ON THE SPACE OF TRAJECTORIES OF A GENERIC GRADIENT LIKE VECTOR FIELD

DAN BURGHELEA, LEONID FRIEDLANDER, AND THOMAS KAPPELER

Dedicated to Dan Papuc on the occasion of his 80th birthday

Abstract. This paper describes the construction of a canonical compactification of the space of trajectories and of the unstable/stable sets of a generic gradient like vector field on a closed manifold as well as a canonical structure of a smooth manifold with corners of these spaces. As an application we discuss the geometric complex associated with a gradient like vector field and show how differential forms can be integrated on its unstable/stable sets. Integration leads to a morphism between the de Rham complex and the geometric complex.

Contents

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1. Introduction

This paper gives a detailed description of the construction of a compactification of the space of trajectories and of the unstable/stable sets of a gradient like Morse-Smale vector field on a closed manifold M as well as of a canonical structure of a smooth manifold with corners on these spaces. The property of a vector field being Morse-Smale is generic.
As an application the paper discusses the geometric complex associated with such a vector field. This complex calculates the cohomology of M and is most commonly known as Morse complex, referring to the case where the vector field is the gradient of a Morse function w.r. to a Riemannian metric of M. Using that the constructed compactification of the unstable sets are a smooth manifold with corners one can show that a canonical integration map from the de-Rham complex to the geometric complex induces an isomorphism in cohomology. For further applications see e.g. [1], [2], [7], [14], [15], [36].
The results presented in this paper are known, compare [8], [18], [19], [20], [28]. The aim of this paper is to give a new and self contained treatment of them.
Due to its comprehensive exposition the paper can be used as part of a course on Morse theory on finite dimensional manifolds. At the beginning of each section we summarize its contents and provide some references. Section 2 and Section 3 can be read independently. This paper is a chapter of a book in preparation on the Witten deformation of the de Rham complex where it will be incorporated.
Professor Dan Papuc is a mathematician interested not only in mathematical research but also in teaching mathematics to interested students. During many years of friendship he has encouraged the first author to give graduate courses on various topics and provided him with a number of opportunities to do this. With this in mind we dedicate this paper to him on the occasion of his 80th birthday.

2. Gradient-like flows

Ideas from dynamical systems can be used to investigate the diffeomorphism type of a closed manifold. Let h be a Morse function on a closed Riemannian manifold M with Riemannian metric g, and let $X = -\mathrm{grad}_g h$ be the gradient vector field of h with respect to the metric g. Note that the rest points of X coincide with the critical points of h. Trajectories $t \mapsto \gamma(t)$ of X originate (as $t \to -\infty$) and terminate (as $t \to +\infty$) at critical points. In physicist's lingo, these trajectories are called instantons. Denote by W_v^- [W_v^+] the unstable [stable] manifold of X at the critical point v of h. They are sets of all points that lie on trajectories that originate [terminate] at v. As any point of M lies on exactly one trajectory of X, and each trajectory originates at a critical point of h, the unstable manifolds W_v^-, $v \in \mathrm{Crit}(h)$ are the cells of a decomposition of M. These cells are open in the sense that they are the image of a smooth embedding of $\mathbb{R}^k$ for some $0 \le k \le n$. Here by $\mathrm{Crit}(h)$ we denote the set of all critical points of h. Notice that the dimension of W_v^- equals the index of the critical point v. To use this decomposition for describing the diffeomorphism type of a manifold, one needs the additional condition that unstable manifolds W_v^- and stable ones W_w^+ intersect transversally. It is called the Morse–Smale condition. In general, the gradient vector field X does not satisfy this condition. However, Smale showed in [Sm1] that one can find an arbitrarily small perturbation g' of the (arbitrarily) given metric g in such a way that $X' = -\mathrm{grad}_{g'} h$ satisfies the Morse–Smale condition.

One can use the cells W_v^- to construct a chain complex of finite-dimensional spaces, called the geometric complex. Typically, they are not compact. To relate the de Rham complex to the geometric complex one has to be able to integrate differential forms over W_v^- and to use Stokes's theorem. For doing that, one needs to compactify the cells. We will discuss a canonical compactification of the unstable manifolds in section 4.
Throughout this section, our approach is based on reducing our investigation to the analysis of the objects under consideration near critical points. In neighborhoods of these points, we will use local coordinates that are convenient for our purposes. Among the many existing references we mention [4], [16], [24], [25], [28] – [31], [35] and references therein.

2.1. **Morse-Smale pairs.** Let M be a smooth, connected manifold of dimension n. A point $v \in M$ is said to be a *critical point* of a given smooth function $h : M \to \mathbb{R}$ if the differential $d_v h$ at v vanishes. The set of critical points $u, v, w, \dots$ of h is denoted by $\mathrm{Crit}(h)$. The function h is a *Morse function* if the Hessian $d_v^2 h$ at any critical point v of h is nondegenerate. According to the Morse Lemma - see [24], [25] and [16] there exist coordinates $x_1, ..., x_n$ around any critical point v of a given Morse function h so that

$$h(x) = h(v) - \frac{1}{2} \sum_{j=1}^{k} x_j^2 + \frac{1}{2} \sum_{j=k+1}^{n} x_j^2. \tag{2.1}$$

Hence, any critical point of a Morse function is isolated. In particular, if M is a compact manifold, a Morse function $h : M \to \mathbb{R}$ has only finitely many critical points. The Hessian $d_v^2 h$ of h at v is a quadratic form on the tangent space $T_v M$ of v at M. We denote by $i(v), 0 \leq i(v) \leq n$, the index of $d_v^2 h$ which is defined to be the maximal dimension of a subspace of $T_v M$ on which $d_v^2 h$ is negative definite. One can read off the index from the representation (2.1) of h, $i(v) = k$.
Let X be a smooth vector field and let $x \in M$. By the existence and uniqueness theorem for the initial value problem of ODE's there exists $T > 0$ so that

$$\frac{d}{dt} \Phi_t(x) = X\left(\Phi_t(x)\right) \ ; \quad \Phi_0(x) = x \tag{2.2}$$

has a unique solution $\Phi_t(x)$, defined for $|t| < T$. By the theorem on the smooth dependence of the solution $\Phi_t(x)$ on the initial data it follows that for any $p \in M$ there exist a neighborhood U of p and $T > 0$ so that for any $x \in U$ the solution $\Phi_t(x)$ exists for $|t| < T$ and that it is smooth in $(x, t) \in U \times (-T, T)$. $\Phi_t(x)$ is referred to as the flow induced by X whereas the solution $t \mapsto \Phi_t(x)$ is referred to as parametrized trajectory of X. The set of points of a parametrized trajectory is sometimes called an unparametrized trajectory or an orbit of the vector field X. In what follows we will often use the term 'trajectory' without further specification within a given context. If not stated otherwise we will always assume in the sequel that X is *complete* , i.e. that the flow induced by X is defined for any time $t \in \mathbb{R}$. In this case $\Phi : M \times \mathbb{R} \to M, (x, t) \mapsto \Phi_t(x)$ is smooth. Using the local existence and uniqueness theorem for ODE's one can show that in the case when M is closed, any smooth vector field is complete - see e.g. [17].
By the uniqueness of a solution of the initial value problem (2.2) one has for any $x \in M$ and $t, s \in \mathbb{R}$

$$\Phi_{t+s}(x) = \Phi_t\left(\Phi_s(x)\right).$$

It follows that for any $t \in \mathbb{R}$, $\Phi_t : M \to M$ is a diffeomorphism with inverse given by Φ_{-t}. In the sequel, the following standard models will be considered. For the manifold M we choose $\mathbb{R}^n$ and the Morse function is given by

$$h_k(x) := -\frac{1}{2}\|x^-\|^2 + \frac{1}{2}\|x^+\|^2 \tag{2.3}$$

where $(x^-, x^+) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ and

$$\|x^-\|^2 = \sum_1^k x_i^2 \quad \text{and} \quad \|x^+\|^2 = \sum_{k+1}^n x_i^2.$$

Note that the origin in $\mathbb{R}^n$ is the only critical point of h_k and that its index is equal to k. The vector field is then chosen to be the gradient vector field of $-h_k$ with respect to the Euclidean metric on $\mathbb{R}^n$,

$$X^{(k)}(x) = \sum_1^k x_j \frac{\partial}{\partial x_j} - \sum_{k+1}^n x_j \frac{\partial}{\partial x_j}. \tag{2.4}$$

Clearly, $X^{(k)}(h_k)(x) = -\|x\|^2 < 0$ for any $x \in \mathbb{R}^n \backslash \{0\}$. These models motivate the following definition.

Definition 2.1. *A vector field X is said to be gradient-like with respect to a Morse function h (in the sense of Milnor* [25] *) if the following properties hold:*

- (GL1) $X(h)(x) < 0 \quad \forall x \in M \backslash \mathrm{Crit}(h)$.
- (GL2) *For any critical point $v \in \mathrm{Crit}(h)$ there exist an open neighborhood U_v of v and a coordinate map $\varphi_v : B_r \to U_v$ from the open ball $B_r = B_r(0; \mathbb{R}^n)$ with center 0 and radius $r = r(v) > 0$ onto U_v so that h and X, when expressed in the coordinates $x_1, \ldots, x_n$ take the form*

$$(h \circ \varphi_v)(x_1, \ldots, x_n) = h(v) - \frac{1}{2}\sum_1^{i(v)} x_j^2 + \frac{1}{2}\sum_{i(v)+1}^n x_j^2 \tag{2.5}$$

and

$$(\varphi_v^* X)(x_1, ..., x_n) = \sum_1^{i(v)} x_j \frac{\partial}{\partial x_j} - \sum_{i(v)+1}^n x_j \frac{\partial}{\partial x_j}. \tag{2.6}$$

We refer to the charts $(U_v, \varphi_v), v \in \mathrm{Crit}(h)$, as standard charts of the pair (h, X) and to the coordinates $x_1, \ldots, x_n$ as standard coordinates. We will always choose $U_v, v \in \mathrm{Crit}(h)$, sufficiently small so that they are pairwise disjoint.

For a gradient-like vector field, h decreases along a trajectory $t \mapsto \Phi_t(x)$ and hence it is a Lyapunov function for the flow. More precisely, for any $x \in M \backslash \mathrm{Crit}(h)$ and any $t \in \mathbb{R}$,

$$\frac{d}{dt} h\left(\Phi_t(x)\right) = X(h)\left(\Phi_t(x)\right) < 0.$$

In particular, it follows that any point x_0 is a zero of X if and only if it is a critical point of h.

As an example we mention the case where the vector field X is given by the gradient vector field $X = -\mathrm{grad}_g h$ with g being a Riemannian metric on M. In local coordinates $x_1, \ldots, x_n$, the components of the gradient of $-h, -\mathrm{grad}_g h$, are given by $X_i = -\sum_{j=1}^n g^{ij} \partial_{x_j} h$ where g^{ij} are the entries of the inverse of the metric tensor

$(g_{k\ell})$ of g. Then $X(h)(x) = -\|d_x h\|^2 < 0$ on $M \backslash \mathrm{Crit}(h)$, i.e. (GL1) is satisfied. To make sure that (GL2) holds as well we need to make an additional assumption. We say that the pair (h, g) is *compatible* or that g is h-compatible if for any critical point v of h there exist a neighborhood U_v of v and a coordinate map $\varphi_v : B_r \to U_v$ so that when expressed in the coordinates $x_1, \ldots, x_n$, h takes the form (2.5) and $\varphi_v^* g$ is given by the standard metric on B_r, i.e.

$$g_{ij}(x) = \delta_{ij} \quad \forall 1 \le i, j \le n.$$

Clearly, if g is h-compatible, then (GL2) is satisfied. Using an appropriate partition of unity for M one can prove that for any given Morse function h, h-compatible metrics can always be constructed. In fact, any gradient-like vector field X is a gradient vector field with respect to an appropriately chosen, h-compatible Riemannian metric g on M.

Lemma 2.1. *Let X be a gradient-like vector field on M with respect to a Morse function $h : M \to \mathbb{R}$. Then there exists an h-compatible Riemannian metric g on M so that $X = -\mathrm{grad}_g h$.*

Proof. Let U be the open neighborhood of $\mathrm{Crit}(h), U = \cup_{v \in \mathrm{Crit}(h)} U_v$, where $(U_v)_{v \in \mathrm{Crit}(h)}$ are pairwise disjoint coordinate charts of the critical points v of h so that for any $v \in \mathrm{Crit}(h), U_v$ satisfies the properties stated in (GL2) of Definition 2.1. Let g' be a Riemannian metric on M so that for any $v \in \mathrm{Crit}(h)$, the pull back of $\varphi_v^* g'$ of g' by the coordinate map $\varphi_v : B_r \to U_v$ of (GL2) is the standard metric on B_r. Furthermore, let N be an open neighborhood of $M \backslash U$ so that $\overline{N} \subseteq X \backslash \mathrm{Crit}(h)$. In particular, $X(h)(x) < 0$ for any $x \in \overline{N}$. Note that for any $x \in N$, the tangent space $T_x M$ decomposes as a direct sum $T_x M = V_x \oplus \langle X(x) \rangle$ where $\langle X(x) \rangle$ denotes the one dimensional $\mathbb{R}$-vector space generated by $X(x)$ and V_x denotes the kernel of $d_x h, V_x = \{\xi \in T_x M | d_x h(\xi) = 0\}$. As X and $-\mathrm{grad}_{g'} h$ agree on U it follows from (GL2) that for any x in $N \cap U$, the positive number $-X(h)(x)$ is the square of the length of $X(x)$ with respect to the inner product $g'(x)$ and $X(x)$ is orthogonal to V_x. Now define a new Riemannian metric g on M as follows. For $u \in U, g(x) := g'(x)$ whereas for $x \in M \backslash U, g(x)$ is determined as follows. The restriction $g(x)|_{V_x}$ is given by $g'(x)|_{V_x}, V_x$ and $\langle X(x) \rangle$ are orthogonal and the length of $X(x)$ is equal to $\sqrt{-X(h)(x)}$. As $-X(h)$ is strictly positive on N, $g(x)$ is positive definite. In a straightforward way one verifies that g is a smooth Riemannian metric on M with $X = -\mathrm{grad}_g h$. $\square$

Many gradient-like vector fields are complete. Indeed it is not hard to show that X is complete if h is a proper function, X a gradient-like vector field with respect to h, and $X(h)$ bounded on M. (Recall that h is said to be proper if the inverse image of any compact set is compact.)
Let us now come back to the standard models introduced earlier where the manifold M is $\mathbb{R}^n$ and the Morse function h is given by $h_k(x) = -\|x^-\|^2/2 + \|x^+\|^2/2$ with $x = (x^-, x^+) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ for some $0 \le k \le n$. The gradient vector field of h_k with respect to the Euclidean metric g_0 on $\mathbb{R}^n$ is then given by

$$X^{(k)}(x) := -\mathrm{grad}_{g_0} h_k(x) = \sum_1^k x_j \frac{\partial}{\partial x_j} - \sum_{k+1}^n x_j \frac{\partial}{\partial x_j} \tag{2.7}$$

and the initial value problem (2.2) takes the form

$$\frac{d}{dt}(x^-(t), x^+(t)) = \big(x^-(t), -x^+(t)\big)\,;\ (x^-(0), x^+(0)) = (x^-, x^+).$$

The corresponding flow $\Phi_t^{(k)}(x) = (x^-(t), x^+(t))$ is then obtained by a simple integration

$$\Phi_t^{(k)}(x) = (e^t x^-, e^{-t} x^+) \tag{2.8}$$

and defined for any $t \in \mathbb{R}$.
Introduce the subsets

$$W_0^{\pm} \equiv W_0^{(k)\pm} := \{x \in \mathbb{R}^n \,\big|\, \lim_{t\to\pm\infty} \Phi_t^{(k)}(x) = 0\}.$$

The subset W_0^+ is referred to as the *stable manifold* of the critical point 0 and is given by

$$W_0^+ = \{(0, x^+) \,\big|\, x^+ \in \mathbb{R}^{n-k}\}$$

whereas W_0^- is the unstable manifold of 0 and given by

$$W_0^- = \{(x^-, 0) \,\big|\, x^- \in \mathbb{R}^k\}.$$

The canonical models are used to describe features of a vector field X which is gradient-like with respect to a Morse function h on the manifold M. First note that whenever the limit $x_\infty := \lim_{t\to\infty} \Phi_t(x)$ exists one has for any $s \in \mathbb{R}$

$$\Phi_s(x_\infty) = \lim_{t\to\infty} \Phi_{t+s}(x) = x_\infty,$$

and it follows from (2.2) that $X(x_\infty) = 0$. As X is gradient-like, this then implies that x_∞ must be a critical point of h. Similarly, one argues that whenever the limit $\lim_{t\to-\infty} \Phi_t(x)$ exists it must be a critical point of h. Denote by W_v^+ the stable and by W_v^- the unstable set of a critical point $v \in \mathrm{Crit}(h)$ with respect to the flow Φ_t,

$$W_v^{\pm} := \{x \in M \,\big|\, \lim_{t\to\pm\infty} \Phi_t(x) = v\}.$$

As in the standard model cases discussed above it turns out that $W_v^{\pm}$ are smooth submanifolds of M. Before we state and prove this result let us introduce the rescaled vector field

$$Y(x) := -\frac{1}{X(h)(x)} X(x) \quad x \in M\backslash\mathrm{Crit}(h). \tag{2.9}$$

As X is gradient-like with respect to h and therefore $X(h)(x) < 0$ for any $x \in M\backslash\mathrm{Crit}(h)$, the vector field Y is well defined on $M\backslash\mathrm{Crit}(h)$ and

$$Y(h)(x) = -1 \quad \forall x \in M\backslash\mathrm{Crit}(h).$$

Denote by Ψ_s the flow of Y, i.e.

$$\frac{d}{ds}\Psi_s(x) = Y(\Psi_s(x));\ \Psi_0(x) = x. \tag{2.10}$$

Note that on $M\backslash\mathrm{Crit}(h)$, the orbits of X and Y are identical, but are traversed at different speeds. We will see that the vector field Y is not complete. For s with $\Psi_s(x)$ defined, one has

$$\frac{d}{ds}h(\Psi_s(x)) = Y(h)(\Psi_s(x)) = -1.$$

Hence, whenever $\Psi_s(x)$ is defined, we have

$$h(\Psi_s(x)) - h(\Psi_0(x)) = \int_0^s \frac{d}{ds'} h(\Psi_{s'}(x))ds' = -s$$

or

$$h(\Psi_s(x)) = h(x) - s. \tag{2.11}$$

The point $\Psi_s(x)$, when regarded as a point on the trajectory of X, coincides with $\Phi_{\tau(s,x)}(x)$ where $\tau(s;x)$ is the solution of the initial value problem $\frac{d\tau}{ds} = -\frac{1}{X(h)(\Psi_s(x))}$ and $\tau(0,x) = 0$ and given by

$$\tau(s;x) = \int_0^s -\big(X(h)(\Psi_{s'}(x))\big)^{-1} ds'. \tag{2.12}$$

Finally recall that a smooth map $f : N_1 \to N_2$ between smooth manifolds N_1 and N_2 is said to be an *immersion* [*submersion*] if $d_x f : T_x N_1 \to T_x N_2$ is $1-1$ [onto] for any $x \in N_1$. An immersion f is said to be an *embedding* if f is $1-1$ and $f^{-1} : f(N_1) \to N_2$ is continuous. The image of a $1-1$ immersion f is a submanifold iff f is an embedding.

