem on a bounded smooth Euclidean domain D and let σ be the surface measure on the boundary ∂D of D. Then the solution of the nonlinear evolution equation on $\overline{D}$,*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} H_t\varphi = \Delta H_t\varphi - \sigma H_t\varphi + \sigma(\sum_{k\geqslant 1} b_k B_k \widehat{H_t\varphi}),\ t \geqslant 0, \\ H_0\varphi = \varphi, \end{cases}$$

has the probabilistic representation (3.1),

$$H_t\varphi(x) = \widehat{\mathbb{E}}^{\delta_x}\widehat{\varphi}(\widehat{X}_t),\ \ x \in E,\ \ t \geqslant 0,$$

where $\widehat{X}$ is the branching process on the set of all finite configurations of the closure of D, having the reflecting Brownian motion X on $\overline{D}$ as base movement and perturbing measure σ. In addition, the formula (3.2) *holds.*

In this way, the statement of Theorem 4.4 from [1] holds for any branching procedure given by a sequence of Markovian kernels $(B_k)_{k\geqslant 1}$ (recall that B_k prescribes the distribution of k descendants), hence we may admit a non-local branching, i.e., the descendant are not forced to start from the point where the parent died. Recall that Theorem 4.4 is a main result used in [1] to represent the vorticity of the 2d Navier-Stokes equation in a bounded planar domain through a stochastic model.

4. CONTINUOUS FLOWS DRIVING BRANCHING PROCESSES

In this chapter we emphasize a class of branching processes which are driven by a right continuous flow Φ on E, in the sense that such a measure-valued process $\widehat{X}$ admits a representation by means of a second branching process $\widehat{X^0}$ and of the flow on measures induced by Φ,

$$\widehat{X}_t = \Phi_t(\widehat{X^0_t}) \text{ for all } t \geqslant 0.$$

It turns out that the branching process $\widehat{X^0}$ has the same branching mechanism as $\widehat{X}$, however $\widehat{X^0}$ has no a spatial motion and therefore we call it a *pure branching* process. This representation holds in the case of a superprocess for which the branching mechanism is independent of the spatial variable.

Recall that a superprocess $\widehat{X}$ provides a stochastic solution for the nonlinear evolution equation written formally as

$$\frac{\mathrm{d}}{\mathrm{d}t} v_t = D v_t + \Psi(v_t),\ t \geqslant 0,$$

with the initial condition $v_0 = f$, where D is the generator of the spatial motion and Ψ is the branching mechanism. The nonlinear evolution equation associated with the pure branching process $\widehat{X^0}$ is the particular case of the above equation with no spatial motion term on the right hand side, that is, with $D \equiv 0$. Actually, we consider the integral version of the equation, or equivalently, we work with the *mild solutions* of it.

As a consequence of the representation above of $\widehat{X}$ in terms of the pure branching process $\widehat{X^0}$ (in particular, D should be the generator of a right continuous flow Φ on E) we obtain a solution v_t, $t \geqslant 0$, to the nonlinear evolution equation having the following two probabilistic representations:

$$v_t(x) = -\ln \widehat{\mathbb{E}^0}^{\delta_x} e_{f\circ\Phi_t}(\widehat{X^0_t}) = -\ln \widehat{\mathbb{E}^0}^{\delta_{\Phi_t(x)}} e_f(\widehat{X^0_t}),\ x \in E,\ t \geqslant 0.$$

With the terminology of [13], page 133, v_t is written as the "log-potential of $\widehat{X^0_t}$ and Φ_t".

If $\mathcal{L}$ and $\mathcal{L}^0$ are the extended weak generators of the superprocesses $\widehat{X}$ and respectively $\widehat{X^0}$ on $M(E)$, then we have

$$\mathcal{L} = \widehat{D} + \mathcal{L}^0,$$

where $\widehat{D}$ is the generator of the continuous flow on $M(E)$ induced by Φ. In this way, the representation we prove may be interpreted as a consequence of regarding $\mathcal{L}$ as a modification of $\mathcal{L}^0$ with the first order operator $\widehat{D}$, which is a substitute for a "drift type" operator acting in the considered infinite dimensional frame. This is an exemplification of a general strategy developed in [3].