Lemma 2.2. *W_v^- and W_v^+ are smooth submanifolds of M which are diffeomorphic to $\mathbb{R}^{i(v)}$ and $\mathbb{R}^{n-i(v)}$ respectively. They are referred to as the unstable and stable manifold of v.*

Proof. We compare $W_v^\pm$ with the model case for $k := i(v)$ introduced above for which $W_0^- = \mathbb{R}^k \times \{0\}$ and $W_0^+ = \{0\} \times \mathbb{R}^{n-k}$. Note that the coordinate map $\varphi_v : B_r \to U_v$ conjugates the flow $\Phi_t^{(k)}$ of the model case with Φ_t when properly restricted. Hence, given any $x^- \in \mathbb{R}^k$, it follows that $\Phi_t\left(\varphi_v(e^{-t}x^-, 0)\right)$ is independent of $t \geq t_-$ where t_- is chosen sufficiently large so that $(e^{-t}x^-, 0) \in B_r$. Similarly, for any $x^+ \in \mathbb{R}^{n-k}$, $\Phi_{-t}\left(\varphi_v(0, e^{-t}x^+)\right)$ is independent of $t \geq t_+$ where t_+ is so large that $(0, e^{-t}x^+) \in B_r$. Hence we can define

$$\Theta_v^- : \mathbb{R}^{i(v)} \to M,\ x^- \mapsto \Phi_t\left(\varphi_v(e^{-t}x^-, 0)\right)$$

and

$$\Theta_v^+ : \mathbb{R}^{n-i(v)} \to M,\ x^+ \mapsto \Phi_{-t}\left(\varphi_v(0, e^{-t}x^+)\right)$$

where for any $x^\pm, t$ is chosen so large that $e^{-t}x^\pm \in B_r$. Note that on $B_r \cap W_0^\pm, \Theta_v^\pm$ coincides with the restriction of φ_v on $B_r \cap W_0^\pm$. Using that Φ_t is a diffeomorphism one concludes that Θ_v^- and Θ_v^+ are smooth immersions which map trajectories of the model flow $\Phi_t^{(k)}$ onto trajectories of the flow Φ_t. Hence Θ_v^- and Θ_v^+ are $1-1$ and the images $\Theta_v^-(\mathbb{R}^{i(v)})$ and $\Theta_v^+(\mathbb{R}^{n-i(v)})$ coincide with W_v^- and W_v^+ respectively. (Note that $\Theta_v^\pm$ but not their images $W_v^\pm$ depend on the choice of the coordinate map $\varphi_v : B_r \to U_v$.)
To see that W_v^+ and W_v^- are submanifolds it is to show that $\Theta_v^\pm$ are embeddings onto $W_v^\pm$. Let us show this for Θ_v^-. the proof for Θ_v^+ is in fact similar. It remains to show that $(\Theta_v^-)^{-1}$ is continuous. Let $(y_n)_{n\geq 1}$ be a sequence in $W_v^- \backslash \{v\}$ which converges to $y \in W_v^-$. As Θ_v^- is an extension of the restriction of φ_v to $B_r \cap (\mathbb{R}^{i(v)} \times \{0\})$ we can assume without loss of generality that $y \neq v$. Choose $c \in \mathbb{R}$ with $c < h(v)$ so that $h^{-1}\{c\} \cap W_v^- \subseteq U_v \cap W_v^-$. Denote by x_n the unique point on the orbit through y_n so that $h(x_n) = c$. Then $y_n = \Phi_{t_n}(x_n)$ for some $t_n \in \mathbb{R}$ and $y = \Phi_t(x)$ for some $x \in U_v \cap W_v^-$ and $t \in \mathbb{R}$. First we show that $\lim_{n\to\infty} x_n = x$. Note that the rescaled vector field Y introduced in (2.9) is defined on all of $W_v^- \backslash \{v\}$. Hence there exists

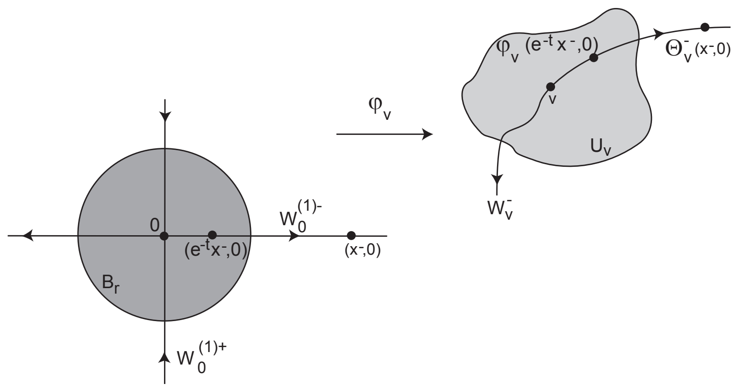

FIGURE 1. Illustration Θ_v^- when $\dim M = 2$ and $k := i(v) = 1$

a unique set of real numbers $s_n, n \geq 1$, and s such that $\Psi_{s_n}(x_n) = \Phi_{t_n}(x_n)$ and $\Psi_s(x) = \Phi_t(x)$. By (2.11) we have $s_n = c - h(y_n)$ and we conclude that

$$\lim_{n\to\infty} s_n = c - h(y) = s$$

Accordingly,

$$x_n = \Psi_{-s_n}(y_n) \underset{n\to\infty}{\longrightarrow} \Psi_{-s}(y) = x.$$

Next we show that $t = \lim_{n\to\infty} t_n$. This follows easily from (2.11) and the convergence of $(s_n)_{n\geq 1}$ and $(x_n)_{n\geq 1}$,

$$\begin{aligned} t &= \int_0^s - \left(X(h)(\Psi_{s'}(x))\right)^{-1} ds' \\ &= \lim_{n\to\infty} \int_0^{s_n} - \left(X(h)(\Psi_{s'}(x_n))\right)^{-1} ds' \\ &= \lim_{n\to\infty} t_n. \end{aligned}$$

Hence we have shown

$$\begin{aligned} (\Theta_v^-)^{-1}(y) = e^t \varphi_v^{-1}(\Phi_{-t} y) &= e^t \varphi_v^{-1}(x) \\ &= \lim_{n\to\infty} e^{t_n} \varphi_v^{-1}(x_n) \\ &= \lim_{n\to\infty} e^{t_n} \varphi_v^{-1}(\Phi_{-t_n} y_n) \\ &= \lim_{n\to\infty} (\Theta_v^-)^{-1}(y_n). \end{aligned}$$

This shows that Θ_v^{-1} is continuous. □

Definition 2.2. *A gradient-like vector field (with respect to the Morse function h) is said to satisfy the Morse-Smale condition if for any pair of critical points, $v, w \in \mathrm{Crit}(h)$, Θ_w^+ and Θ_v^- are transversal or, equivalently, the submanifolds W_v^- and W_w^+ intersect transversally,*

$$W_v^- \pitchfork W_w^+, \tag{2.13}$$

i.e. for any $x \in W_v^- \cap W_w^+$, the tangent space T_xM at x is given by the span of $T_x(W_v^-) \cup T_x(W_w^+)$.

The Morse-Smale condition implies that $W_v^- \cap W_w^+$ is a submanifold of M. Note that for any $x \in W_v^- \cap W_w^+$,

$$\lim_{t\to-\infty} \Phi_t(x) = v \text{ and } \lim_{t\to\infty} \Phi_t(x) = w.$$

In particular, for $v = w$ one has $W_v^- \cap W_v^+ = \{v\}$. In fact, the flow Φ acts on this submanifold,

$$\Phi : \mathbb{R} \times \left(W_v^- \cap W_w^+\right) \to W_v^- \cap W_w^+,\ (t, x) \mapsto \Phi_t(x). \tag{2.14}$$

For $v \neq w$ with $W_v^- \cap W_w^+ \neq \emptyset$, this action is free and we denote by $\mathcal{T}(v, w)$ the quotient,

$$\mathcal{T}(v, w) := \left(W_v^- \cap W_w^+\right)/\mathbb{R} \tag{2.15}$$

with its induced differentiable structure. By slight abuse of notation, elements in $\mathcal{T}(v, w)$ are called trajectories or, more specifically, unbroken trajectories from v to w. They are denoted by $\gamma, \gamma_1, \gamma_2, \ldots$. The trajectory corresponding to a solution $(\Phi_t(x))_{-\infty<t<\infty}$ of (2.2) is sometimes denoted by $[\Phi_\cdot(x)]$. Note that $\mathcal{T}(v, w)$ is a manifold and, for any a with $h(w) < a < h(v)$, it can be canonically embedded into the level set $L_a = h^{-1}(\{a\})$ by assigning to a trajectory in $\mathcal{T}(v, w)$ its intersection with the level set L_a.

Definition 2.3. *A pair (h, X), consisting of a smooth, proper Morse function h and a smooth vector field X, is said to be Morse-Smale or a Morse-Smale pair if*

(MS1) *X is gradient-like with respect to h;*
(MS2) *X satisfies the Morse-Smale condition;*

A vector field X satisfying (MS1) - (MS2) is also referred to as being Morse-Smale with respect to h.

Two Morse-Smale pairs (h_1, X_1) and (h_2, X_2) are said to be equivalent, $(h_1, X_1) \sim (h_2, X_2)$ if

(EQ1) $\mathrm{Crit}(h_1) = \mathrm{Crit}(h_2)$;
(EQ2) for any $v \in \mathrm{Crit}(h_1)$, the unstable manifolds corresponding to X_1 and X_2 coincide.

Definition 2.4. *A Morse cellular structure τ of a compact manifold is an equivalence class of Morse-Smale pairs.*

The reason to call an equivalence class of Morse-Smale pairs a Morse cellular structure is that according to [31], the collection of unstable manifolds of X can be viewed as the cells of a cell partition of M. We will say more on this later.
One can also consider compact manifolds with boundaries, or more generally, with corners as well as noncompact manifolds. In these cases one has to make further assumptions on a Morse-Smale pair. For example if M is not compact one typically imposes the additional condition

(MS3) h is proper and bounded from below.

In the sequel we will not distinguish between a Morse pair (h, X) and its equivalence class $[(h, X)]$ and by a slight abuse of terminology refer to (h, X) as a Morse cellular structure as well. Instead of (h, X), in view of Lemma 2.1 we will also use (h, g) to denote a Morse cellular structure where g is an h-compatible Riemannian metric on M so that $X = -\mathrm{grad}_g h$.
Throughout this chapter, we always assume that (h, X) is Morse-Smale and we fix a collection of pairwise disjoint neighborhoods U_v and coordinate maps φ_v as above so that (2.5) and (2.6) hold. Let M be a smooth manifold (not necessarily compact). A number $c \in \mathbb{R}$ is said to be a *critical value* of h if there exists $v \in \mathrm{Crit}(h)$ with $h(v) = c$. As h is assumed to be a proper Morse function its critical values form a sequence (c_j) of isolated numbers which we list in descending order,

$$\ldots < c_{j+1} < c_j < c_{j-1} < \ldots$$

Note that this sequence can be bounded from below or above, or unbounded on both sides. If the sequence $(c_j)_j$ is bounded - which holds e.g. if M is compact - there are only finitely many critical values, which we denote by

$$c_{K+N} < c_{K+N-1} < \ldots < c_K.$$

For any critical value c_j introduce

$$M_j^{\pm} \equiv M_{j,\varepsilon_j}^{\pm} := L_{c_j \pm \epsilon_j}$$

with $\varepsilon_j > 0$ sufficiently small so that

$$c_j + \varepsilon_j < c_{j-1} - \varepsilon_{j-1}$$

and where L_a is the a-level set

$$L_a := \{x \in M \mid h(x) = a\}.$$

Throughout this chapter we will use a collection $(U_v, \varphi_v), v \in \mathrm{Crit}(h)$, of canonical charts of $M, \varphi_v : B_r \to U_v$ so that for any critical value c_j of h, r corresponding to v is denoted by $r_j > 0$, is taken to be the same for any of the finitely many critical points $v \in \mathrm{Crit}(h)$ with $h(v) = c_j$ and

$$c_j + r_j^2 < c_{j-1} - r_{j-1}^2.$$

For convenience we then choose $0 < \varepsilon_j < (r_j/2)^2$. With this choice one has for any $v \in \mathrm{Crit}(h)$ with $h(v) = c_j$

$$W_v^{\pm} \neq \{v\} \text{ iff } \varphi_v(B_{r_j}) \cap M_j^{\pm} \neq \emptyset \tag{2.16}$$

(The condition $0 < \varepsilon_j < (r_j/2)^2$ makes sure that $\varphi_v(B_{r_j}) \cap M_j^{\pm}$ is not empty if $W_v^{\pm} \neq \{v\}$.) To investigate the level sets $M_j^{\pm}$ we use the rescaled vector field $Y(x)$ introduced in (2.9). Recall that it is defined on $M \backslash \mathrm{Crit}(h)$.

Lemma 2.3. *Let $a, b \in \mathbb{R}$ with $c_{j+1} < b < a < c_j$. Then $\Psi_{a-b}(\cdot)$ is a diffeomorphism from L_a to L_b.*

Proof. We have already noticed that $\frac{d}{ds} h(\Psi_s(x)) = -1$ whenever $\Psi_s(x)$ is defined. For any $x \in L_a$, $\Psi_s(x)$ exists at least for $0 \leq s < a - c_{j+1}$ and

$$h(\Psi_{a-b}(x)) - h(\Psi_0(x)) = -\int_0^{a-b} ds = b - a$$

or

$$h(\Psi_{a-b}(x)) = b.$$

By the uniqueness of a solution of the initial value problem (2.10) it follows that $\Psi_{a-b} : L_a \to L_b$ has $\Psi_{-(a-b)}$ as an inverse. By the smooth dependence of the solution of (2.10) on the initial data one concludes that $\Psi_{a-b} : L_a \to L_b$ is a diffeomorphism. □

Lemma 2.3 can be partially extended. Precisely, if $b = c_{j+1}$, Ψ_{a-b} is still well defined but only a continuous map.
First note that in the case $a = b$, the map $\Psi_0 : L_a \to L_a$ is always the identity map. To go further we analyze the rescaled vector fields $Y^{(k)}(0 \le k \le n)$ of the standard vector fields $X^{(k)}$ and verify the above statement in the case of standard model. According to (2.3) and (2.7) one has for any $y \in \mathbb{R}^n \backslash \{0\}$

$$Y^{(k)}(y) := -\frac{1}{X^{(k)}(h_k)(y)} X^{(k)}(y) = \sum_1^k \frac{y_j}{\|y\|^2} \frac{\partial}{\partial y_j} - \sum_{k+1}^n \frac{y_j}{\|y\|^2} \frac{\partial}{\partial y_j}. \tag{2.17}$$

The solution of the initial value problem

$$\frac{d}{ds} \Psi_s^{(k)}(z) = \frac{1}{\|y(s)\|^2} \left(y^-(s), -y^+(s)\right); \; \Psi_0^{(k)}(0) = (z^-, z^+) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$$

can be explicitly computed. For initial data $z = (0, z^+) \in \{0\} \times \mathbb{R}^{n-k} \backslash \{0\}$, one has $y^-(s) \equiv 0$ and $y^+(s)$ is of the form $f(s) z^+$ where $f(s) > 0$ satisfies

$$\frac{d}{ds} f(s) = -\left(f(s) \|z^+\|^2\right)^{-1}; \; f(0) = 1.$$

Hence $f(s)^2 = 1 - 2s/\|z^+\|^2$ and

$$y(s) = \left(1 - 2s/\|z^+\|^2\right)^{1/2} (0, z^+). \tag{2.18}$$

This solution exists for $0 \le s < \|z^+\|^2/2$ and has a limit

$$\lim_{s \to \|z^+\|^2/2} y(s) = 0. \tag{2.19}$$

For initial data $z = (z^-, z^+)$ with $z^- \neq 0$, a solution $y(s)$ of (2.17) can be found by reparametrizing the solution $x(t) = (z^- e^t, z^+ e^{-t})$ given by (2.8). In view of the definition of the rescaled vector field $Y^{(k)}$ in (2.17) the function $s(t) \equiv s(t; z)$, determined by $y\,(s(t)) = x(t)$, then satisfies

$$\frac{ds}{dt} = \|x(t)\|^2 \quad ; \; s(0) = 0.$$

As $\|x(t)\|^2 = \|z^-\|^2 e^{2t} + \|z^+\|^2 e^{-2t}$ this leads to

$$s(t) = \|z^-\|^2 (e^{2t} - 1)/2 + \|z^+\|^2 (1 - e^{-2t})/2. \tag{2.20}$$

For any $0 \le k \le n$ and $0 < b < a$, the diffeomorphism $\Psi_{a-b}^{(k)}$ from the level set $h_k^{-1}(a)$ to the level set $h_k^{-1}(b)$ has an extension for $b = 0$: For $z = (z^-, z^+)$ with $z^- \neq 0$, $s := a - b = a$ is given by $s = h_k((z^-, z^+)) = \left(\|z^+\|^2 - \|z^-\|^2\right)/2$ and thus by (2.20)

$$\left(\|z^+\|^2 - \|z^-\|^2\right)/2 = \|z^-\|^2 (e^{2t} - 1)/2 + \|z^+\|^2 (1 - e^{-2t})/2.$$

Hence $e^t = (\|z^+\|/\|z^-\|)^{1/2}$. Substituting this expression into $\Psi_a^{(k)}(z^-,z^+) = (z^-e^t, z^+e^{-t})$ we obtain the map $\Psi_a^{(k)} : h_k^{-1}(a) \to h_k^{-1}(0)$,

$$\Psi_a^{(k)}(z^-,z^+) = \begin{cases} \left(\left(\|z^+\|/\|z^-\|\right)^{1/2} z^-, \left(\|z^-\|/\|z^+\|\right)^{1/2} z^+ \right) & \text{if } z^- \neq 0 \\ (0,0) & \text{if } z^- = 0. \end{cases} \tag{2.21}$$

Clearly, this extension is continuous. The next lemma shows that a similar result as for the standard models holds in the general situation. See Figure 2 for illustration.