It turns out analogous results hold for non-local branching processes also, provided that the flow and the distributions of the branching are compatible.

This chapter is based on the results obtained in [11].

4.1. Preliminaries on measure-valued branching processes

In the first section, we recall the general construction of branching processes in both the superprocess and non-local branching case.

Superprocesses. Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}^x)$ be a fixed right Markov process with state space E and infinite lifetime. Let $(T_t)_{t\geqslant 0}$ be its transition function, $\mathcal{U}$ its resolvent, and D its generator. We also fix a *branching mechanism*, that is, a function $\Psi : E \times [0, \infty) \longrightarrow \mathbb{R}$ of the form

$$\Psi(x, \lambda) = -b(x)\lambda - c(x)\lambda^2 + \int_0^\infty (1 - e^{-\lambda s} - \lambda s)N(x, \mathrm{d}s),$$

where $c \geqslant 0$ and b are bounded $\mathcal{B}(E)$-measurable functions, and $N : \mathcal{B}_+((0, \infty)) \longrightarrow \mathcal{B}_+(E)$ is a kernel such that $N(u \wedge u^2) \in \mathrm{b}\mathcal{B}_+(E)$. The (X, Ψ)-*superprocess* is constructed as follows, cf. [16], [19], and [2].

Equation

$$\frac{\mathrm{d}}{\mathrm{d}t} v_t = Dv_t + \Psi(v_t), \quad t \geqslant 0, \quad v_0 = f \tag{4.1}$$

has a unique mild solution, more precisely, for each $f \in \mathrm{b}\mathcal{B}_+(E)$ the integral evolution equation

$$v_t(x) = T_t f(x) + \int_0^t T_s(x, \Psi(\cdot, v_{t-s}))\mathrm{d}s, \quad t \geqslant 0, \quad x \in E, \tag{4.2}$$

has a unique jointly measurable solution $(t, x) \longmapsto V_t f(x)$ such that $\sup_{0\leqslant s\leqslant t} ||V_s f||_\infty < \infty$ for all $t > 0$ and $[0, \infty) \ni t \longmapsto V_t f(x)$ is right continuous for all $x \in E$, provided that $t \longmapsto T_t f(x)$ has this property. The mappings $f \longmapsto V_t f$ form a nonlinear semigroup of operators on $\mathrm{b}\mathcal{B}_+(E)$.

For each $t \geqslant 0$ there exists a unique Markovian kernel $\widehat{T}_t$ on $(M(E), \mathcal{M}(E))$ such that

$$\widehat{T}_t(e_f) = e_{V_t f}, \quad f \in \mathrm{b}\mathcal{B}_+(E). \tag{4.3}$$

Since the family $(V_t)_{t\geqslant 0}$ is a (nonlinear) semigroup on $\mathrm{b}\mathcal{B}_+(E)$, it follows that $\widehat{\mathbb{T}} = (\widehat{T}_t)_{t\geqslant 0}$ is a transition function $(M(E), \mathcal{M}(E))$. By Theorem 4.9 from [2], under a Feller-type regularity condition, there exists a Borel right process $\widehat{X}$, with state space $M(E)$, having the transition function $\widehat{\mathbb{T}}$, called (X, Ψ)-*superprocess*.

Non-local branching processes. We work in the framework of [6]. Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}^x)$ be a fixed right Markov process with state space E and transition function $(T_t)_{t\geqslant 0}$. As in the superprocess case, we assume that X has infinite lifetime. Let $(b_k)_{k\geqslant 1}$ be a sequence of functions from $\mathrm{b}\mathcal{B}_+(E)$ such that $\sum_{k\geqslant 1} b_k \leqslant 1$, let $m_1 := \|\sum_{k\geqslant 1} kb_k\|_\infty$ and assume that $1 < m_1 < \infty$.

We fix also a constant c such that $0 < c \leqslant \frac{m_1}{m_1-1}$. For each $k \geqslant 1$, let B_k be a Markovian kernel from $E^{(k)}$ to E.