Lemma 2.4. *(i) Let $a, b \in \mathbb{R}$ with $c_{j+1} = b < a < c_j$. Then $\Psi_{a-c_{j+1}}$ is a diffeomorphism from $L_a \backslash \bigcup_{h(v)=c_{j+1}} W_v^+$ onto $L_{c_{j+1}} \backslash \mathrm{Crit}(h)$ and admits a continuous extension from L_a onto $L_{c_{j+1}}$ which, for any critical point v with $h(v) = c_{j+1}$, maps $L_a \cap W_v^+$ to v .*
(ii) Let $a, b \in \mathbb{R}$ with $c_{j+1} < b < a = c_j$. Then Ψ_{b-c_j} is a diffeomorphism from $L_b \backslash \bigcup_{h(v)=c_j} W_v^-$ onto $L_{c_j} \backslash \mathrm{Crit}(h)$ and admits a continuous extension from L_b onto L_{c_j} which, for any critical point v with $h(v) = c_j$, maps $L_b \cap W_v^-$ to v .

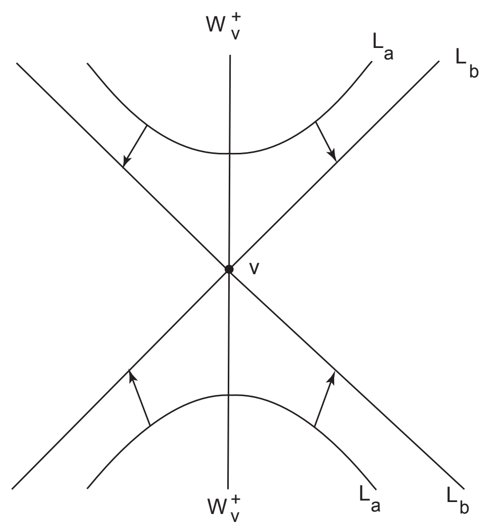

FIGURE 2. Illustration of the map $\Psi_{a-b} : L_a \to L_b$ (indicated by arrows); $v \in \mathrm{Crit}(h)$ with $i(v) = 1$, a is a regular value of h near $b = h(v)$ with $b < a$.

Proof. Statement (i) and (ii) are proved in the same way, so we concentrate on (i). For $x \in L_a \backslash \bigcup_{h(v)=c_{j+1}} W_v^+$, the trajectory $\Psi_s(x)$ exists for s in the compact interval $[0, a - c_{j+1}]$. Hence $\Psi_{a-c_{j+1}}(x)$ is a well defined point of $L_{c_{j+1}}$. For $x \in L_a \cap W_v^+$, one has by the definition of the stable manifold W_v^+ that $\lim_{t\to\infty} \Phi_t(x) = v$. Hence $\Psi_s(x)$ exists for $0 \le s < a - c_{j+1}$ and $\lim_{s\to a-c_{j+1}} \Psi_s(x) = v$. In this case we define

$\Psi_{a-c_{j+1}}(x) := v$. Using the properties of the flow Ψ_s, the fact that X is a gradient-like vector field w.r.t. to h and the investigations above of $\Psi_a^{(k)}$ one concludes that

$$\Psi_{a-c_{j+1}} : L_a \backslash \bigcup_{h(v)=c_{j+1}} W_v^+ \to L_{c_{j+1}} \backslash \mathrm{Crit}(h)$$

is a diffeomorphism. By definition, $\Psi_{a-c_{j+1}} : L_a \cap W_v^+ \to L_{c_{j+1}}$ is the constant map with value v. Hence to prove that $\Psi_{a-c_j} : L_a \to L_{c_j}$ is a continuous map it suffices to show that the restriction of Ψ_{a-b} to a neighborhood of $L_a \cap W_v^+$ is continuous where v is one of finitely many critical points with critical value b. In view of Lemma 2.3 we may assume without loss of any generality that $a - b < \rho^2$ so that $L_a \cap U_v$ is a neighborhood of $L_a \cap W_v^+$. The continuity of Ψ_{a-b} on $L_a \cap U_v$ then follows from formula (2.21). □

As an application of Lemma 2.3 and Lemma 2.4 we get the following

Corollary 2.5. *Assume that M is closed, $h : M \to \mathbb{R}$ a Morse function and X a gradient-like vector field with respect to h. Then, for any $x \in M$, both limits, $\lim_{t\to\pm\infty} \Phi_t(x)$ exist and they are critical points of h. As a consequence, $M = \bigcup_{v\in\mathrm{Crit}(h)} W_v^-$, and the unstable manifolds $(W_v^-)_{v\in\mathrm{Crit}(h)}$ are a decomposition of M into pairwise disjoint submanifolds of M, each diffeomorphic to some $\mathbb{R}^k$, $0 \le k \le \dim M$.*

2.2. **Smale's Theorem.** In this subsection we prove that for any given Morse function $h : M \to \mathbb{R}$ with M closed, i.e. compact and without boundary, there exists a Morse cellular structure (h, g) or (h, X). More precisely we show the following result due to Smale [29], [30].

Theorem 2.6. *Let M be closed, (h, g) be a compatible pair, and let $\ell \in \mathbb{N}$. Then, in any neighborhood of g in the space of smooth Riemannian metrics on M, equipped with the C^ℓ-topology, there is a metric g' so that (h, g') is a Morse cellular structure. The metric g' can be chosen in such a way that it coincides with g outside shells contained in the standard charts $(U_v, \varphi_v : B_r \to U_v), v \in \mathrm{Crit}(h)$. Here a shell in U_v is an open subset of the form $\varphi_v\left(B_{r_2} \backslash \overline{B}_{r_1}\right)$ with $0 < r_1 < r_2 < r$.*

By Lemma 2.1 we know that for any gradient-like vector field X w.r. to h there exists a Riemannian metric g so that $X = -\mathrm{grad}_g h$ and (h, g) is compatible. Hence Theorem 2.6 implies the following result on h-compatible vector fields.

Theorem 2.7. *Let M be closed, X be an h-compatible vector field, and $\ell \in \mathbb{N}$. Then, in any neighborhood of X in the space of smooth vector fields on M, equipped with the C^ℓ-topology, there exists a vector field X' so that (h, X') is Morse-Smale. The vector field X' can be chosen in such a way that it coincides with X outside shells contained in the standard charts (U_v, φ_v), $v \in \mathrm{Crit}(h)$.*

Remark 2.1. For versions of both previous theorems in the case where M is not compact but the set of critical values of h is bounded from below see e.g. [16].

Proof. (Proof of Theorem 2.6) We essentially follow the proof given by Smale [29]. Let $c_N < \ldots < c_1$ be the critical values of h. For any h-compatible Riemannian metric g' denote by $W_v^{+\prime}$ and $W_v^{-\prime}$ the stable and unstable manifolds of $-grad_{g'}h$ at v. To start, we first observe that whenever $x \in (W_v^- \cap W_w^+)\backslash\{v,w\}$ satisfies

$$\dim W_v^- + \dim W_w^+ = n + \dim(T_x W_v^- \cap T_x W_w^+)$$

then the same holds for any point on the orbit $[\Phi_\bullet(x)]$ through x. This suggests that it might suffice to change the metric g near v to achieve that $W_v^{-\prime}$ and $W_w^{+\prime}$ intersect transversally and leads to the formulation of the following statement $\mathcal{H}(i)$ which we will prove by induction starting at i corresponding to the lowest critical value.

$\mathcal{H}(i)$: in any C^ℓ-neighborhood of an arbitrary h-compatible Riemannian metric g, there exists a smooth Riemannian metric g' so that

$\mathcal{H}(i)_1$ $W_v^{-\prime} \pitchfork W_w^{+\prime} \quad \forall v, w \in \mathrm{Crit}(h)$ with $h(v) \leq c_i$;

$\mathcal{H}(i)_2$ g and g' coincide outside the union of shells each of which is contained in a standard neighborhood of a critical point v with $h(v) \leq c_i$. In particular, g' is h-compatible.

Notice that $\mathcal{H}(1)$ coincides with the statement of Theorem 2.6. Further, as $h^{-1}(c_N)$ consists of absolute minima only, one has $h^{-1}(c_N) \subseteq \mathrm{Crit}(h)$, hence $W_v^- = \{v\}$ for any $v \in h^{-1}(c_N)$ and for any $w \in \mathrm{Crit}(h)$ with $w \neq v$, one has $W_w^+ \cap W_v^- = \emptyset$. Thus $\mathcal{H}(N)$ is always satisfied and we might choose $g' = g$. It remains to prove the induction step $\mathcal{H}(i+1) \Longrightarrow \mathcal{H}(i)$. To this end it suffices to consider any Riemannian metric g satisfying $\mathcal{H}(i+1)$. Property $\mathcal{H}(i)$ then follows by successively applying Proposition 2.8 below to the finitely many critical points v with $h(v) = c_i$. □

Proposition 2.8. *Let (h, g) be a compatible pair, $v \in \mathrm{Crit}(h)$, and $\ell \in \mathbb{N}$. Then, in any C^ℓ-neighborhood of g in the space of smooth metrics on M, there exists a Riemannian metric g' so that*

(i) $W_v^{-\prime} \pitchfork W_w^{+\prime} \quad \forall w \in \mathrm{Crit}(h)$*;*

(ii) g and g' coincide outside of a shell, contained in a standard neighborhood of v. In particular, (h, g') is a compatible pair.

Here $W_v^{-\prime}$ $[W_v^{+\prime}]$ denotes the unstable [stable] manifold of v with respect to the vector field $-\mathrm{grad}_{g'}h$.

We will derive Proposition 2.8 from the following model problem: For any $0 \leq k \leq n$ given let

$$\begin{aligned} M_0 :=& \mathbb{R} \times \mathbb{S}_\rho^{k-1} \times \mathbb{R}^{n-k} \\ h_0 :=& M \to \mathbb{R}, (s, p, \xi) \mapsto s \\ Y_0 :=& -\frac{\partial}{\partial s} \end{aligned}$$

where $\mathbb{S}_\rho^{k-1} \subseteq \mathbb{R}^k$ is the $(k-1)$ dimensional sphere of radius $\rho > 0$ centered at 0 and $0 \leq k \leq n$. Let g_0 be an arbitrary Riemannian metric on M_0 so that $Y_0 = -\mathrm{grad}_{g_0} h_0$. Further let

$$V^- := \mathbb{S}_\rho^{k-1} \times \{0\} \subseteq \mathbb{S}_\rho^{k-1} \times \mathbb{R}^{n-k}$$

and let V^+ denote a smooth submanifold of $\mathbb{S}_\rho^{k-1} \times \mathbb{R}^{n-k}$. In the proof of Proposition 2.8, k will be the index of the critical point $v \in \mathrm{Crit}(h), k = i(v), V^-$

the set $W_v^- \cap L_{h(v)-\rho^2}$ and V^+ will be formed from $\sqcup_w (W_w^+ \cap L_{h(v)-\rho^2})$. For an arbitrary smooth vector field Z on M_0 with the property that the support of $Z - Y_0, \mathrm{supp}(Z - Y_0)$, is compact introduce the auxiliary sets $W_Z^\pm$ defined as follows: Choose $s_0 > 0$ so that the support of $Z - Y_0$ is contained in the strip $(-s_0, s_0) \times \mathbb{S}_\rho^{k-1} \times \mathbb{R}^{n-k}$. Then W_Z^- is defined as the set of all points of M_0 which lie on a trajectory of Z, *originating* in $(s_0, \infty) \times V^-$. Similarly, W_Z^+ is defined as the set of all points which lie on a trajectory ending up in $(-\infty, -s_0) \times V^+$. As the trajectories of Z outside $\mathrm{supp}(Z - Y_0)$ coincide with those of Y_0 and $\mathrm{supp}(Z - Y_0)$ is compact, Z is a complete vector field. It follows that $W_Z^\pm \cong \mathbb{R} \times V^\pm$. In fact $W_Z^\pm$ are submanifolds of M. To see it, define

$$\Theta_Z^\pm : \mathbb{R} \times V^\pm \to W_Z^\pm;\ (s, x) \mapsto \Phi_{\pm s_0 + s}^Z(\mp s_0, x)$$

where Φ_s^Z denotes the flow of Z. By the properties of a flow, one sees that $\Theta_Z^\pm$ are immersions. Arguing as in the proof of Lemma 2.2 one concludes that $\Theta_Z^\pm$ are embeddings and therefore, $W_Z^\pm$ are submanifolds. Notice that for $Z = Y_0$, one has $W_{Y_0}^\pm = \mathbb{R} \times V^\pm$. Our aim is to find a metric g_0' on M_0 which is close to g_0 and coincides with g_0 outside a compact set so that for the gradient vector field

$$Y_0' := -\mathrm{grad}_{g_0'} h_0$$

the manifolds $W_{Y_0'}^+$ and $W_{Y_0'}^-$ intersect transversally. To make a more precise statement, introduce the box

$$\mathcal{B} := (-s_0, s_0) \times \mathbb{S}_\rho^{k-1} \times B_\rho^{n-k}$$

where B_ρ^{n-k} is the open ball of radius ρ in $\mathbb{R}^{n-k}$ centered at 0. The notations introduced above are illustrated in Figure 3.

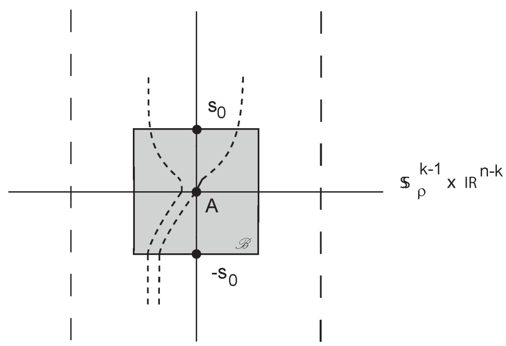

FIGURE 3. Trajectories of Z; $A := \mathbb{S}_\rho^{k-1} \times \{0\}$

Lemma 2.9. *In any C^ℓ-neighborhood of g_0 with $\ell \geq 1$ there exists a smooth metric g_0' with the following properties:*

(i) $g_0 = g_0'$ *on an open neighborhood of* $M_0 \backslash \mathcal{B}$.
(ii) $W_{Y_0'}^+ \pitchfork W_{Y_0'}^-$ *where* $Y_0' := -grad_{g_0'} h_0$.

Before proving Lemma 2.9 let us show how it is used to prove Proposition 2.8.

Proof. (Proof of Proposition 2.8) Let $v \in \mathrm{Crit}(h)$ with $h(v) = c_i$ and $i(v) = k$. Following [29] one gets a diffeomorphism Θ from $\mathcal{B} := (-s_0, s_0) \times \mathbb{S}_\rho^{k-1} \times B_\rho^{n-k}$ into U_v where $s_0 > 0$ will be chosen sufficiently small to insure that in the construction below, $\mathcal{B}$ is indeed mapped into U_v. Denote by M_i^- the level set $L_{c_i-\rho^2}$ where $0 < \rho < r_i/4$ and $r_i > 0$ is the radius of the ball B_{r_i} of the domain of the coordinate map $\varphi_v : B_{r_i} \to U_v$. The diffeomorphism Θ is chosen in such a way that

(Θ1) $\Theta(\{0\} \times \mathbb{S}_\rho^{k-1} \times B_\rho^{n-k}) \subseteq M_i^-$
(Θ2) $\Theta(\{0\} \times V^-) = M_i^- \cap W_v^-$ where $V^- = \mathbb{S}_\rho^{k-1} \times \{0\}$
(Θ3) $\Theta(\mathcal{B}) \subseteq U_v \cap \{x \in M | h(x) < c_i - \rho^2/2\}$
(Θ4) $\Theta_*(-\frac{\partial}{\partial s}\big|_{\mathcal{B}}) = -\mathrm{grad}_{\|d_x h\|^2 g} h\big|_{\Theta(\mathcal{B})}$.

To satisfy (Θ4) the map Θ is defined in terms of the flow of the rescaled vector field $-\mathrm{grad}_{\|d_x h\|^2 g} h$. More precisely we set

$$\Theta : (-s_0, s_0) \times \mathbb{S}_\rho^{k-1} \times B_\rho^{n-k} \to U_v,\ (s, p, \xi) \mapsto \varphi_v(y(-s)).$$

Here $y(t) = (y^-(t), y^+(t)) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ is the solution of the initial value problem

$$\dot{y}(t) = Y^{(k)}(y(t)) = \frac{1}{\|y(t)\|^2}(y^-(t), -y^+(s)), \quad y(0) = (\lambda p, \xi),$$

where $Y^{(k)}$ is the rescaled standard vector field defined by (2.17) and the scalar $\lambda = \lambda(\xi, \rho)$ appearing in the initial condition $y(0)$ is determined in such a way that (Θ1) holds, i.e. $\varphi_v(\lambda p, \xi) \in M_i^-$. As $M_i^- = h^{-1}(c_i - \rho^2)$ and

$$(h \circ \varphi_v)(\lambda p, \xi) = c_i - \frac{1}{2}\|\lambda p\|^2 + \frac{1}{2}\|\xi\|^2$$

one has

$$\lambda(\xi, \rho) = (2 + \|\xi\|^2/\rho^2)^{1/2}.$$

By construction, (Θ2) holds. To verify (Θ3), note that for $(p, \xi) \in \mathbb{S}_\rho^{k-1} \times B_\rho^{n-k}$, one has $\Theta(0, p, \xi) = \varphi_v(\lambda p, \xi)$ and

$$\|(\lambda p, \xi)\|^2 = (2 + \|\xi\|^2/\rho^2)\rho^2 + \|\xi\|^2 < 4\rho^2 < (r_i/2)^2$$

as $0 \le \rho < r_i/4$. Hence $(\lambda p, \xi) \in B_{r_i}$ and therefore $\varphi_v(\lambda p, \xi) \in U_v$. Moreover, as by the definition of Θ, the set $\Theta(\{s\} \times \mathbb{S}_\rho^{k-1} \times B_\rho^{n-k})$ is contained in $h^{-1}(c_i - \rho^2 - s)$ it follows that (Θ3) is satisfied if $s_0 > 0$ is chosen sufficiently small.
We now apply Lemma 2.9 with V^+ given by

$$\{0\} \times V^+ = \Theta^{-1}\left(M_i^- \cap \bigsqcup_w W_w^+\right)$$

and the metric g_0 on $M_0 = \mathbb{R} \times \mathbb{S}_\rho^{k-1} \times \mathbb{R}^{n-k}$ chosen in such a way that its restriction to $\mathcal{B}$ coincides with the pullback $\Theta^*(\|d_x h\|^2 g\big|_{\Theta(\mathcal{B})})$ and $-\mathrm{grad}_{g_0} h_0 = -\frac{\partial}{\partial s}$. In view of the property (Θ4) and the assumption that U_v is a standard coordinate chart such a metric g_0 exists.
Denote by g' the metric on M given by g on $M \backslash \Theta(\mathcal{B})$ and on $\Theta(\mathcal{B})$ by $\|d_x h\|^{-2}\Theta_*(g_0'\big|_{\mathcal{B}})$ where $\Theta_*(g_0'\big|_{\mathcal{B}})$ is the push forward by Θ of the metric $g_0'\big|_{\mathcal{B}}$ provided by Lemma 2.9. Then g' is a smooth metric on M. As g_0' can be chosen arbitrarily close to g_0 in C^ℓ-topology, g' can be chosen arbitrarily close to g in C^ℓ-topology as well. By