If $\varphi \in \mathcal{B}_+(E)$, $0 \leqslant \varphi \leqslant 1$, then by Proposition 4.1 from [6] (see also Theorem 2.3 from Chapter 2), the integral evolution equation

$$h_t = e^{-ct}T_t\varphi + c\int_0^t e^{-c(t-s)}T_{t-s}\sum_{k\geqslant 1} b_k B_k(h_s^{(k)})\mathrm{d}s,\ t \geqslant 0, \tag{4.4}$$

has a unique solution $t \longmapsto H_t\varphi$, jointly measurable in $(t,x) \in \mathbb{R}_+ \times E$, such that $0 \leqslant H_t\varphi \leqslant 1$. Here, for a function $h \in \mathrm{b}\mathcal{B}_+(E)$ we have denoted by $h^{(k)}$, $k \geqslant 1$, the function on $E^{(k)}$ defined as $h^{(k)}(\mathbf{x}) := h(x_1)\cdots h(x_k)$ for all $\mathbf{x} = (x_1, \ldots, x_k) \in E^{(k)}$.

The integral evolution equation (4.4) associated with a non-local branching process on $\widehat{E}$ is formally equivalent to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}h_t = (L-c)h_t + c\sum_{k\geqslant 1} b_k B_k(h_t^{(k)}),\ t \geqslant 0, \tag{4.5}$$

with the initial condition $h_0 = \varphi$, where L is the generator of the spatial motion X; see e.g. Remark 4.2 (ii) from [6].

It turns out that the nonlinear semigroup $(H_t)_{t\geqslant 0}$ induces a branching semigroup of kernels $(\widehat{\mathbf{H}}_t)_{t\geqslant 0}$ on $\widehat{E}$ such that $\widehat{\mathbf{H}}_t\widehat{\varphi} = \widehat{H_t\varphi}$ for all $\varphi \in \mathrm{b}\mathcal{B}_+(E)$, $0 \leqslant \varphi \leqslant 1$. According to Theorem 4.10 from [6], under some additional assumptions, there exists a branching right Markov process $\widehat{X}$ with state space $\widehat{E}$ (depending on the spatial motion X, $(b_k)_{k\geqslant 1}$, the sequence of kernels $(B_k)_{k\geqslant 1}$, and c) having the transition function $(\widehat{\mathbf{H}}_t)_{t\geqslant 0}$.

4.2. A result on solving nonlinear evolution equations

In the second section, we present a unifying argument, which is applicable on both spaces of measures ($M(E)$ and $\widehat{E}$).

Lemma 4.1. *([11]) Let $J : E \longrightarrow E$ be a $\mathcal{B}(E)/\mathcal{B}(E)$-measurable map and denote by $\mathcal{K}(J)$ the set of all operators $(W, \mathcal{D}(W))$ on $\mathrm{b}\mathcal{B}_+(E)$, $W : \mathcal{D}(W) \longrightarrow \mathrm{b}\mathcal{B}_+(E)$, such that for all $f \in \mathrm{b}\mathcal{D}_+(W)$ we have $f \circ J \in \mathcal{D}(W)$ and $W(f \circ J) = Wf \circ J$. Convention:*

If the domain $\mathcal{D}(W)$ *of* W *is not indicated, then this means that* $\mathcal{D}(W) = \mathrm{b}\mathcal{B}_+(E)$.

Then the following assertions hold.

(i) *The set* $\mathcal{K}(J)$ *has the following properties.*

(p1) If $(W, \mathcal{D}(W))$ *is an operator on* $\mathrm{b}\mathcal{B}_+(E)$ *such that there exists a sequence* $(W_n)_n$ *in* $\mathcal{K}(J)$ *with* $\mathcal{D}(W_n) = \mathcal{D}(W)$ *for all* n*, which is converging pointwise to* W *(i.e.* $\lim_n W_n f(x) = W f(x)$ *for all* $f \in \mathcal{D}(W)$ *and* $x \in E$*), then* W *belongs to* $\mathcal{K}(J)$*;*

(p2) If $(W, \mathcal{D}(W))$ *and* $(V, \mathcal{D}(V))$ *are two operators on* $\mathrm{b}\mathcal{B}_+(E)$ *such that* $W(\mathcal{D}(W)) \subset \mathcal{D}(V)$ *and* $V, W \in \mathcal{K}(J)$*, then* $V \circ W \in \mathcal{K}(J)$*;*