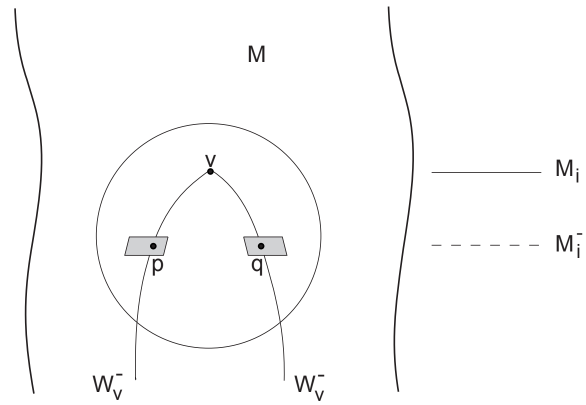

FIGURE 4. Illustration of $\Theta(\mathcal{B})$ (shaded area) in the case $k = 1$. Note that $W_v^- \cap M_i^- = \{p, q\}$

construction, $-\mathrm{grad}_{g'}h$ coincides with $-\mathrm{grad}_g h$ on $M \backslash \Theta(\mathcal{B})$ whereas on $\Theta(\mathcal{B})$ it is given by $\|d_x h\|^2 \Theta_*(-\mathrm{grad}_{g_0'} h_0)$ and

$$W_{Y_0'}^- \cap h_0^{-1}(\{0\}) = \Theta^{-1}(W_v^{-\prime} \cap M_i^-)$$

$$W_{Y_0'}^+ \cap h_0^{-1}\{0\} = \Theta^{-1}\left(\bigsqcup_w (W_w^{+\prime} \cap M_i^-)\right)$$

where $W_{Y_0'}^{\pm}$ are the submanifolds given by Lemma 2.9 and $W_w^{\pm\prime}$ denote the stable/unstable manifolds corresponding to $-\mathrm{grad}_{g'}h$. As $W_{Y_0'}^- \cap h_0^{-1}(\{s_0\}) = \{s_0\} \times V^-$ one concludes that $W_v^{-\prime} \cap M_i^-$ is completely contained in the image of Θ

$$\Theta(W_{Y_0'}^- \cap h_0^{-1}\{0\}) = W_v^{-\prime} \cap M_i^-.$$

By Lemma 2.9, it follows that $W_{Y_0'}^- \cap h_0^{-1}(\{0\}) \pitchfork W_{Y_0'}^+ \cap h_0^{-1}(\{0\})$ and hence

$$W_v^{-\prime} \cap M_i^- \pitchfork \bigsqcup_w (W_w^{+\prime} \cap M_i^-).$$

We therefore have proved that $W_v^{-\prime} \pitchfork W_w^{+\prime}$ for any $w \in \mathrm{Crit}(h)$. This completes the proof of Proposition 2.8. $\square$

In the remainder of this section we prove Lemma 2.9. The construction of g_0' involves two cut-off functions, introduced in [29] whose properties are stated in the following lemmas. Denote by $(g_{ij}(x))$ the $n \times n$ matrix that represents in local coordinates the metric g_0; as usual $(g^{ij}(x))$ denotes the inverse of $(g_{ij}(x))$. Choose $\eta_0 \equiv \eta(g_0) > 0$ so small that for any symmetric $n \times n$ matrix $\big(G^{ij}(x)\big)$ with support in $\mathcal{B} := (-s_0, s_0) \times \mathbb{S}_\rho^{k-1} \times B_\rho^{n-k}$ and $\sup_{x \in M_0} \left(\sum_{i,j} (G^{ij}(x))^2\right)^{1/2} \leq \eta_0$, the matrix $\big(g^{ij}(x) + G^{ij}(x)\big)$ is positive definite for any $x \in M_0$; then its inverse defines a Riemannian metric on M_0.

Lemma 2.10. *Let $s_0 > 0, \ell \in \mathbb{Z}_{\geq 1}$ and $0 < \eta \leq \eta_0$. Then there exists $\delta > 0$ depending on s_0, ℓ, and η such that for any $0 < \alpha \leq \delta$ there is a C^∞-function $\beta \equiv \beta_\alpha : \mathbb{R} \to \mathbb{R}$ with support in the open interval $(-s_0, s_0)$ and the property that β and its derivatives $d_t^j \beta$ $(1 \leq j \leq \ell)$ satisfy the estimates $0 \leq \beta \leq \eta$, $|d_t^j \beta| \leq \eta$, and*

$$\int_{-s_0}^{s_0} \beta(t) dt = \alpha.$$

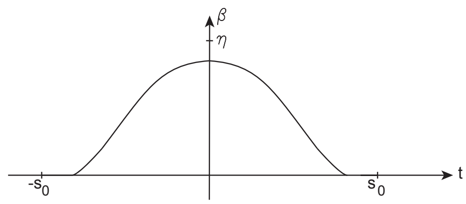

FIGURE 5. Graph of β

Proof. (Proof of Lemma 2.10) Choose a smooth cut-off function $\zeta : \mathbb{R} \to \mathbb{R}_{\geq 0}$ with $\operatorname{supp}(\zeta) \subseteq (-s_0, s_0)$ so that $\int_{-s_0}^{s_0} \zeta(s) ds = 1$ and let $\delta := \eta/(1 + \|\zeta\|_{C^\ell})$ where $\|\zeta\|_{C^\ell} = \sup_{\substack{s \in \mathbb{R} \\ 0 \leq j \leq \ell}} |d_s^j \zeta|$. Then for any $0 < \alpha \leq \delta$, the cut-off function $\beta_\alpha := \alpha \zeta$ has the desired properties. □

Lemma 2.11. *For any given $\ell \in \mathbb{Z}_{\geq 1}$ there is a constant $C_\ell > 0$ so that for any $\rho > 0$ there exists a C^∞-function $\gamma : \mathbb{R} \to [0, 1]$ with support in the open interval $(-\rho, \rho)$ satisfying $\sup_t |d_t^j \gamma| \leq C_r (\rho/2)^{-j}$ for $1 \leq j \leq \ell$ and $\gamma(t) = 1$ for $|t| \leq \rho/3$.*

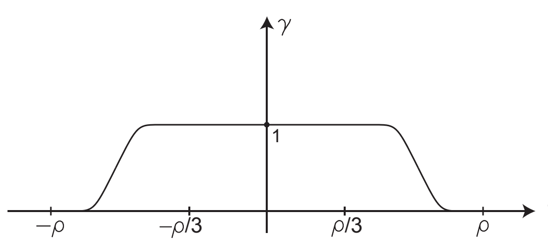

FIGURE 6. Graph of γ

Proof. (Proof of Lemma 2.11) Let $f : \mathbb{R} \to [0, 1]$ be a smooth increasing function with $f(t) = 0$ for $t \leq 0$ and $f(t) = 1$ for $t \geq 1$ and set $C_\ell := \|f\|_{C^\ell} = \sup_{\substack{t \in \mathbb{R} \\ 0 \leq j \leq \ell}} |d_t^j f|$. Then define γ to be the even function determined by

$$\gamma(t) := \begin{cases} 0 & t \leq -\frac{11}{12}\rho \\ f\left(\frac{\rho}{2}(t - \frac{11}{12}\rho)\right) & -\frac{11}{12}\rho < t \leq -\frac{5}{12}\rho \\ 1 & -\frac{5}{12}\rho < t \leq 0. \end{cases}$$

The function γ has all the required properties. $\square$

Proof. (Proof of Lemma 2.9) The proof consists of three parts: the construction of Y_0', the verification of the transversality property (ii), and the construction of g_0'.

Construction of Y_0': Choose $0 < \eta \leq \eta_0$ arbitrarily small. Let π denote the projection of the submanifold $V_0^+ \subseteq \mathbb{S}_\rho^{k-1} \times \mathbb{R}^{n-k}$ on the second component, $\pi : V_0^+ \to \mathbb{R}^{n-k}$. By Sard's theorem [27] there exists a regular value a^+ of π with $0 < \|a^+\| \leq \min(\delta, \rho/3)$. Here $\delta > 0$ is given by Lemma 2.10 and depends on the choice of η. Choose an orthonormal basis of $\mathbb{R}^{n-k}$ so that the corresponding coordinates of a^+ are given by $(\alpha, 0, \ldots, 0)$. Note that $0 < \alpha < \rho/3$. Given these data, define the following vector field on M_0

$$Y_0'(s,p,\xi) = -\frac{\partial}{\partial s} - \beta(s)\gamma(\|\xi\|)\frac{\partial}{\partial \xi_1}$$

where $\beta \equiv \beta_\alpha$ and γ are the cut-off functions given in Lemma 2.10 and Lemma 2.11 respectively.

Transversality property: By the definition of γ, $\gamma(\|\xi\|) = 1$ for $\|\xi\| \leq \rho/3$. As $\alpha < \rho/3$ and $\int_{-s_0}^{s_0} \beta(\tau)d\tau = \alpha$, the solution $\Psi_t^{(0)}(s_0,p,0)$ of $\frac{d}{dt}\Psi_t^{(0)} = Y_0'$ with initial data $(s_0,p,0) \in \mathbb{R} \times S_p^{k-1} \times \{0\}$, can be easily computed. Note that $s(t) = s_0 - \int_0^t dt = s_0 - t$. Hence for $t = 2s_0$ one gets

$$\Psi_{2s_0}^{(0)}(s_0,p,0) = (-s_0, p, \xi(s_0,p))$$

where

$$\xi(s_0,p) = \left(\int_0^{2s_0} \beta(s_0 - t)dt, 0, \ldots, 0\right) = (\alpha, 0, \ldots, 0).$$

As $Y_0' = Y_0$ on $M_0 \backslash \mathcal{B}$ one has

$$W_{Y_0'}^- \cap h_0^{-1}\{s_0\} = W_{Y_0}^- \cap h_0^{-1}\{s_0\} = \{s_0\} \times V^-$$

and hence

$$W_{Y_0'}^- \cap h_0^{-1}\{-s_0\} = \Psi_{2s_0}^{(0)}\left(W_{Y_0'}^- \cap h^{-1}\{s_0\}\right).$$

Combined with $V^- = \mathbb{S}_\rho^{n-k} \times \{0\}$ one sees that

$$W_{Y_0'}^- \cap h_0^{-1}\{-s_0\} = \{-s_0\} \times \mathbb{S}_\rho^{k-1} \times \{a^+\}.$$

Similarly, one has

$$W_{Y_0'}^+ \cap h_0^{-1}\{-s_0\} = W_{Y_0}^+ \cap h_0^{-1}\{-s_0\} = \{-s_0\} \times V^+.$$

As a^+ is a regular value of $\pi : V^+ \to \mathbb{R}^{n-k}$ one concludes that $W_{Y_0'}^- \cap h_0^{-1}\{-s_0\}$ and $W_{Y_0'}^+ \cap h_0^{-1}\{-s_0\}$ intersect transversally inside $h_0^{-1}(\{-s_0\})$, hence $W_{Y_0'}^-$ and $W_{Y_0'}^+$ intersect transversally as well.

Construction of g_0': To describe g_0', it is convenient to reorder the coordinates $(s, p, \xi_1, \ldots, \xi_{n-k})$ so that in the new coordinates $\zeta = (\zeta_1, \ldots, \zeta_n)$ one has $\zeta_1 = s$ and $\zeta_2 = \xi_1$. With respect to these coordinates, the coefficients $g_0^{'ij}$ are defined as follows

$$g_0^{'ij}(\zeta) := \begin{cases} g_0^{ij}(\zeta) & \text{if } (i,j) \neq (1,2) \text{ or } (2,1) \\ g_0^{ij}(\zeta) + \beta(\zeta_1)\gamma(\|\xi\|) & \text{if } (i,j) = (1,2) \text{ or } (2,1). \end{cases}$$

By Lemma 2.10, $\beta \leq \eta$ and as $\eta \leq \eta_0$, the matrix $\left(g_0'^{ij}\right)$ is positive definite, hence has an inverse (g'_{0ij}) which defines a Riemannian metric on M. As $\beta \leq \eta, |\dot{\beta}| \leq \eta$, and $0 < \eta \leq \eta_0$ can be chosen arbitrarily small, g'_0 is arbitrarily close to g_0 in the C^ℓ-topology. The gradient $\mathrm{grad}_{g'_0} h_0$ can be easily computed. By definition,

$$\mathrm{grad}_{g'_0} h_0(\zeta) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} g_0'^{ij} \frac{\partial h_0}{\partial \zeta_j} \right) \frac{\partial}{\partial \zeta_i}$$

and $h_0(\zeta) = \zeta_1$ $(= s)$. From $\mathrm{grad}_{g_0} h_0 = \frac{\partial}{\partial s}$ we read off that $g_0^{i1} = \delta_{1i}$. Hence

$$\mathrm{grad}_{g'} h(\zeta) = \frac{\partial}{\partial \zeta_1} + \beta(\zeta_1)\gamma(\|\xi\|)\frac{\partial}{\partial \zeta_2}$$

or

$$-\mathrm{grad}_{g'_0} h_0(\zeta) = -\frac{\partial}{\partial s} - \beta(s)\gamma(\|\xi\|)\frac{\partial}{\partial \xi_1} = Y'_0(\zeta)$$

as claimed. Further note that g'_0 coincides with g_0 in a neighborhood of $M_0 \backslash \mathcal{B}$. This completes the proof of Lemma 2.9. $\square$

Remark 2.2. Comments on the proof of Theorem 2.6: (i) The hypothesis of being h-admissible for the metric g is not used in the proof of Theorem 2.6. (ii) The proof of Theorem 2.6 could be shortened by applying transversality theorems to make $W_v^- \cap M_k^-$ transversal to $W_w^+ \cap M_k^-$. However, this has to be done with care as $W_w^+ \cap M_k^-$ is not necessarily a closed subset of M_k^-. (iii) A conceptually different proof of Theorem 2.6, based on Fredholm theory, can be found in [28].

2.3. **Spaces of broken trajectories.** Let M be a smooth manifold and (h, X) a Morse-Smale pair. In particular this means that h is proper (cf Definition 2.3). It is useful to define the following partial ordering for critical points $w, v \in \mathrm{Crit}(h)$

$$w < v \;\text{ iff }\; i(w) < i(v) \text{ and } h(w) < h(v)$$

and

$$w \leq v \;\text{ iff }\; w < v \text{ or } w = v.$$

According to (2.15), $\mathcal{T}(v, w) = (W_v^- \cap W_w^+)/\mathbb{R}$ denotes the space of unbroken trajectories from v to w. For $v, w \in \mathrm{Crit}(h)$ with $w < v$ introduce

$$\mathcal{B}(v, w) := \bigcup_{w < v_\ell < \ldots < v_1 < v} \mathcal{T}(v, v_1) \times \ldots \times \mathcal{T}(v_\ell, w)$$

$$\hat{W}_v^- := \bigcup_{\substack{w \in \mathrm{Crit}(h) \\ w \leq v}} \mathcal{B}(v, w) \times W_w^-$$

where $\mathcal{B}(v, v) := \{v\}$. Further let $\hat{i}_v : \hat{W}_v^- \to M$ be the map whose restriction to $\mathcal{B}(v, w) \times W_w^-$ is given by the projection onto the second component, composed with the inclusion $i_w : W_w^- \hookrightarrow M$,

$$\hat{i}_v : \mathcal{B}(v, w) \times W_w^- \to W_w^- \hookrightarrow M.$$

Note that $\hat{i}_v$ is an extension of the inclusion $i_v : W_v^- \hookrightarrow M$ as $\mathcal{B}(v, v) \times W_v^- = \{v\} \times W_v^-$. Elements in $\mathcal{B}(v, w)$ are called trajectories connecting v and w whereas an element in $\mathcal{B}(v, w) \backslash \mathcal{T}(v, w)$ is referred to as a broken trajectory. Note that an

element in $\hat{W}_v^-$ is a (possibly broken) trajectory from the critical point v to a point x on M which is the image of that element by the map $\hat{i}_v$.
Our goal is to prove that $\hat{W}_v^-$ and $\mathcal{B}(v,w)$ have a canonical differentiable structure of a manifold with corners so that the unstable manifold W_v^- is the interior of $\hat{W}_v^-$, $\mathcal{T}(v,w)$ is the interior of $\mathcal{B}(v,w)$ and $\hat{i}_v : \hat{W}_v^- \to M$ is smooth and proper. As h is assumed to be smooth and proper it then follows that

$$\hat{h}_v := h \circ \hat{i}_v$$

is smooth and proper as well. In this subsection, as a first step, we describe for any given $v \in \mathrm{Crit}(h)$ the topology of the set $\hat{W}_v^-$ and then verify that $\hat{W}_v^-$ is a Hausdorff space and $\hat{i}_v$ is continuous and proper. Let us briefly outline how we will do this.

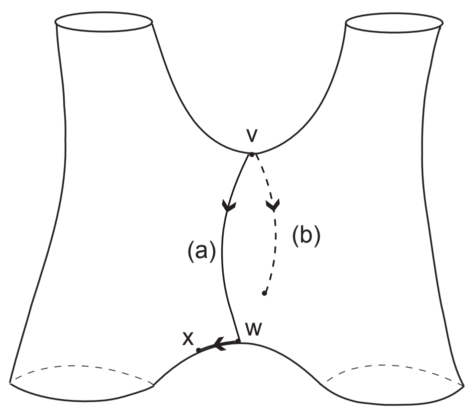

FIGURE 7. Examples of elements of $\hat{W}_v^-$: $(a), (b)$

First observe that $\mathcal{T}(v,w) \subseteq \mathcal{B}(v,w)$ and for any v, w with $v < w$ one has $\mathcal{T}(v,w) = \emptyset$. The canonical parametrization of a trajectory $\gamma \in \mathcal{B}(v,w)$, denoted also by γ, is defined to be a continuous map $\gamma : [h(w), h(v)] \to M$ so that $h(\gamma(s)) = s$ for any $h(w) \le s \le h(v)$. We note that away from the critical points, $\gamma(h(v) - t)$ is a smooth solution for the rescaled vector field Y introduced in (2.9). Similarly, an element $(\gamma, x) \in \mathcal{B}(v,w) \times W_w^- \subseteq \hat{W}_v^-$ can be viewed as a broken trajectory connecting v and x and its canonical, continuous parametrization

$$\gamma_x : [h(x), h(v)] \to M \tag{2.22}$$

is determined by the property that $h(\gamma(s)) = s$ for any $h(x) \le s \le h(v)$. Recall that the critical values of h have been denoted by $\ldots < c_\ell < c_{\ell-1} < \ldots$. Assume that $h(v) = c_k$. The topology of $\hat{W}_v^-$ will be defined by the covering

$$\hat{h}_v^{-1}\left([c_{\ell+1} + \delta_\ell, c_{\ell-1} - \delta_\ell]\right), \quad \ell = k, k+1, \ldots$$

where for any $\ell \ge k$, the positive number δ_ℓ is chosen sufficiently small — see below. The spaces $\hat{h}_v^{-1}([c_{\ell+1} + \delta_\ell, c_{\ell-1} - \delta_\ell])$ are endowed with a topology so that they become compact Hausdorff spaces as follows: for $\ell = k$, it is identified with a compact subset of W_v^- whereas for $\ell \geq k+1$ it is identified with a compact subset in

$$\left(\prod_{j=k}^{\ell-1} h^{-1}(\{c_j - \varepsilon\})\right) \times h^{-1}\left([c_{\ell+1} + \delta_\ell, c_{\ell-1} - \delta_\ell]\right)$$

by associating to an element $(\gamma, x) \in \hat{h}_v^{-1}\left([c_{\ell+1} + \delta_\ell, c_{\ell-1} - \delta_\ell]\right)$ the sequence of points $\left((x_j^-)_j, x\right)$ on M with x_j^- being the (unique) point on γ with $h(x_j) = c_j - \varepsilon$. The parameter $\varepsilon > 0$ is chosen sufficiently small so that $c_j - 2\varepsilon > c_{j+1}$ for any j. We will show that the topology on $\hat{h}_v^{-1}\left([c_{\ell+1} + \delta_\ell, c_{\ell-1} - \delta_\ell]\right)$ is independent of the choice of ε and thus canonically defined.