(p3) Let $(\mathbf{K}, \mathcal{D}(\mathbf{K}))$ *be an operator on* $\mathrm{b}\mathcal{B}_+(E)$ *such that* $\mathbf{K} \in \mathcal{K}(J)$*. If* $Wf = \int_0^t \mathbf{K}(W_s f)\nu(\mathrm{d}s)$*, where* ν *is a finite measure on* $[0, t]$ *and* $W_s \in \mathcal{K}(J)$*,* $\mathcal{D}(W_s) = \mathcal{D}(\mathbf{K})$*, and* $W_s(\mathcal{D}(W_s)) \subset \mathcal{D}(\mathbf{K})$ *for all* s*, then* W *also belongs to* $\mathcal{K}(J)$*.*

(ii) *Assume that* $(\mathbf{K}, \mathcal{D}(\mathbf{K}))$ *is Lipschitz with respect to the supremum norm and* $\mathbf{K}(0) = 0$*. Let* $a > 0, t_o > 0$ *and consider the integral equation*

$$w_t = e^{-at} f + \int_0^t e^{-a(t-s)} \mathbf{K}(w_s)\mathrm{d}s, \ 0 \leqslant t \leqslant t_o, \tag{4.6}$$

where $f \in \mathcal{D}(\mathbf{K})$*.*

$(ii.1)$ *The equation* (4.6) *has a unique solution* $[0, t_o] \ni t \longmapsto W_t f \in \mathcal{D}(\mathbf{K})$ *such that the function* $(t, x) \longmapsto W_t f(x)$ *is jointly measurable, provided that one of the following two conditions is verified:*

(I) $\mathcal{D}(\mathbf{K}) = \mathrm{b}\mathcal{B}_+(E)$*;*

(II) $\mathcal{D}(\mathbf{K}) = \{f \in \mathrm{b}\mathcal{B}_+(E) : f \leqslant 1\}$ *and* $\mathbf{K}f \leqslant a$ *for all* $f \in \mathcal{D}(\mathbf{K})$*.*

$(ii.2)$ *Assume that* $\mathbf{K}$ *preserves the pointwise convergence, that is, if* $(f_n)_n \subset \mathcal{D}(\mathbf{K})$ *is pointwise converging to* $f \in \mathcal{D}(\mathbf{K})$*, then* $\mathbf{K}(f_n) \longrightarrow \mathbf{K}(f)$ *pointwise on* E*. Then the function* $[0, t_o) \ni t \longmapsto W_t f(x)$ *is differentiable for each* $x \in E$ *and* $u_t := W_t f$ *is the unique solution of the nonlinear evolution equation*

$$\frac{\mathrm{d}u_t}{\mathrm{d}t} = -a u_t + \mathbf{K}(u_t), \ 0 \leqslant t < t_o, \tag{4.7}$$

with the initial condition $u_o = f$*.*

(iii) *Suppose in addition that* $\mathbf{K} \in \mathcal{K}(J)$*, then* $W_t \in \mathcal{K}(J)$ *for all* $t \in [0, t_o]$*.*

4.3. Pure branching processes

In the third section, we study the pure branching processes and their nonlinear evolution equations, in the context of both superprocesses and non-local branching processes.

Pure branching superprocesses. Pure branching processes arise by considering, in the construction from Section 4.1, branching processes with no spatial motion. Let $X^0 = (X^0_t)_{t\geqslant 0}$ be the trivial Markov process on E, where each point is a trap, i.e., $\mathbb{P}^x(X^0_t = x) = 1$ for all $t \geqslant 0$ and $x \in E$. The (X^0, Ψ)-superprocess is named *pure branching.*

Corollary 4.2. *([11]) Let Ψ be a branching mechanism. Then the following assertions hold.*

(i) Consider the pure branching (X^0, Ψ)-superprocess, the right Markov process $\widehat{X^0} = (\widehat{X^0_t}, \widehat{\mathbb{P}^0}^{\mu})$ with state space $M(E)$, and let $u \in \mathrm{b}\mathcal{B}_+(E)$. Let

$$V^0_t u(x) := -\ln \widehat{\mathbb{E}^0}^{\delta_x} e_u(\widehat{X^0_t}),\ t \geqslant 0, x \in E,$$

then $w_t := V^0_t u$, $t \geqslant 0$, solves the following nonlinear evolution equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} w_t = \Psi(\cdot, w_t),\ t \geqslant 0, \tag{4.8}$$

with the initial condition $w_0 = u$.