Let us now treat the above outlined construction in detail. In a first step consider the set $\mathcal{B}(v, w)$ for given critical points v, w with $w < v$. Let $c_m \equiv h(w) < \ldots < c_k = h(v)$ be the set of all critical values of h between $h(w)$ and $h(v)$. For any $k \leq j \leq m$ introduce the level sets $M_j^- \equiv M_{j,\varepsilon}^- = h^{-1}(\{c_j - \varepsilon\})$ with $\varepsilon > 0$ chosen as above. Given any $0 < \varepsilon' < \varepsilon$, the flow $\Psi_t(x)$ defined in (2.10) then provides a diffeomorphism $\Psi_{\varepsilon - \varepsilon'} : M_{j,\varepsilon'}^- \to M_{j,\varepsilon}^-$. We define

$$\begin{aligned} J_\varepsilon : \mathcal{B}(v, w) &\to M_{k,\varepsilon}^- \times \ldots \times M_{m-1,\varepsilon}^- \\ \gamma &\mapsto \left(\gamma(c_k - \varepsilon), \ldots, \gamma(c_{m-1} - \varepsilon)\right). \end{aligned}$$

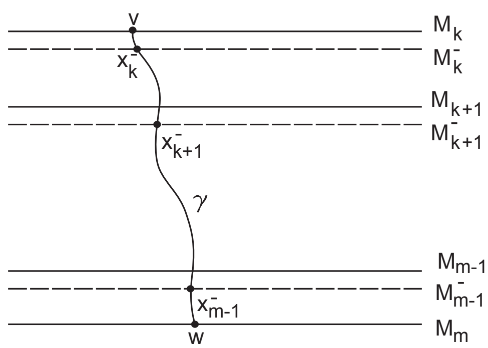

FIGURE 8. $J_\varepsilon(\gamma) = (x_k^-, \ldots, x_{m-1}^-)$

Using the flow Ψ_t one sees that J_ε is injective and for any $0 < \varepsilon' < \varepsilon$, we have

$$J_\varepsilon = (\Psi_{\varepsilon-\varepsilon'} \times \ldots \times \Psi_{\varepsilon-\varepsilon'}) \circ J_{\varepsilon'}.$$

Hence via the identification of $\mathcal{B}(v, w)$ with a subset in $M_{k,\varepsilon}^- \times \ldots \times M_{m-1,\varepsilon}^-$ by J_ε, the set $\mathcal{B}(v, w)$ becomes a Hausdorff space whose topology is *independent* of ε and thus canonically defined. As h is assumed to be proper, the level sets $M_{j,\varepsilon}^-$ are compact and hence $M_{k,\varepsilon}^- \times \ldots \times M_{m-1,\varepsilon}^-$ is compact as well. The compactness of $\mathcal{B}(v, w)$ then follows from the following

Proposition 2.12. *Let $v, w \in \mathrm{Crit}(h)$ with $h(v) = c_k > h(w) = c_m$ and let $\varepsilon > 0$ be as above. Then $J_\varepsilon(\mathcal{B}(v, w))$ is a closed subset of $M_{k,\varepsilon}^- \times \ldots \times M_{m-1,\varepsilon}^-$.*

Proof. To prove that $J_\varepsilon\left(\mathcal{B}(v, w)\right)$ is closed consider a sequence $(\gamma_i)_{i\geq 1}$ in $\mathcal{B}(v, w)$ so that $\left(J_\varepsilon(\gamma_i)\right)_{i\geq 1}$ is a convergent sequence in $M_{k,\varepsilon}^- \times \ldots \times M_{m-1,\varepsilon}^-$ with limit $(a_k, \ldots,$

a_{m-1}). By the chosen parametrization of the curves γ_i and the extension of the flow Ψ_t of Lemma 2.4 one has

$$\Psi_{-\varepsilon}\left(\gamma_i(c_k-\varepsilon)\right)=v; \quad \Psi_{(c_{m-1}-\varepsilon)-c_m}\left(\gamma_i(c_{m-1}-\varepsilon)\right)=w$$

and, for any $k \leq j \leq m-2$,

$$\Psi_{(c_j-\varepsilon)-c_{j+1}}\left(\gamma_i(c_j-\varepsilon)\right)=\Psi_{-\varepsilon}\left(\gamma_i(c_{j+1}-\varepsilon)\right).$$

Hence, using the continuity of $\Psi_t(x)$ in x and taking the limit $i \to \infty$, one obtains for any $k \leq j \leq m-2$

$$b_j := \Psi_{(c_j-\varepsilon)-c_{j+1}}(a_j)=\Psi_{-\varepsilon}(a_{j+1})$$

which is a point on the level set $L_{c_{j+1}}$ as well as

$$\Psi_{-\varepsilon}(a_k)=v;\ \Psi_{(c_{m-1}-\varepsilon)-c_m}(a_{m-1})=w.$$

Denote by $v_1,\dots,v_\ell$ the critical points among the elements $b_1,\dots,b_{m-1}$ ordered so that $h(w)<h(v_\ell)<\dots<h(v_1)<h(v)$. Then $(a_k,\dots,a_{m-1})$ defines a unique trajectory $\gamma \in \mathcal{T}(v,v_1)\times\mathcal{T}(v_1,v_2)\times\dots\times\mathcal{T}(v_\ell,w)$ with $J_\varepsilon(\gamma)=(a_k,\dots,a_{m-1})$. $\square$

Let us now turn our attention to $\hat{W}_v^-$. Assume again that $h(v)=c_k$. Choose for any $j \geq k$,

$$0<\delta_j<\frac{1}{2}\min(c_j-c_{j+1},c_{j-1}-c_j) \tag{2.23}$$

and introduce

$$\hat{W}_{v,j}^- \equiv \hat{W}_{v,j,\delta_j}^- := \hat{h}_v^{-1}([c_{j+1}+\delta_j,c_{j-1}-\delta_j]).$$

Note that $\hat{W}_{v,k,\delta_k}^-$ is contained in $\{v\}\times W_v^-$ whereas for $j \geq k+1, \hat{W}_{v,j,\delta_j}$ is the subset of $\hat{W}_v^-$ of elements (γ,x) consisting of a (possibly broken) trajectory $\gamma \in \mathcal{B}(v,w)$ for some $w \in \mathrm{Crit}(h)$ with $h(w)\leq h(v)$ and $x \in W_w^-$ with x satisfying $c_{j+1}+\delta_j \leq h(x) \leq c_{j-1}-\delta_j$. Further define the map

$$\begin{aligned}\hat{J}_{\varepsilon,j} : \hat{W}_{v,j}^- &\to M_{k,\varepsilon}^-\times\dots\times M_{j-1,\varepsilon}^-\times h^{-1}\left([c_{j+1}+\delta_j,c_{j-1}-\delta_j]\right)\\ (\gamma,x) &\mapsto (J_{\varepsilon,j}(\gamma_x),x),\end{aligned}$$

where

$$J_{\varepsilon,j}(\gamma_x) := (\gamma_x(c_k-\varepsilon),\dots,\gamma_x(c_{j-1}-\varepsilon))$$

with γ_x denoting the (possibly broken) trajectory from v to x, defined by (2.22). Again one easily sees that $\hat{J}_{\varepsilon,j}$ is injective and that for any $0<\varepsilon'<\varepsilon$

$$\hat{J}_{\varepsilon,j}=(\Psi_{\varepsilon-\varepsilon'}\times\dots\times\Psi_{\varepsilon-\varepsilon'}\times Id)\circ\hat{J}_{\varepsilon',j}.$$

Hence via the identification by $\hat{J}_{\varepsilon,j}, \hat{W}_{v,j}^-$ becomes a compact Hausdorff space whose topology is *independent* of ε and hence canonically defined. The sets $\hat{W}_{v,j}^-$ will be used to define a Hausdorff topology on $\hat{W}_v^-$.

Proposition 2.13. *Let $v \in Crit(h)$ with $h(v)=c_k$. Then for any $j \geq k$, the set $\hat{J}_{\varepsilon,j}(\hat{W}_{v,j})$ is a closed subset of $M_{k,\varepsilon}^-\times\dots\times M_{j-1,\varepsilon}^-\times h^{-1}\left([c_{j+1}+\delta_j,c_{j-1}-\delta_j]\right)$ and the restriction of $\hat{i}_v$ to $\hat{W}_{v,j}^-$ is continuous. For any $k \leq j,\ j'$ with $j \neq j'$, the topologies induced on $\hat{W}_{v,j}^-\cap\hat{W}_{v,j'}^-$ by $\hat{W}_{v,j}^-$ and $\hat{W}_{v,j'}^-$ coincide and the intersection is closed in both $\hat{W}_{v,j}^-$ and $\hat{W}_{v,j'}^-$.*

Proof. Note that $\hat{J}_{\varepsilon,k}(\hat{W}_{v,k})$ is a closed subset of $\{v\} \times h^{-1}([c_{k+1}+\delta_k, c_{k-1}-\delta_k])$. To prove that for $j \geq k+1$, $\hat{J}_{\varepsilon,j}(\hat{W}_{v,j})$ is closed consider a sequence $(\gamma_i, x_i)_{i\geq 1}$ in $\hat{W}^-_{v,j}$ so that $\left(\hat{J}_{\varepsilon,j}(\gamma_i, x_i)\right)_{i\geq 1}$ is a convergent sequence in

$$M^-_{k,\varepsilon} \times \ldots \times M^-_{j-1,\varepsilon} \times h^{-1}\big([c_{j+1}+\delta_j, c_{j-1}-\delta_j]\big)$$

with limit $(a_k, \ldots, a_{j-1}, x)$. As h is continuous,

$$c_{j+1}+\delta_j \leq h(x) \leq c_{j-1}-\delta_j.$$

Arguing as in the proof of Proposition 2.12 one sees that there exists $(\gamma, x) \in \hat{W}^-_v$ with $\gamma \in \mathcal{B}(v,w)$ for some $w \in \mathrm{Crit}(h)$ with $h(x) \leq h(w) \leq h(v)$ so that

$$\hat{J}_{\varepsilon,j}(\gamma, x) = (a_k, \ldots, a_{j-1}, x).$$

From the definition of $\hat{i}_v$ it follows that the restriction of $\hat{i}_v$ to $\hat{W}^-_{v,j}$ is continuous. Finally, let us consider the intersection $\hat{W}_{v,j} \cap \hat{W}^-_{v,j'}$. Let $j, j' \geq k$. For $\big| j - j' \big| \geq 2$, one has $\hat{W}^-_{v,j} \cap \hat{W}^-_{v,j'} = \emptyset$, hence it remains to consider the case $j' = j+1$. As by (2.23) $c_{j+1}+\delta_j < c_j - \delta_{j+1}$, it follows that $D_{j,j+1} := \hat{W}^-_{v,j} \cap \hat{W}^-_{v,j+1}$ is the set of elements (γ, x) in $\hat{W}^-_v$ with $c_{j+1}+\delta_j \leq h(x) \leq c_j - \delta_{j+1}$. Note that $\hat{J}_{\varepsilon,j}(\gamma, x) = (J_{\varepsilon,j}(\gamma_x), x)$ and

$$\hat{J}_{\varepsilon,j+1}(\gamma, x) = (J_{\varepsilon,j}(\gamma_x), \gamma_x(c_j - \varepsilon), x)\,.$$

As $\gamma_x(c_j - \varepsilon)$ and x are on the same trajectory, one has

$$\gamma_x(c_j - \varepsilon) = \Psi_{(c_j-\varepsilon)-h(x)}(x),$$

thus

$$\hat{J}_{\varepsilon,j}(\gamma, x) \mapsto \big(J_{\varepsilon,j}(\gamma_x), \Psi_{(c_j-\varepsilon)-h(x)}(x), x\big)$$

is a homeomorphism from $\hat{J}_{\varepsilon,j}\ (D_{j,j+1})$ onto $\hat{J}_{\varepsilon,j+1}(D_{j,j+1})$.
As the intersection $D_{j,j+1}$ is equal to $\hat{i}_v^{-1}([c_{j+1}+\delta_j, c_j - \delta_{j+1}])$ and the restrictions of $\hat{i}_v$ to $\hat{W}^-_{v,j}$ and $\hat{W}^-_{v,j+1}$ are both continuous, $D_{j,j+1}$ is closed in both of these spaces. □

Notice that $\hat{W}^-_v = \cup_{j\geq k} \hat{W}^-_{v,j}$. By Proposition 2.13, the covering $(\hat{W}^-_{v,j})_{j\geq k}$ then defines a Hausdorff topology on $\hat{W}^-_v$, and $\hat{i}_v : \hat{W}^-_v \to M$ is continuous. We leave it to the reader to verify that the topology on $\hat{W}^-_v$ defined in this way is independent of the choice of the δ_j $(j \geq k)$. It can be done in a way similar to how we proved that the topology is independent of ε. Proposition 2.12 and 2.13 then lead to the following

Theorem 2.14. *Assume that M is a smooth manifold, (h, X) a Morse-Smale pair and v, w arbitrary critical points of h with $w < v$. Then*

(i) *$\mathcal{B}(v,w)$ is a compact Hausdorff space.*

(ii) *$\hat{W}^-_v$ is a Hausdorff space and both $\hat{i}_v$ and $\hat{h}_v = h \circ \hat{i}_v$ are proper continuous maps. In particular, if in addition, M is compact so is $\hat{W}^-_v$.*

Proof. (i) follows from Proposition 2.12. By Proposition 2.13, $\hat{W}^-_v$ is a Hausdorff space and $\hat{i}_v$, and therefore $\hat{h}_v = h \circ \hat{i}_v$, are continuous. If $\hat{i}_v$ is proper, so is $\hat{h}_v$. To show that $\hat{i}_v$ is proper it remains to prove that for any compact set $K \subseteq M, \hat{i}_v^{-1}(K)$ is *contained* in a compact subset of $\hat{W}^-_v$. As h is proper, $h^{-1}(h(K))$ is a compact set. Note that $K \subseteq h^{-1}(h(K))$ and $\hat{i}_v^{-1}(K) \subseteq \hat{h}_v^{-1}(h(K))$. By the definition of the compact sets $\hat{W}^-_{v,j}$, the preimage $\hat{h}_v^{-1}(h(K))$ is contained in the union of finitely many $\hat{W}^-_{v,j}$ and hence contained in a compact subset of $\hat{W}^-_v$. If, in addition, M is compact then $\hat{W}^-_v = \hat{i}_v^{-1}(M)$ is compact by the properness of $\hat{i}_v$. □

3. Manifold with corners

The notion of a manifold with corners is a generalization of the notion of a smooth manifold with boundary in the sense that the boundary of such a manifold is not required to be a smooth manifold. One of the main reasons to consider such a generalization is the fact that the product of two manifolds with boundary is not a manifold with boundary. The local model proposed for such a generalization is the positive quadrant $\mathbb{R}^n_{\geq 0}$, hence we first study smooth $\mathbb{R}^n_{\geq 0}$-manifolds – see Subsection 3.1 below. In Subsection 3.2 we study manifolds with corners, a special class of $\mathbb{R}^n_{\geq 0}$-manifolds having the property that all their faces (see below for a precise definition) are again $\mathbb{R}^k_{\geq 0}$-manifolds for appropriate k. It turns out that the concepts, results and methods of the analysis on manifolds with boundary can be extended in a natural way to this class of manifolds. In Section 4 we will show that the canonical compactification of the unstable manifolds and of the space of trajectories associated to a Morse-Smale pair (h, X) on a closed manifold have the structure of oriented manifolds with corners.
For further information on manifolds with corners and related topics see e.g. [9], [11], [12], [22], [23], [26].

3.1. $\mathbb{R}^n_{\geq 0}$**-manifolds.** Let us denote by $\mathbb{R}^n_{\geq 0}$ the positive quadrant in $\mathbb{R}^n$,

$$\mathbb{R}^n_{\geq 0} = \mathbb{R}_{\geq 0} \times \ldots \times \mathbb{R}_{\geq 0} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \,\big|\, x_i \geq 0 \quad \forall i\}$$

endowed with the topology induced from $\mathbb{R}^n$. Recall that a map $f : U \to \mathbb{R}^m$ from an open subset U of $\mathbb{R}^n_{\geq 0}$ into $\mathbb{R}^m$ is said to be C^∞-smooth (or smooth, for short) if there exists an open neighborhood V of U in $\mathbb{R}^n$ and a smooth map $g : V \to \mathbb{R}^m$ such that the restriction of g to U is f. For any $x \in U$, the differential $d_x f := d_x g : \mathbb{R}^n \to \mathbb{R}^m$ is well defined, i.e. does *not* depend on the choice of the extension g of f. Let U, V be open subsets of $\mathbb{R}^n_{\geq 0}$. We say that $f : U \to V$ is a C^∞-diffeomorphism (or, diffeomorphism for short) if f is bijective and f as well as f^{-1} are smooth. For such a map, the Jacobian $d_x f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective for any $x \in U$. More generally, a smooth map $f : U \to \mathbb{R}^m$ is said to be an immersion if $d_x f$ is $1-1$ for any $x \in U$ and it is an embedding if in addition, f is a homeomorphism onto its image. Further we recall that a topological space is said to be paracompact if any covering by open sets has a locally finite refinement.
A family $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}$ of charts $(U_\alpha, \varphi_\alpha)$ is said to be an $\mathbb{R}^n_{\geq 0}$-atlas of a paracompact Hausdorff space M if $\{U_\alpha\}$ is an open cover of M and $\varphi_\alpha : U_\alpha \to V_\alpha$ is a homeomorphism onto an open subset V_α of $\mathbb{R}^n_{\geq 0}$ so that any two charts $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ in $\mathcal{U}$ are C^∞-compatible, i.e. $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a C^∞-diffeomorphism. Adding additional compatible charts one obtain larger atlases. It is rather straightforward to verify that any atlas can be enlarged to a unique maximal atlas.