(ii) Let $(\mathcal{L}^0, \mathcal{D}(\mathcal{L}^0))$ be the extended weak generator of the pure branching (X^0, Ψ)-superprocess on $M(E)$. Let $u \in \mathrm{b}\mathcal{B}^0_+$, then $F := e_u$ belongs to $\mathcal{D}(\mathcal{L}^0)$, $\mathcal{L}^0 F = -F \cdot l_{\Psi(\cdot,u)}$, and for $\mu \in M(E)$ we have

$$\mathcal{L}^0 F(\mu) = \int_E c(x) F''(\mu, x)\mu(\mathrm{d}x) - \int_E b(x) F'(\mu, x)\mu(\mathrm{d}x) + \tag{4.9}$$

$$\int_E \int_0^\infty [F(\mu + s\delta_x) - F(\mu) - sF'(\mu, x)] N(x, \mathrm{d}s)\mu(\mathrm{d}x).$$

Non-local pure branching processes. Similarly to the superprocess case, we consider now the *non-local pure branching process $\widehat{X^0}$ on $\widehat{E}$* for which the base process is the trivial Markov process X^0. In this case, equation (4.4) becomes

$$h_t = e^{-ct}\varphi + c\int_0^t e^{-c(t-s)} \sum_{k\geqslant 1} b_k B_k(h^{(k)}_s)\mathrm{d}s,\ t \geqslant 0. \tag{4.10}$$

Let $(H^0_t \varphi)_{t\geqslant 0}$ be the solution to equation (4.10), then the transition function of $\widehat{X^0}$ is $(\widehat{\mathbf{H}^0_t})_{t\geqslant 0}$.

Proposition 4.3. *Consider the solution of equation (4.10) as a family of operators*

$(H_t^0)_{t\geqslant 0}$, $H_t^0 : \mathcal{B}_{\mathsf{u}} \longrightarrow \mathcal{B}_{\mathsf{u}}$. Then $(H_t^0)_{t\geqslant 0}$ is (nonlinear) C_0-semigroup on $\mathcal{B}_{\mathsf{u}}$, viewed as a closed subset of the Banach space $\mathrm{b}\mathcal{B}(E)$ endowed with the supremum norm.

Proposition 4.4. *([11]) Let $M \in \mathcal{B}(E)$ and suppose that*

$$b_k(x)B_{k,x}(\widehat{M} \cap E^{(k)}) = 0 \text{ for every } k \geqslant 1 \text{ and } x \in E \setminus M, \tag{4.11}$$

that is, the sub-probability measure on $E^{(k)}$ induced by the sub-Markovian kernel b_kB_k at x is carried by $E^{(k)} \setminus \widehat{M}$. Then $\widehat{E} \setminus \widehat{M}$ is a finely closed absorbing subset of $\widehat{E}$ with respect to the pure branching process $\widehat{X^0}$. The restriction of the pure branching process $\widehat{X^0}$ to $\widehat{M}$ is still a pure branching process, it is induced by the trivial process on M and the restrictions of B_k, $k \geqslant 1$, to M.

4.4. Continuous flows driving superprocesses

In the fourth section, we deal with superprocesses that are driven by a right continuous flow. The first main result is Theorem 4.6. It is preceded by the description of the extended weak generator of a superprocess (Proposition 4.5), which completes results from [15], [19], [16], and [2]. Results related to the nonlinear evolution equation and the log potential formula are gathered in assertion (*iii*) of Theorem 4.6.

Let $(L, \mathcal{D}(L))$ be the extended weak generator of the spatial motion X on E and $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ the extended weak generator of the (X, Ψ)-superprocess on $M(E)$.

Let $(\widehat{D}, \mathcal{D}(\widehat{D}))$ be the extended weak generator of the right continuous flow $\widehat{\Phi}$ on $M(E)$ induced by the right Markov process X, $\widehat{\Phi}_t(\mu) := X_t(\mathbb{P}^\mu)$, $\mu \in M(E)$, that is, the extended weak generator of the transition function $\mathbb{Q}^0 = (Q_t^0)_{t\geqslant 0}$, $Q_t^0 F(\mu) := F(\mu \circ T_t)$ for all $F \in \mathrm{b}\mathcal{M}_+(E)$, $\mu \in M(E)$, and $t \geqslant 0$.

Proposition 4.5. *([11]) (i) If $u \in \mathrm{b}\mathcal{D}_o^c(L)$ is such that Lu is a bounded function then l_u belongs to the domain $\mathcal{D}(\mathcal{L})$ of the extended weak generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ of $\widehat{X}$ and*

$$\mathcal{L}(l_u) = l_{Lu-bu}.$$

(ii) Let $n \geqslant 1$, $u_1, \ldots, u_n \in \mathrm{b}\mathcal{D}(L)$ such that all Lu_i are bounded functions, $\psi \in C_b^1(\mathbb{R}^n)$, and consider the function F on $M(E)$, defined as $F(\mu) := \psi(\langle \mu, u_1\rangle, \ldots, \langle \mu, u_n\rangle)$, $\mu \in M(E)$. Then F belongs to $\mathcal{D}(\widehat{D})$, $F'(\mu, \cdot) \in \mathcal{D}(L)$ and we have

$$\widehat{D}F(\mu) := \int_E LF'(\mu, x)\mu(\mathrm{d}x) \quad \text{for all } \mu \in M(E), \tag{4.12}$$

where recall that the variational derivative *of a function* $F : M(E) \longrightarrow \mathbb{R}$ *is* $F'(\mu, x) := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}(F(\mu + \varepsilon\delta_x) - F(\mu)),\ \mu \in M(E),\ x \in E.$

Theorem 4.6. *([11]) Consider a branching mechanism* Ψ *which is independent of the spatial variable, that is, with* b *and* c *constant functions and* $N(x, \mathrm{d}s) = N(\mathrm{d}s)$ *for all* $x \in E$*. Assume that the spatial motion* $X = (X_t)_{t \geqslant 0}$ *is the deterministic process associated with a right continuous flow* $\Phi = (\Phi_t)_{t \geqslant 0}$ *on* E *and suppose that the mapping* $[0, \infty) \times E \ni (t, x) \longmapsto \Phi_t(x)$ *is continuous.*

Let $\widehat{X} = (\widehat{X}_t)_{t \geqslant 0}$ *be the* (X, Ψ)*-superprocess induced by the spatial motion* X *and the branching mechanism* Ψ *and let* $\widehat{X^0} = (\widehat{X^0_t})_{t \geqslant 0}$ *be the pure branching* (X^0, Ψ)*-superprocess. Then the following assertions hold.*

(i) The (X, Ψ)*-superprocess* $\widehat{X} = (\widehat{X}_t, \widehat{\mathbb{P}}^\mu)$ *has the following representation, as the pure branching process* $\widehat{X^0} = (\widehat{X^0_t}, \widehat{\mathbb{P}^0}^\mu)$ *driven by the continuous flow* Φ *on* $M(E)$*,*

$$\widehat{X}_t = \Phi_t(\widehat{X^0_t}) \text{ for all } t \geqslant 0, \tag{4.13}$$

where the equality is in the distribution sense and we also have

$$\widehat{\mathbb{P}}^\mu(\widehat{X}_t \in \Gamma) = \widehat{\mathbb{P}^0}^{\Phi_t(\mu)}(\widehat{X^0_t} \in \Gamma) \text{ for all } \mu \in M(E) \text{ and } \Gamma \in \mathcal{M}(E). \tag{4.14}$$

(ii) Let $(D, \mathcal{D}(D))$ *(resp.* $(\widehat{D}, \mathcal{D}(\widehat{D}))$, $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$*, and* $(\mathcal{L}^0, \mathcal{D}(\mathcal{L}^0))$*) be the extended weak generator of the spatial motion* X *(resp. of the flow* Φ *on* $M(E)$*,* $\widehat{X}$*, and* $\widehat{X^0}$*). Let further* $\mathcal{D}_o := U_\alpha V_\beta(\mathrm{b}C(M(E)))$*,* $\alpha, \beta > 0$*, where* $(U_\alpha)_{\alpha > 0}$ *(resp.* $(V_\alpha)_{\alpha > 0}$*) is the resolvent of* $\widehat{X^0}$ *(resp. the resolvent of the flow* Φ *on* $M(E)$*). Then* $\mathcal{D}_o \subset \mathcal{D}^c_o(\mathcal{L}) \cap \mathcal{D}_o(\widehat{D}) \cap \mathcal{D}(\mathcal{L}^0)$ *and*