Definition 3.1. *A pair* $(M, \mathcal{U})$ *of a paracompact Hausdorff space* M *equipped with a maximal* $\mathbb{R}^n_{\geq 0}$*-atlas is called a smooth* $\mathbb{R}^n_{\geq 0}$*-manifold.*

In view of the above observation, a pair $(M, \mathcal{U})$ with M a paracompact Hausdorff space and $\mathcal{U}$ an atlas, not necessarily maximal, will specify a smooth $\mathbb{R}^n_{\geq 0}$-manifold structure. The $\mathbb{R}^n_{\geq 0}-$ manifold structure is defined by the maximal atlas which contains $\mathcal{U}$.
In the sequel, we often write M instead of $(M, \mathcal{U})$ and refer to $\mathcal{U}$ as a $\mathbb{R}^n_{\geq 0}$-smooth or differential structure of M.
A natural class of $\mathbb{R}^n_{\geq 0}$-manifold is defined in terms of a regular system of inequalities as follows. Let $\tilde{M}$ be a smooth manifold (without boundary) of dimension n, let $g_i : \tilde{M} \to \mathbb{R}, 1 \leq i \leq N$, be a family of $N \geq 1$ smooth functions and set

$$M := \{x \in \tilde{M} \,\big|\, g_i(x) \geq 0 \quad \forall 1 \leq i \leq N\}. \tag{3.1}$$

For any $x \in M$ define $J(x) = \{1 \leq i \leq N \mid g_i(x) = 0\}$ and assume that the differentials

$$(d_x g_i)_{i \in J(x)} \text{ are linearly independent in } T^*_x \tilde{M}. \tag{3.2}$$

If the interior $\overset{\circ}{M}$ is not empty then M is a smooth $\mathbb{R}^n_{\geq 0}$-manifold when endowed with the $\mathbb{R}^n_{\geq 0}$-differentiable structure induced by the following atlas: for any $x \in M$, choose a (sufficiently small) coordinate chart (U, φ) of $\tilde{M}$ so that U is an open neighborhood of x in $\tilde{M}$ satisfying for any $y \in U$

$$g_i(y) > 0 \quad \forall i \notin J(x)$$

and

$$(d_y g_i)_{i \in J(x)} \text{ linearly independent.}$$

Notice that $|J(x)| \geq n$. We renumber the functions g_i so that $J(x) = \{1, \ldots, m\}$ with $m \leq n$. Using a coordinate map $\varphi : U \to V \subseteq \mathbb{R}^n$ one can construct a family of smooth functions $h_i : U \to \mathbb{R}_{>0}, i = m+1, \ldots, n$ so that

$$(g_i)_{1 \leq i \leq m} \times (h_i)_{m+1 \leq i \leq n} : U \to \mathbb{R}^n$$

is a smooth embedding. In this way one obtains a smooth coordinate chart (U_x, φ_x) where $U_x = U \cap M$ and $\varphi_x : U_x \to \mathbb{R}^n_{\geq 0}$ is given by the restriction of $(g_i)_{i \in J(x)} \times (h_i)_{i \notin J(x)}$ to U_x. One then verifies that $(U_x, \varphi_x)_{x \in M}$ is a $\mathbb{R}^n_{\geq 0}$-atlas for M.
Figure 9 shows an example of a $\mathbb{R}^2_{\geq 0}$-manifold of this type. The triangle ABC on the sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$ can be thought of as the intersection of half spaces $\{g_\alpha \geq 0\}, \alpha \in \{a, b, c\}$ where the smooth functions $g_\alpha : U \subseteq \mathbb{S}^2 \to \mathbb{R}$, defined on an open neighborhood U of the triangle, are conveniently chosen so that the g_α's satisfy the regularity condition introduced above, the intersection $\bigcap_{\alpha \in \{a,b,c\}} \{g_\alpha \geq 0\}$ is the triangle ABC and for any α in $\{a, b, c\}$, the zero set $\{g_\alpha = 0\}$ contains the side α of the triangle ABC.
Using coordinate charts one defines the notion of a smooth map, a diffeomorphism, an embedding, an immersion, etc. of a $\mathbb{R}^n_{\geq 0}$-manifold into a $\mathbb{R}^m_{\geq 0}$-manifold in the usual manner as well as the notion of a smooth $\mathbb{C}$-vector bundle (or $\mathbb{R}$-vector bundle) $E \to M$ over a $\mathbb{R}^n_{\geq 0}$-manifold and the space of smooth sections, $s : M \to E$.
Next we want to introduce the notion of a tangent space for $\mathbb{R}^n_{\geq 0}$-manifolds. Let M be a smooth $\mathbb{R}^n_{\geq 0}$-manifold and $\varphi_\alpha : U_\alpha \to V_\alpha$ a chart. For $x \in U_\alpha$, denote by $J_\alpha(x)$ the subset of $\{1, \ldots, n\}$ given by

$$J_\alpha(x) := \{1 \leq i \leq n \,\big|\, \varphi^i_\alpha(x) = 0\}$$

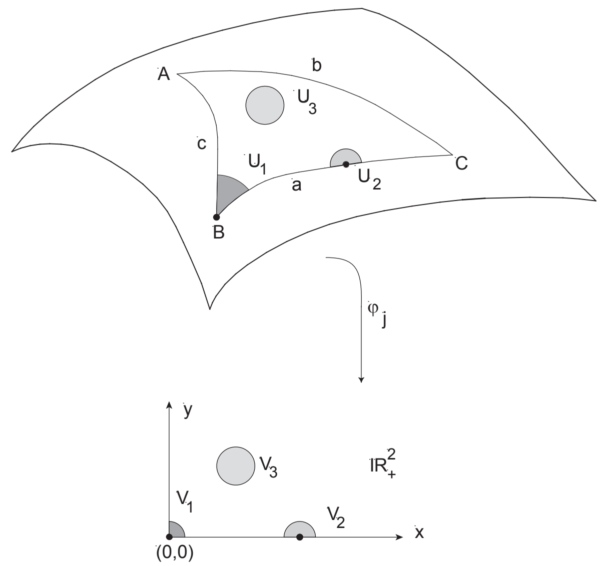

FIGURE 9. Triangle ABC; a $\mathbb{R}^2_{\geq 0}$-manifold on a piece of $S^2 \subset \mathbb{R}^3$. $\partial_1 M = a \cup b \cup c$; $\varphi_j : U_j \to V_j$ coordinate maps

where $\varphi^1_\alpha(x), \dots, \varphi^n_\alpha(x)$ denote the components of $\varphi_\alpha(x)$. Introduce

$$\mathcal{C}_\alpha(x) := \{\xi \in \mathbb{R}^n \,\big|\, \xi_i \geq 0 \quad \forall i \in J_\alpha(x)\}$$
$$T_\alpha(x) := \{\xi \in \mathbb{R}^n \,\big|\, \xi_i = 0 \quad \forall i \in J_\alpha(x)\}.$$

Then $\mathcal{C}_\alpha(x)$ is a closed, positive, convex cone and $T_\alpha(x)$ is a maximal linear subspace contained in $\mathcal{C}_\alpha(x)$. Its dimension is given by $n - \sharp J_\alpha(x)$. Now let $\varphi_\beta : U_\beta \to V_\beta$ be another chart of M with $x \in U_\beta$. By definition, $d_{\varphi_\alpha(x)}(\varphi_\beta \circ \varphi_\alpha^{-1}) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism. One easily verifies that it maps $\mathcal{C}_\alpha(x)$ bijectively onto $\mathcal{C}_\beta(x)$ and that its restriction to $T_\alpha(x)$ is a linear isomorphism onto $T_\beta(x)$. In particular one has

$$\sharp J_\beta(x) = \sharp J_\alpha(x) \tag{3.3}$$

and we write $j(x) = \sharp J_\alpha(x)$. To define the cone $\mathcal{C}(x)$ of directions at $x \in M$, tangent to M, we introduce an equivalence relation $\sim$ on the space Γ_x of smooth paths $\gamma : [0, a] \to M$ issuing at x, i.e. $\gamma(0) = x$. Choose a coordinate map $\varphi_\alpha : U_\alpha \to V_\alpha$. We say that $\gamma_1 \sim \gamma_2$ if $\frac{d}{dt}\big|_{t=0}\varphi_\alpha(\gamma_1(t)) = \frac{d}{dt}\big|_{t=0}\varphi_\alpha(\gamma_2(t))$. It is easy to verify that this is indeed an equivalence relation and it does not depend on the choice of the coordinate map φ_α. Then $\mathcal{C}(x)$ is defined as the set of equivalence classes $[\gamma] \subseteq \Gamma_x$. Note that φ_α defines a bijective map

$$\mathcal{C}(x) \to \mathcal{C}_\alpha(x), \quad [\gamma] \mapsto \frac{d}{dt}\Big|_{t=0}\varphi_\alpha(\gamma(t))$$

which we denote by $d_x\varphi_\alpha$. Then we define T_xM to be the $\mathbb{R}$-vector space defined as the linear span of the elements $(d_x\varphi_\alpha)^{-1}(e_i) \in \mathcal{C}(x)$ $(1 \leq i \leq n)$. Hence T_xM is a $\mathbb{R}$-vector space of dimension n. Again it is easy to verify that the construction of T_xM is independent of the choice of the coordinate map φ_α. Moreover, $d_x\varphi_\alpha$ extends to a linear isomorphism between T_xM and $\mathbb{R}^n$ and that $\mathcal{C}(x)$ is a closed,

positive convex cone contained in T_xM, referred to as the cone of tangent directions to M.
Using local coordinates it is easy to see that the vector spaces T_xM give rise to a smooth vector bundle over M with fiber isomorphic to $\mathbb{R}^n$. It is referred to as the tangent bundle of M and denoted by TM with projection map $p : TM \to M$. In the usual way one then defines the cotangent bundle $T^*M \to M$. For any $x \in M$, the fiber of $T^*M \to M$ above x is given by the dual T_x^*M of T_xM. In particular it follows that exterior differential forms can be defined on a smooth $\mathbb{R}^n_{\geq 0}$-manifold and that the exterior calculus remains valid.
In the case when M is given as the subset of a smooth manifold $\tilde{M}$ satisfying a finite number of inequalities – see (3.1), (**??**) above – T_xM coincides with the tangent space $T_x\tilde{M}$ of $\tilde{M}$ at x and $\mathcal{C}(x)$ is the closed, positive convex cone defined by

$$\{\xi \in T_x\tilde{M} \,\big|\, \langle d_xg_i, \xi\rangle \geq 0 \quad \forall i \in J(x)\}$$

where $g_1, \dots, g_N$ are the smooth functions in (3.1) and $\langle\cdot,\cdot\rangle$ denotes the dual pairing between T_x^*M and T_xM. With the notion of tangent space introduced as above it follows that a smooth map $f : M_1 \to M_2$ between $\mathbb{R}^{m_i}_{\geq 0}$-manifolds M_i a linear map $d_xf : T_xM_1 \to T_{f(x)}M_2$ satisfying $d_xf(\mathcal{C}(x)) \subseteq \mathcal{C}(f(x))$.
Let us now take a closer look at the structure of the set of points of a $\mathbb{R}^n_{\geq 0}$-manifold M which are at the boundary. For any $0 \leq k \leq n$, define

$$\partial_kM := \{x \in M \,\big|\, j(x) = k\} \tag{3.4}$$

where $j(x)$ has been introduced above. Note that the subsets ∂_kM are pairwise disjoint and $M = \bigcup_k \partial_kM$. Using local coordinates it is easy to see that for any $0 \leq k \leq n, \partial_kM$ is a smooth manifold of dimension $n-k$ with

$$T_x(\partial_kM) \subseteq \mathcal{C}(x) \subseteq T_xM \quad \forall x \in \partial_kM.$$

In particular, ∂_nM is a discrete set of points. The n-dimensional manifold ∂_0M is referred to as the interior of M whereas the manifold ∂_kM $(1 \leq k \leq n)$ is called the k-boundary of M, k being the codimension of ∂_kM. The union $\partial M = \bigcup_{1\leq k\leq n} \partial_kM$ is referred to as the boundary of M. In the case where M is given as a subset of a smooth manifold $\tilde{M}$ satisfying a finite number of inequalities (cf (3.1) - (**??**)), $\partial_0M = \overset{\circ}{M}$ is the interior of M when viewed as a subset of $\tilde{M}$ and $\partial M \subseteq \tilde{M}$ the boundary of M. In the example depicted in Figure 9, ∂_0M is the interior of the triangle ABC, ∂_1M the union of the sides a, b, c of the triangles without the end points A, B, C and the 2-boundary ∂_2M the set $\{A, B, C\}$.

Definition 3.2. *The closure F of a connected component of ∂_kM in M is called a k-face of M. The integer $0 \leq k \leq n$ is referred to as the codimension of F.*

In the $\mathbb{R}^2_{\geq 0}$-manifold depicted in Figure 10, there is one 1-face. It coincides with ∂M. Note that it is not a smooth manifold. The origin is the only 2-face. In the $\mathbb{R}^2_{\geq 0}$-manifold depicted in Figure 9, there are three 1-faces. They are given by the three sides (with end points) of the triangle and are manifolds with boundary. More generally, for any smooth $\mathbb{R}^n_{\geq 0}$-manifold M given as a subset of a smooth manifold $\tilde{M}$,

$$M = \{x \in \tilde{M} \,\big|\, g_i(x) \geq 0 \quad \forall 1 \leq i \leq N\}$$

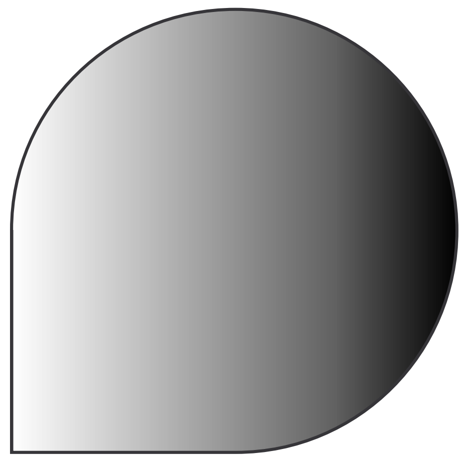

FIGURE 10. Example of a $\mathbb{R}^2_{\geq 0}$-manifold

where $(g_i)_{1\leq i\leq N}$ satisfy (**??**), it can easily be shown that any k-face of M is given by a connected component of

$$M \cap g_{i_1}^{-1}(\{0\}) \cap \ldots \cap g_{i_k}^{-1}(\{0\})$$

where $1 \leq i_1 < i_2 < \ldots < i_k \leq N$. It can be shown that this is a $\mathbb{R}^{n-k}_{\geq 0}$-manifold. This illustrates how restrictive the class of $\mathbb{R}^n_{\geq 0}$-manifolds is. Let F be an arbitrary $(k+1)$-face of an $\mathbb{R}^n_{\geq 0}$-manifold. By definition, F is the closure of a connected component F_0 of $\partial_{k+1}M$. Using local coordinates one sees that there exists a k-face F' of M (not necessarily unique) so that F is a 1-face of F'.
For $i = 1, 2$, let M_i be a $\mathbb{R}^{n_i}_{\geq 0}$-manifold with $\mathbb{R}^{n_i}_{\geq 0}$-atlas $\mathcal{U}_i = \{(U^{(i)}_\alpha, \varphi^{(i)}_\alpha)\}$. Denote by $\mathcal{U}_1 \times \mathcal{U}_2$ the atlas given by the collection of charts $(U^{(1)}_\alpha \times U^{(2)}_\beta, \varphi^{(1)}_\alpha \times \varphi^{(2)}_\beta)$. In a straightforward way one obtains the following result.

Lemma 3.1. (i) *$\mathcal{U}_1 \times \mathcal{U}_2$ is a $\mathbb{R}^{n_1+n_2}_{\geq 0}$-atlas for $M_1 \times M_2$.*

(ii) *For any $0 \leq k \leq n$*

$$\partial_k(M_1 \times M_2) = \bigsqcup_{k=i+j} \partial_i M_1 \times \partial_j M_2.$$

In particular, $\partial_0(M_1 \times M_2) = \partial_0 M_1 \times \partial_0 M_2$ and

$$\partial_1(M_1 \times M_2) = (\partial_1 M_1 \times \partial_0 M_2) \cup (\partial_0 M_1 \times \partial_1 M_2).$$

(iii) *For $i = 1, 2$, let F_i be a k_i-face of M_i. Then $F_1 \times F_2$ is a $(k_1 + k_2)$-face of $M_1 \times M_2$. Any k-face of $M_1 \times M_2$ is of this type.*

In the sequel, $M_1 \times M_2$ will always be endowed with the differentiable structure induced by $\mathcal{U}_1 \times \mathcal{U}_2$ and referred to as the (Cartesian) product of M_1 and M_2.

3.2. **Manifolds with corners.** In this subsection we study a useful class of $\mathbb{R}^n_{\geq 0}$-manifolds whose faces satisfy an additional condition.

Definition 3.3. *A smooth $\mathbb{R}^n_{\geq 0}$-manifold is said to be a n-dimensional manifold with corners if any k-face, $0 \leq k \leq n$, is a smooth $\mathbb{R}^{n-k}_{\geq 0}$-manifold.*

We have already observed that any $\mathbb{R}^n_{\geq 0}$-manifold given as a subset of points of a smooth manifold satisfying a finite number of inequalities (cf (3.1) - (3.2)) is a manifold with corners whereas the $\mathbb{R}^2_{\geq 0}$-manifold depicted in Figure 10 is *not* a manifold with corners.
An important class of manifolds with corners is obtained by taking Cartesian products. From Lemma 3.1 the following result can be easily deduced.

Corollary 3.2. *Let M_1 and M_2 be smooth manifolds with corners. Then $M_1 \times M_2$ is a smooth manifold with corners.*

In the next subsection we will use Corollary 3.2 to confirm that a finite Cartesian product of manifolds with boundary is a manifold with corners.
In the category of manifolds with corners, the natural notion of a submanifold is the notion of a neat submanifold with corners [12]. It is an extension of the notion of a neat submanifold with boundary, introduced by Hirsch [16]. Let M be a n-dimensional manifold with corners. A subset $N \subseteq M$ is said to be a *topological submanifold of* M of codimension s if for every $x \in N$, there exists a coordinate chart (U_x, φ_x) of M with $x \in U_x$ where $\varphi_x : U_x \to V_x$ is a coordinate map between the open subsets $U_x \subseteq M$ and $V_x \subseteq \mathbb{R}^n_{\geq 0}$ so that

$$\varphi_x(U_x \cap N) = V_x \cap (\mathbb{R}^{n-s}_{\geq 0} \times \{(0, \ldots, 0)\}). \tag{3.5}$$

The topological submanifold N of codimension 1 of $\mathbb{R}^2_{\geq 0}$ depicted in Figure (11) is not a $\mathbb{R}^1_{\geq 0}$-manifold. The property of being "neat" is a *sufficient* condition for a topological submanifold of a manifold with corners to be a manifold with corners.