$$\mathcal{L} = \widehat{D} + \mathcal{L}^0 \text{ on } \mathcal{D}_o.$$

(iii) Equation (4.1)*,*

$$\frac{\mathrm{d}}{\mathrm{d}t} v_t = D v_t + \Psi(v_t),\ t \geqslant 0,\ v_0 = f,$$

has a mild solution v_t*,* $t \geqslant 0$*, which is given by the log-potential type formula*

$$v_t(x) = -\ln \widehat{\mathbb{E}^0}^{\delta_x} e_{f \circ \Phi_t}(\widehat{X^0_t}) = -\ln \widehat{\mathbb{E}^0}^{\delta_{\Phi_t(x)}} e_f(\widehat{X^0_t}),\ x \in E,\ t \geqslant 0. \tag{4.15}$$

More precisely, v_t *is the solution of the nonlinear integral evolution equation* (4.2)*.*

4.5. Continuous flows driving non-local branching processes

The non-local branching processes on configuration spaces, driven by right continuous flows, are investigated in the fifth section. Theorem 4.7 is the second main statement of this chapter. Results similar to the log potential formula, but valid on $\widehat{E}$, are collected in this section.

In the context of non-local branching processes described in Section 4.1, we additionally assume that the functions $b_k \in b\mathcal{B}_+(E)$, $k \geqslant 1$, are constants. Let c be a constant such that $0 < c \leqslant \frac{m_1}{m_1 - 1}$ and let B_k be a Markovian kernel from $E^{(k)}$ to E, $k \geqslant 1$.

For each $k \geqslant 1$, let $(\mathbf{B}_k, \mathcal{D}(\mathbf{B}_k))$ be the operator on $b\mathcal{B}_+(E)$ defined as $\mathbf{B}_k \varphi := B_k(\varphi^{(k)})$, with $\mathcal{D}(\mathbf{B}_k) := \{\varphi \in \mathcal{B}_+(E) : \varphi \leqslant 1\}$.

Theorem 4.7. *([11]) Let $\widehat{X} = (\widehat{X}_t)_{t \geqslant 0}$ be the non-local branching process on $\widehat{E}$, depending on a spatial motion given by a right continuous flow $\Phi = (\Phi_t)_{t \geqslant 0}$ on E, $(b_k)_{k \geqslant 1}$, the sequence of kernels $(B_k)_{k \geqslant 1}$, and c. Assume that the mapping $[0, \infty) \times E \ni (t, x) \longmapsto \Phi_t(x)$ is continuous and suppose that*

$$\mathbf{B}_k \in \mathcal{K}(\Phi_t) \text{ for all } t \geqslant 0 \text{ and } k \geqslant 1. \tag{4.16}$$

Then the following assertions hold.

(i) The branching process $\widehat{X} = (\widehat{X_t}, \widehat{\mathbb{P}}^\mu)$ has the representation (4.13) on $\widehat{E}$ as the pure branching process $\widehat{X^0} = (\widehat{X^0_t}, \widehat{\mathbb{P}^0}^\mu)$ driven by the continuous flow Φ and (4.14) holds for all $\mu \in \widehat{E}$ and $\Gamma \in \mathcal{B}(\widehat{E})$.

(ii) Equation (4.5), with L as the generator of the right continuous flow Φ, has a mild solution, namely, the unique solution h_t, $t \geqslant 0$, to equation (4.4) given by

$$h_t(x) = \widehat{\mathbb{E}^0}^{\delta_x} \prod_{k=1}^{N(t)} \varphi(\Phi_t(x_t^k)) = \widehat{\mathbb{E}^0}^{\delta_{\Phi_t(x)}} \prod_{k=1}^{N(t)} \varphi(x_t^k), \; x \in E, \tag{4.17}$$

with the notation $\widehat{X^0_t} = (x_t^1, \ldots, x_t^{N(t)}) \in E^{(N(t))}$, where $N(t)$ is the number of the particles at time t.

4.6. Application. A nonlinear evolution equation on measures

The last section is an application of the results obtained in this chapter and consists of solving a nonlinear parabolic equation on measures. Let E be a Lusin topological space and $(b_k)_{k\geqslant 1}$ be a sequence of positive numbers such that $\sum_{k\geqslant 1} b_k \leqslant 1$ and $1 < \sum_{k\geqslant 1} kb_k < \infty$. Let $(L, \mathcal{D}(L))$ be the extended weak generator of a spatial motion X on E and let $\mathbb{T} = (T_t)_{t\geqslant 0}$ be its transition function.

We consider the following nonlinear evolution equation on measures

$$\frac{\mathrm{d}}{\mathrm{d}t} F_t = LF_t' - cF_t + c\sum_{k\geqslant 1} b_k F_t^k \text{ on } M(E),\, t \geqslant 0, \tag{4.18}$$

with the initial condition $F_0 = F$, where $c > 0$ for a function $F : M(E) \longrightarrow [0,1]$ we denoted by F' its variational derivative and $LF'(\mu) := \int_E LF'(\mu, x)\mu(\mathrm{d}x)$.

Let $\widehat{\Phi} = (\widehat{\Phi}_t)_{t\geqslant 0}$ be the right continuous flow on $M(E)$ induced by the right Markov process X.

Further, we need to consider the following condition:

$$\text{the mapping } [0,\infty) \times M(E) \ni (t,\mu) \longmapsto \mu \circ T_t \in M(E) \text{ is continuous.} \tag{4.19}$$

It turns out that if E is locally compact with countable base and X is Feller then (4.19) holds.

Corollary 4.8. *([11]) If we take $E = \mathbb{R}^d$, then equation*

$$\frac{\mathrm{d}}{\mathrm{d}t} F_t = \Delta F_t' - cF_t + c\sum_{k\geqslant 1} b_k F_t^k \text{ on } M(\mathbb{R}^d),\, t \geqslant 0, \tag{4.20}$$

has a mild solution F_t, $t \geqslant 0$, with $F_0 = F \in \mathcal{M}_+(\mathbb{R}^d)$, $F \leqslant 1$, uniquely determined by the pure branching process $\widehat{Y^0}$ on $\widehat{M(\mathbb{R}^d)}$, the d-dimensional Brownian motion $(B_t)_{t\geqslant 0}$, and the constant c,

$$F_t(\mu) = \widehat{\mathbb{E}^0}^{\delta_\mu} \prod_{k=1}^{N(t)} F(B_t(\mathbb{P}^{y_t^k})) = \widehat{\mathbb{E}^0}^{\delta_{B_t(\mathbb{P}^\mu)}} \prod_{k=1}^{N(t)} F(y_t^k),\ \mu \in M(\mathbb{R}^d), \tag{4.21}$$

where $\widehat{Y_t^0} = (y_t^1, \ldots, y_t^{N(t)})$ and $N(t)$ is the number of the particles in $M(\mathbb{R}^d)$ at time t.

More generally, if we return to the general case of the base space E and assume that condition (4.19) holds then equation (4.18) has a mild solution uniquely determined by the pure branching process $\widehat{Y^0}$ on $\widehat{M(E)}$, the spatial motion X, and the constant c, namely, for every $F \in \mathcal{M}_+(E)$, $F \leqslant 1$, the nonlinear evolution equation

$$F_t = e^{-ct} F \circ \widehat{\Phi}_t + c\int_0^t e^{-c(t-s)} \sum_{k\geqslant 1} b_k F_s^k \mathrm{d}s \text{ on } M(E),\ t \geqslant 0,$$

has a unique solution $t \longmapsto F_t$*, jointly measurable in* $(t, \mu) \in \mathbb{R}_+ \times M(E)$*, such that* $0 \leqslant F_t \leqslant 1$ *and* (4.21) *holds with* X_t *instead of* B_t *and* $M(E)$ *instead of* $M(\mathbb{R}^d)$.

APPENDIX

We present additional topics required for the understanding of this thesis; we describe the different variants of infinitesimal generators of Markov processes; we introduce multiplicative and additive functionals, as well as the Revuz correspondence, an essential ingredient for Chapters 2 and 3; we present the preliminary notions regarding the right continuous flows, used in Chapter 4.

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