Definition 3.4. *A subset N of a n-dimensional manifold with corners M is said to be a neat submanifold with corners of codimension $0 \leq s \leq n$ if for any $k > \dim N = n - s$ $\quad N \cap \partial_k M = \emptyset$ and for any $0 \leq k \leq n - s$ and $x \in N \cap \partial_k M$, there exists a chart (U_x, φ_x) of M, $\varphi_x : U_x \to V_x$, where U_x is an open neighborhood of x in M and V_x is an open neighborhood in $\mathbb{R}^n_{\geq 0}$, diffeomorphic to $\mathbb{R}^k_{\geq 0} \times \mathbb{R}^{n-k}_{>0}$ so that $\varphi_x(U_x \cap N)$ is diffeomorphic to $\mathbb{R}^k_{\geq 0} \times \mathbb{R}^{n-s-k}_{>0}$.*

Denote by U_N the $\mathbb{R}^{n-s}_{\geq 0}$-atlas $\{(U_x \cap N, \varphi_x \big|_{U_x \cap N})_{x \in N}\}$. Thus (N, U_N) is a $\mathbb{R}^{n-s}_{\geq 0}$-manifold. Actually, more is true.

Lemma 3.3. *Assume that N is a neat submanifold with corners of (M, U). Then (N, U_N) is a manifold with corners.*

Proof. We have already seen that N is a smooth $\mathbb{R}^{n-s}_{\geq 0}$-manifold. Further, any k-face F_N of N is a connected component of a set of the form $F \cap N$ where F is a k-face of M. $\square$

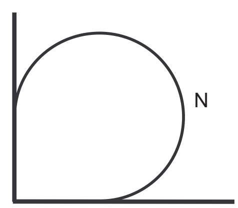

FIGURE 11. $N \subseteq \mathbb{R}^2_{\geq 0}$ not $\mathbb{R}^1_{\geq 0}$-manifold

Note that among the examples depicted in Figure 13, only in Figure 13 A is a neat submanifold with corners (of codimension 1) of the unit square, whereas in the examples depicted in Figure **??** A, only the cylinder in Figure **??** B is a neat submanifold with corners of codimension 1 of the unit cube in $\mathbb{R}^3_{\geq 0}$.
Another way of constructing manifolds with corners is based on the transversality theorem, properly extended to the situation at hand. Let $f : P \to M$ be a smooth

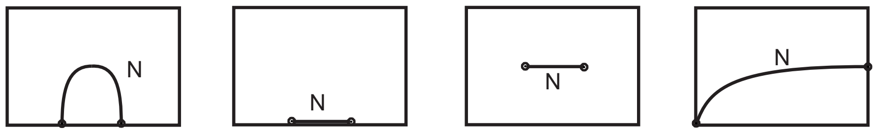

FIGURE 12. Figures A - D

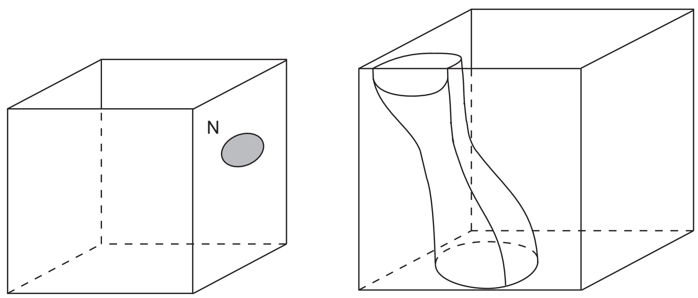

FIGURE 13. Figures A - B

map from a manifold with corners P to a manifold M. The map f is said to be *transversal* to a submanifold N of M if for any $x \in \partial_k P$ with $0 \leq k \leq \dim P$ and $f(x) \in N$

$$T_{f(x)}M = T_{f(x)}N + d_x f(T_x \partial_k P).$$

In words, it means that there exists a complement of $T_{f(x)}N$ in $T_{f(x)}M$ spanned by certain elements which are the image of elements in the tangent space at x to the k-boundary $\partial_k P$ of P.

Lemma 3.4. *Let $f : P \to M$ be a smooth map from a p-dimensional manifold with corners to an n-dimensional manifold M. If f is transversal to a submanifold N of M of codimension s, then $f^{-1}(N)$, if not empty, is a topological submanifold of P of codimension s with the property that for any $0 \leq k \leq p - s$*

$$\partial_k f^{-1}(N) = f^{-1}(N) \cap \partial_k P.$$

Hence $f^{-1}(N)$ is a neat submanifold with corners of P. In particular, for any k with $p - s + 1 \leq k \leq p$

$$f^{-1}(N) \cap \partial_k P = \emptyset.$$

Proof. First we show that $f^{-1}(N)$ is a topological submanifold of P of codimension s. Without loss of generality we may assume that P is the open subset $U \subseteq \mathbb{R}^p_{\geq 0}$, M is the open subset $V \subseteq \mathbb{R}^n$ and N is given by

$$W := V \cap (\{0\} \times \mathbb{R}^{n-s}) \subseteq \mathbb{R}^s \times \mathbb{R}^{n-s}.$$

This means that

$$f^{-1}(W) = \{x \in U \,\big|\, f_j(x) = 0 \quad \forall 1 \leq j \leq s\}$$

where $f = (f_1, \ldots, f_n)$. We want to apply the implicit function theorem to construct a $\mathbb{R}^{p-s}_{\geq 0}$-atlas of $f^{-1}(W)$. The assumption of $f : P \to M$ being transversal to N says that for any $x \in \partial_k U$ with $f(x) \in W$,

$$\mathbb{R}^n = \{0\} \times \mathbb{R}^{n-s} + d_x f(T_x \partial_k U). \tag{3.6}$$

Hence $\dim(d_x f(T_x \partial_k U)) \geq s$. On the other hand, $\dim(T_x \partial_k U) = p - k$, so that $p - k \geq s$, i.e. $k \leq p - s$. It means that $f^{-1}(W) \cap \partial_k U = \emptyset$ if $p - s + 1 \leq k \leq p$. Let $z \in f^{-1}(W) \cap \partial_k U$ be given. By renumbering the coordinates if needed we may assume that z is of the form $(z_1, \ldots, z_{p-k}, 0, \ldots, 0)$. In view of (3.6) we may further assume that $\left(\frac{\partial f_j}{\partial x_i}(z)\right)_{1 \leq j, i \leq s}$ is invertible. By the implicit function theorem applied to the system of s equations $f_j(x) = 0$ $(1 \leq j \leq s)$ with p unknowns $x_1, \ldots, x_p$ near $x = z$, the first s components $x_1, \ldots, x_s$ can be expressed in terms of $(x_{s+1}, \ldots, x_p)$. More precisely, there exist an open neighborhood $U_z = U_1 \times U_2 \times U_3$ of $z = (z^{(1)}, z^{(2)}, z^{(3)})$ in $U \subseteq \mathbb{R}^s_{>0} \times \mathbb{R}^{p-s-k}_{>0} \times \mathbb{R}^k_{\geq 0}$ and a smooth map

$$g : U_2 \times U_3 \to U_1, \quad (x^{(2)}, x^{(3)}) \mapsto x^{(1)} = g(x^{(2)}, x^{(3)})$$

so that $z^{(1)} = g(z^{(2)}, z^{(3)})$ and

$$f^{-1}(W) \cap U_z = \{(g(x^{(2)}, x^{(3)}), x^{(2)}, x^{(3)}) \,\big|\, (x^{(2)}, x^{(3)}) \in U_2 \times U_3\}.$$

Now define

$$\begin{aligned} \varphi_z : & f^{-1}(W) \cap U_z \to V_z := U_2 \times U_3 \subseteq \mathbb{R}^{p-s-k}_{>0} \times \mathbb{R}^k_{\geq 0} \\ & (g(x^{(2)}, x^{(3)}), x^{(2)}, x^{(3)}) \mapsto (x^{(2)}, x^{(3)}). \end{aligned}$$

Then $(f^{-1}(W) \cap U_z, \varphi_z)$ is a coordinate chart of $f^{-1}(W)$ containing the point $z \in \partial_k f^{-1}(W)$. Hence $\{(U_z, \varphi_z)_{z \in f^{-1}(W)}\}$ is an $\mathbb{R}^{p-s}_{\geq 0}$-atlas for $f^{-1}(W)$ making $f^{-1}(W)$ into a topological submanifold of P of codimension s. Further, for z as above,

$$\begin{aligned} (\partial_k f^{-1}(W)) \cap U_z &= \{(g(x^{(2)}, 0), x^{(2)}, 0) \,\big|\, x^{(2)} \in U_2\} \\ &= f^{-1}(W) \cap \partial_k U_z. \end{aligned}$$

As the point z is arbitrary it then follows that $f^{-1}(W)$ is a neat submanifold of U of codimension s as claimed. $\square$

Next we introduce the notion of orientation of a manifold with corners. To do so one could use local coordinates, extending the familiar definition of orientation given in [16] for smooth manifolds to manifolds with corners. For convenience we consider here the following equivalent definition. Let M be a n-dimensional manifold with corners. Denote by $\det(M) \to M$ the vector bundle of rank 1 whose fibre at $x \in M$ is the n'th exterior product $\Lambda^n T_x M$ of the tangent space $T_x M$.

Definition 3.5. *The manifold M with corners is said to be orientable if* $\det(M) \to M$ *admits a smooth nowhere vanishing section* $\sigma : M \to \det(M)$. *An orientation* $\mathcal{O}$ *of* M *is an equivalence class of nowhere vanishing sections where two smooth sections* $\sigma_j : M \to \det(M)$ $(j = 1, 2)$ *are equivalent if there exists a smooth function* $\lambda : M \to \mathbb{R}_{>0}$ *so that* $\sigma_1(x) = \lambda(x)\sigma_2(x)$ *for any* $x \in M$.

Given a smooth metric g on M, an orientation $\mathcal{O}$ contains a unique normalized section, i.e. a section $\sigma : M \to \det(M)$ with $\|\sigma(x)\| = 1 \; \forall x \in M$ where $\|\sigma(x)\|^2 = \langle \sigma(x), \sigma(x)\rangle$ and $\langle \cdot, \cdot \rangle$ denotes the fiberwise scalar product on $\det(M)$ induced by g. Given any orthonormal basis $e_1(x), \dots, e_n(x)$ of $T_xM, \sigma(x)$ is of the form

$$\sigma(x) = \pm e_1(x) \wedge \dots \wedge e_n(x).$$

For later reference we state a few elementary facts about the orientation of a manifold with corners.

Lemma 3.5. *Assume that M is a manifold with corners.*

(i) *If M is orientable and connected, then M has two different orientations.*

(ii) *M is orientable if and only if the interior $\partial_0 M$ of M is orientable; the orientations of M and $\partial_0 M$ are in bijective correspondence.*

(iii) *An orientation of M determines in a canonical way an orientation on any 1-face of M.*

(iv) *If M is orientable so is any k-face of M.*

Proof. For the whole proof fix an arbitrary Riemannian metric on M.

(i) As M is orientable, there exists a normalized smooth section $\sigma : M \to \det(M)$ in the sense defined as above. Any other normalized smooth section $\sigma' : M \to \det(M)$ is then of the form $\sigma'(x) = \lambda(x)\sigma(x)$ where $\lambda : M \to \mathbb{R}$ is smooth and satisfies $\lambda(x) \in \{\pm 1\}$. As M is connected the claim follows.

(ii) By restriction, the orientability of M implies the orientability of $\partial_0 M$. Conversely, assume that $\partial_0 M$ is orientable. Hence there exists a normalized section $\sigma : \partial_0 M \to \det(\partial_0 M)$. On a chart $(U_\alpha, \varphi_\alpha)$ of M σ takes the form

$$\sigma(x) = \varepsilon_\alpha e_1^{(\alpha)}(x) \wedge \dots \wedge e_n^{(\alpha)}(x) \quad \forall x \in U_\alpha \cap \partial_0 M$$

where $\varepsilon_\alpha \in \{\pm 1\}$ and $(e_j^{(\alpha)}(x))_{1 \le j \le n}$ is an orthonormal basis of T_xM smoothly varying with $x \in U_\alpha$. In this way one sees that σ has a unique smooth extension $\overline{\sigma} : M \to \det M$ with $\|\overline{\sigma}(x)\| = 1$ for any $x \in M$ hence M is orientable. By the same token, the second part of claim (ii) is proved.

(iii) Let $\mathcal{O}$ be the orientation of M. For any $x \in \partial_1 M$ denote by $\nu(x)$ the unique element of norm 1 which is orthogonal to $T_x\partial_1 M$ and contained in the cone $\mathcal{C}(x)$ of tangent directions to M at x. Further denote by $\nu^*(x)$ the unique element in T_x^*M so that $\langle \nu^*(x), \nu(x)\rangle = 1$ and the restriction $\nu^*(x)$ to $T_x\partial_1 M$ vanishes where $\langle \cdot, \cdot \rangle$ denotes the dual pairing. Using local coordinates one sees that both $\nu : \partial_1 M \to TM\big|_{\partial_1 M}$ and $\nu^* : \partial_1 M \to T^*M\big|_{\partial_M}$ are smooth sections. Let σ be a smooth normalized section representing the orientation $\mathcal{O}$. For any $x_0 \in \partial_1 M$, choose a chart (U, φ) of M with $x_0 \in U$ and for any $x \in U \cap \partial_1 M$ an orthonormal basis $(e_j(x))_{1 \le j \le n}$ of T_xM with $e_n(x) = \nu(x)$ varying smoothly with x. Then $(e_j(x))_{1 \le j \le n-1}$ is an orthonormal basis of $T_x\partial_1 M$. Now define for any $x \in U \cap \partial_1 M$,

$$\sigma_1(x) := e_1(x) \wedge \dots \wedge e_{n-1}(x) \in \Lambda^{n-1}(T_x\partial_1 M).$$

Note that $\sigma_1(x)$ is a smooth normalized section, $\sigma_1 : U \cap \partial_1 M \to \Lambda^{n-1}(T_x\partial_1 M)\big|_{U \cap \partial_1 M}$. As

$$\sigma_1(x) = \iota_{\nu^*(x)}((-1)^{n-1}\sigma(x))$$

where $\iota_{\nu^*(x)}$ is the contraction by $\nu^*(x)$, it follows that $\sigma_1(x)$ is well defined i.e. it does not depend on the choice of the orthonormal basis $(e_j(x))_{1\leq j\leq n-1}$ of $T_x\partial_1 M$ used to represent $\sigma(x), \sigma(x) = e_1(x)\wedge\ldots\wedge e_{n-1}(x)\wedge\nu(x)$. Since the point $x_0 \in \partial_1 M$ is arbitrary, we conclude that σ_1 defines a normalized smooth section of $\det(\partial_1 M)$ and hence an orientation of $\partial_1 M$ in a canonical way. By (ii) and the fact that M is a manifold with corners it then follows that any 1-face of M is oriented in a canonical way.

(iv) The claimed statement is proved by induction. The statement for $k = 1$ is implied by the statement in (iii). So let us assume that F is an orientable $(k+1)$-face where $1 \leq k \leq n$. Then there exists a k-face F' (not necessarily unique) so that $\partial_0 F \subseteq \partial_1 F'$. By the induction hypothesis, F' is orientable. Hence it follows from (iii) that $\partial_0 F$ and thus by (ii) F itself are orientable. □

We remark that it follows from the proof of statement (iii) in Lemma 3.4 that the normal bundle on $\partial_1 M$ whose fibre at $x \in \partial_1 M$ is the linear span of $\mathcal{C}(x)/T_x\partial_1 M$ is trivial. Further we point out that statement (iv) of Lemma 3.4 is no longer true for smooth $\mathbb{R}^n_{\geq 0}$-manifolds as the following example of a smooth orientable $\mathbb{R}^4_{\geq 0}$-manifold M with a non-orientable 2-face illustrates.
In the sequel, we will also consider products of *oriented* manifolds with corners. Let M_j $(j = 1,2)$ be oriented manifolds with corners of dimension n_j. Let g_j be a Riemannian metric on M_j and denote by $\sigma_j : M_j \to \det M_j$ the normalized smooth section in $\mathcal{O}_j$. As $T(M_1 \times M_2) \cong TM_1 \times TM_2$ one concludes that $\det(M_1)\otimes \det(M_2) \cong \det(M_1 \times M_2)$ by the fusion isomorphism defined for $v_i \in \Lambda^{n_1}TM_1$ $(1 \leq i \leq n_1), w_i \in \Lambda^{n_2}(TM_2)$ $(1 \leq i \leq n_2)$

$$(v_1 \wedge \ldots \wedge v_{n_1}) \otimes (w_1 \wedge \ldots \wedge w_{n_2}) \mapsto (v_1, 0) \wedge \ldots \wedge (v_{n_1}, 0) \wedge (0, w_1) \wedge \ldots \wedge (0, w_{n_2}).$$

Hence

$$\sigma_1 \otimes \sigma_2 : M_1 \times M_2 \to \det M_1 \otimes \det M_2,\ (x, y) \mapsto \sigma_1(x) \otimes \sigma_2(y)$$

defines a smooth section with

$$\|\sigma_1 \otimes \sigma_2(x, y)\| = \|\sigma_1(x)\|\|\sigma_2(y)\| = 1.$$

The orientation determined by this normalized section is referred to as the product orientation and denoted by $\mathcal{O}_1 \otimes \mathcal{O}_2$.
By the same arguments used for oriented manifolds with boundary – see [21] – can prove a version of Stokes's theorem for oriented manifolds with corner.

Theorem 3.6. *(Stokes's theorem) Assume that M is a compact orientable manifold with corners of dimension n. Then for any smooth $(n-1)$-form ω on M,*

$$\int_M d\omega = \int_{\partial_1 M} \iota^*\omega$$

where the n-form $d\omega$ denotes the exterior differential of ω and $\iota^*\omega$ is the pull back of ω by the inclusion $\iota : \partial_1 M \hookrightarrow M$. Here $\partial_1 M$ is endowed with the canonical orientation induced by the orientation on M (cf Lemma 3.5 (ii)).

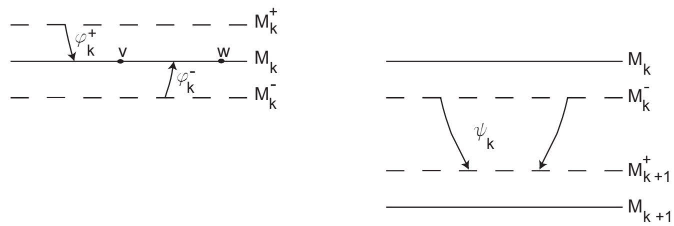

FIGURE 14. Illustration of the maps $\varphi_k^{\pm}$ and ψ_k

4. Smooth structure on $\hat{W}_v^-$ and $\hat{\mathcal{B}}(v,w)$

Let (h,X) be a Morse-Smale pair and v a critical point of h. In this section our aim is to prove that the Hausdorff spaces $\mathcal{B}(v,w)$ and $\hat{W}_v^-$ (cf Theorem 2.14) have a canonical structure of smooth manifolds with corners with $\mathcal{T}(v,w)$ and, respectively, the unstable manifold W_v^- as their interiors.
We will do this by realizing $\mathcal{B}(v,w)$ as a subset of a smooth manifold with corners and realizing $\hat{W}_v^-$ locally as a subset of a smooth manifold with corners, both much simpler to describe. The smooth manifold with corners in the first case will be a product of smooth manifolds with boundary of type P_k and in the second a product of several manifolds with boundary of type P_k and one of type Q_k. The manifold with boundary P_k will be defined as a smooth submanifold with boundary of $M_k^+ \times M_k^-$ while Q_k as a smooth submanifold with boundary of $M_k^+ \times h^{-1}(c_{k+1}, c_{k-1})$.

4.1. **Preliminary constructions.** In this subsection we introduce some notation and analyze two collections $\{P_k\}$ and $\{Q_k\}$ of manifolds with boundary which will be used to prove that $\mathcal{B}(v,w)$ and $\hat{W}_v^-$ are manifolds with corners.
Let (h,X) be a Morse-Smale pair and $(U_v,\varphi_v), v \in \mathrm{Crit}(h)$, a collection of standard charts. For any k, let M_k and $M_k^{\pm}$ denote the level sets

$$M_k := h^{-1}(c_k);\ M_k^{\pm} := h^{-1}\{c_k \pm \varepsilon\}$$

where $\varepsilon > 0$ is chosen sufficiently small (cf (2.16)). Note that $M_k^{\pm}$ and $M_k \backslash \mathrm{Crit}(h)$, if not empty, are smooth manifolds and of dimension $n-1$. On the other hand, M_k is not a smooth manifold. The flow Ψ_t corresponding to the rescaled vector field $Y = -\frac{1}{X(h)}X$, introduced in (2.9), defines the maps

$$\varphi_k^{\pm} : M_k^{\pm} \to M_k, \qquad x \mapsto \Psi_{\pm\varepsilon}(x)$$
$$\psi_k : M_k^- \to M_{k+1}^+, \quad x \mapsto \Psi_b(x)$$

where $b := c_k - c_{k+1} - 2\varepsilon$. By Lemma 2.4, $\varphi_k^{\pm}$ are continuous and ψ_k are diffeomorphisms. For any $v \in \mathrm{Crit}(h) \cap M_k$, define

$$S_v^{\pm} := W_v^{\pm} \cap M_k^{\pm};\ S_v := S_v^+ \times S_v^-$$

and let

$$S_k^{\pm} := \bigsqcup_{h(v)=c_k} S_v^{\pm};\ S_k := \bigsqcup_{h(v)=c_k} S_v.$$

Note that $S_v^{\pm}$ are smooth spheres with $\dim(S_v^-) = i(v)-1$ and $\dim(S_v^+) = n-i(v)-1$. As $\varepsilon > 0$ has been chosen sufficiently small they are contained in the standard chart U_v. The product $S_v = S_v^+ \times S_v^-$ and hence S_k are smooth submanifolds of dimension $n-2$ of $M_k^+ \times M_k^-$. For any $0 \le k \le n$, define

$$P_k := \{(x^+, x^-) \in M_k^+ \times M_k^- \mid \varphi_k^+(x^+) = \varphi_k^-(x^-)\}$$

together with the subset $P_k' \subseteq P_k$,

$$P_k' := \{(x^+, x^-) \in P_k \mid x^\pm \in M_k^\pm \backslash S_k^\pm\}.$$

Notice that $P_k = P_k' \cup S_k$ and that an element $(x^+, x^-) \in M_k^+ \times M_k^-$ is in P_k iff x^+ and x^- are connected by a (possibly broken) trajectory. More precisely, (x^+, x^-) is in P_k' iff x^+ and x^- are connected by an *unbroken* trajectory whereas (x^+, x^-) is in S_k iff x^+ and x^- are connected by a *broken* trajectory. As P_k' is the graph of the diffeomorphism

$$\varphi_k : M_k^+ \backslash S_k^+ \to M_k^- \backslash S_k^-, \ x \mapsto \Psi_{2\varepsilon}(x), \tag{4.1}$$

it is a manifold of dimension $n-1$. As already mentioned above, S_k is a manifold of dimension $n-2$.

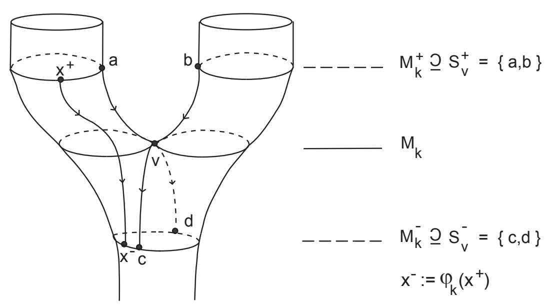

FIGURE 15. Illustration of data used in definition of P_k: $M_k^+ \cong \mathbb{S}^1 \sqcup \mathbb{S}^1$; $M_k^- = \mathbb{S}^1$

Lemma 4.1. *For any $0 \le k \le n$, P_k is a $(n-1)$-dimensional manifold with boundary whose interior $\partial_0 P_k$ is given by P_k' and whose boundary $\partial_1 P_k$ is S_k, i.e.*

$$\partial_0 P_k = P_k'; \ \partial_1 P_k = S_k.$$

If $p_k^\pm : M_k^+ \times M_k^- \to M_k^\pm$ denote the canonical projections, then the restrictions $p_k^\pm : \partial_0 P_k \to M_k^\pm \backslash S_k^\pm$ are diffeomorphisms and $p_k^+ \times p_k^- : \partial_1 P_k \to S_k$ is the identity.

Proof. Let us first verify the statement of Lemma 4.1 for the standard model, defined as follows. Let $0 \le \ell \le n$ and let M be $\mathbb{R}^{n-\ell} \times \mathbb{R}^\ell$, endowed with the Euclidean metric g and define for $y = (y^+, y^-) \in \mathbb{R}^{n-\ell} \times \mathbb{R}^\ell$,

$$h_\ell(y) = \frac{1}{2}\left(\|y^+\|^2 - \|y^-\|^2\right).$$

Clearly, in this model $0 \in \mathbb{R}$ is the only critical value of h_ℓ and the origin in $\mathbb{R}^{n-\ell} \times \mathbb{R}^\ell$ its only critical point. Its index is given by ℓ. Let $S^\pm$ be the spheres

$$S^+ := \{x^+ = (y^+, 0) \mid \|y^+\|^2 = 2\varepsilon\}; \ S^- := \{x^- = (0, y^-) \mid \|y^-\|^2 = 2\varepsilon\}$$

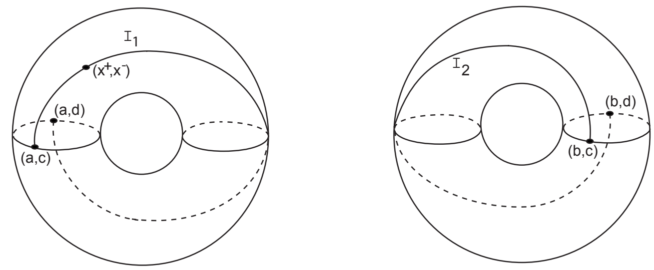

FIGURE 16. Illustration of $P_k \subseteq M_k^+ \times M_k^- : M_k^+ \times M_k^- \cong (\mathbb{S}^1 \times \mathbb{S}^1) \sqcup (\mathbb{S}^1 \times \mathbb{S}^1)$, $P_k = I_1 \sqcup I_2$; $\partial_1 P_k = \partial I_1 \sqcup \partial I_2$ with $\partial I_1 = \{(a,c),(a,d)\}, \partial I_2 = \{(b,c),(b,d)\}$ (cf Figure 15)

and the subsets of $\mathbb{R}^n \times \mathbb{R}^n$,

$$P := \{(x^+, x^-) \in \mathbb{R}^n \times \mathbb{R}^n \,\big|\, h_\ell(x^\pm) = \pm\varepsilon;\ \varphi_\ell^+(x^+) = \varphi_\ell^-(x^-)\}$$
$$P' := \{(x^+, x^-) \in P \,\big|\, x^\pm \notin S^\pm\}$$

where $\varphi_\ell^\pm = \Psi_{\pm\varepsilon}$ with Ψ_t denoting the flow corresponding to the normalized vector field (cf (2.17))

$$Y^{(\ell)} = \sum_{j=1}^{\ell} \frac{y_j}{\|y\|^2}\frac{\partial}{\partial y_j} - \sum_{j=\ell+1}^{n} \frac{y_j}{\|y\|^2}\frac{\partial}{\partial y_j}.$$

Being a graph with base $\{x^+ \in \mathbb{R}^n \backslash S^+ \,\big|\, h_\ell(x^+) = \varepsilon\}$, P' is a $(n-1)$ dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}^n$. To show that P' is the interior of P and $S := S^+ \times S^-$ its boundary we provide a collar of S in P. For this purpose define

$$\theta : S \times [0, 1/2) \to \mathbb{R}^n \times \mathbb{R}^n, \big((y^+, 0), (0, y^-), s\big) \mapsto (x^+, x^-)$$

with $x^\pm \equiv x^\pm(s; y^+, y^-)$ given by

$$x^+ := (1 - s^2)^{-1/2}(y^+, sy^-)\ ; \quad x^- := (1 - s^2)^{-1/2}(sy^+, y^-).$$

The scaling factor $(1 - s^2)^{-1/2}$ has been chosen in such a way that $h(x^\pm) = \pm\varepsilon$. According to (2.8), the point x^- is on the trajectory $\Phi_t(x^+)$ of the gradient vector field $-\mathrm{grad}_g h_\ell$. This shows that the range of θ is contained in P. Clearly, θ is a smooth embedding into $\mathbb{R}^n \times \mathbb{R}^n$, the restriction of θ to $S \times (0, 1/2)$ is a diffeomorphism onto its image in P', and the restriction of θ to $S \times \{0\}$ is the standard inclusion. This proves the statement of Lemma 4.1 for the standard model. To prove Lemma 4.1 in the general case, we proceed in a similar fashion. Let $0 \le k \le n$. We already know that P_k' and $S_k = \sqcup_{h(v)=c_k} S_v$ are smooth submanifolds of $M_k^+ \times M_k^-$ of dimension $n-1$ and $n-2$, respectively. To show that P_k' is the interior of P_k and S_k its boundary, we provide for any $v \in \mathrm{Crit}(h)$ with $h(v) = c_k$, a smooth embedding $\theta_v : S_v \times [0, 1/2) \to M_k^+ \times M_k^-$ so that

(i) $\theta_v \big|_{S_v \times \{0\}}$ is the standard inclusion,
(ii) $\theta_v(S_v \times [0, 1/2)) \subseteq P_k$
(iii) $\theta_v(S_v \times (0, 1/2)) \subseteq P_k'$.

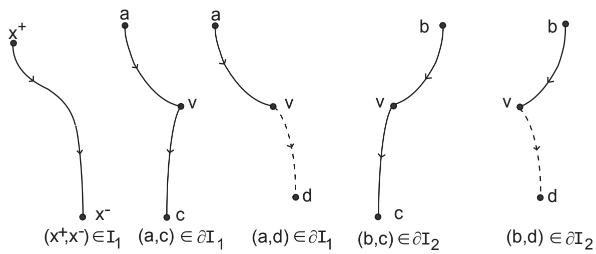

FIGURE 17. Trajectories corresponding to points in P_k (cf Figure 15)

Recall that we have chosen $\varepsilon > 0$ sufficiently small so that $S_v^{\pm}$ are contained in the standard chart U_v Hence the map θ_v can be defined in terms of the standard coordinates. Note that for $S^{\pm}$ given as above with $\ell = i(v)$, $\varphi_v : B_r \to U_v$ maps $S^{\pm}$ onto $S_v^{\pm}$ and $x^{\pm}(s)$, defined as above, are elements in B_r as for $\varepsilon > 0$ sufficiently small,

$$\|x^{\pm}(s)\|^2 = 2\varepsilon \frac{1+s^2}{1-s^2} \leq 2\varepsilon \frac{5}{3} < r^2$$

for any $(y^+, 0) \in S^+, (0, y^-) \in S^-$, and $0 \leq s < 1/2$. Hence for y^+, y^-, and s as above one can define

$$\theta_v \left(\varphi_v(y^+, 0), \varphi_v(0, y^-), s \right) := \left(\varphi_v(x^+(s)), \varphi_v(x^-(s)) \right).$$

The map θ_v then satisfies the claimed properties (i) - (iii) as by construction, θ satisfies the corresponding ones for the standard model. The statements on the projections $p_k^{\pm}$ are verified in a straight forward way. □

To introduce the second collection $\{Q_k\}$ denote for any $k < \ell$ by $M_{\ell,k}$ the inverse image of the open interval (c_ℓ, c_k) by h

$$M_{\ell,k} := \{x \in M \mid c_\ell < h(x) < c_k\}.$$

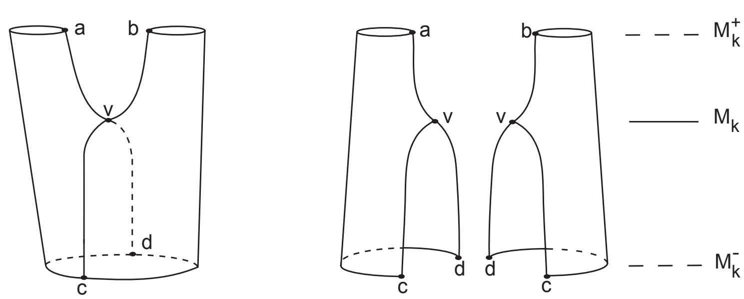

FIGURE 18. Illustration of $Q_k : M_k^+ = S^1 \sqcup S^1;\ M_k^- = S^1$

For any k let

$$Q_k := \{(x^+, x) \in M_k^+ \times M_{k+1,k-1} \,\big|\, x^+ \sim x\}$$

where $x^+ \sim x$ means that x^+ and x lie on the same (possibly broken) trajectory. Further let $W_k^- := \bigsqcup_{h(v)=c_k} W_v^-$, and define

$$Q_k' := \{(x^+, x) \in Q_k \,\big|\, x \in M_{k+1,k-1} \backslash W_k^-\}$$

and $T_k := \bigsqcup_{h(v)=c_k} T_v$ where

$$T_v := S_v^+ \times (W_v^- \cap M_{k+1,k-1}).$$

Notice that $Q_k = Q_k' \cup T_k$ and an element (x^+, x) $M_k^+ \times M_{k+1,k-1}$ is in Q_k' iff x^+ and x are connected by an *unbroken* trajectory and x is not a critical point of h whereas (x^+, x) is in T_k iff x^+ and x are connected by a *broken* trajectory or $x \in \mathrm{Crit}(h) \cap M_k$. Note that Q_k' is the graph of the smooth map

$$\eta_k^+ : M_{k+1,k-1} \backslash W_k^- \to M_k^+, x \mapsto x_k^+ \tag{4.2}$$

where x_k^+ is defined to be the unique point of M_k^+ on the trajectory $\Phi_\cdot(x)$. Hence it is a manifold of dimension n. Clearly, T_k is a manifold of dimension $n - 1$.

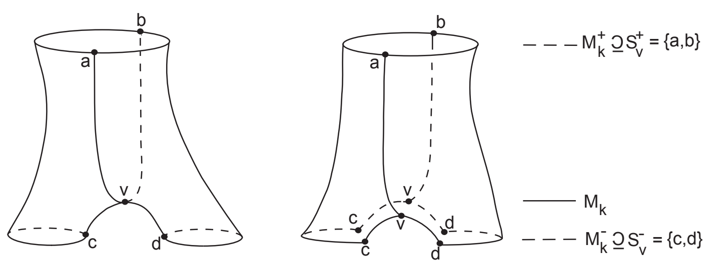

FIGURE 19. Illustration of $Q_k : M_k^+ = S^1;\ M_k^- = S^1 \sqcup S^1$

Lemma 4.2. *For any k, Q_k is a n-dimensional manifold with boundary whose interior is given by Q_k' and whose boundary is T_k,*

$$\partial_0 Q_k = Q_k';\ \partial_1 Q_k = T_k.$$

If $p_k^+ : M_k^+ \times M_{k+1,k-1} \to M_k^+$ and $q_k : M_k^+ \times M_{k+1,k-1} \to M_{k+1,k-1}$ denote the canonical projections, then the restriction $p_k^+ : Q_k' \to M_k^+ \backslash S_k^+$ is a smooth bundle map with fibre diffeomorphic to $(0,1)$, the restriction $q_k : Q_k' \to M_{k+1,k-1} \backslash W_k^-$ is a diffeomorphism, and the restriction $p_k^+ \times q_k : T_k \to \sqcup_{h(v)=c_k} S_v^+ \times W_v^-$ is the identity.

Proof. First note that $M_{k+1,k-1} = U_1 \cup U_2 \cup U_3$ with U_j being the open subsets of M given by

$$U_1 := M_{k+1,k};\ U_2 := M_{k,k-1};\ U_3 := h^{-1}\big((c_k - \varepsilon, c_k + \varepsilon)\big).$$

It suffices to show that for any $1 \leq j \leq 3, Q_k \cap (M_k^+ \times U_j)$ is a submanifold with boundary of $M_k^+ \times M_{k+1,k-1}$ where its boundary is given by $T_k \cap (M_k^+ \times U_j)$.

$\underline{Q_k \cap (M_k^+ \times M_{k+1,k})}$: Consider the diffeomorphism

$$\Theta : M_k^+ \times M_k^- \times (c_{k+1}, c_k) \to M_k^+ \times M_{k+1,k},$$

defined by $\Theta(x^+, x^-, s) := (x^+, \Psi_{s-c_k+\varepsilon}(x^-))$ where $\Psi_s(x)$ denotes the flow of the normalized vector field Y, defined in (2.9) . It is easy to see that Θ maps $P_k \times (c_{k+1}, c_k)$ diffeomorphically onto

$$Q_k \cap \left(M_k^+ \times M_{k+1,k}\right)$$

and $S_k \times (c_{k+1}, c_k)$ onto $T_k \cap (M_k^+ \times M_{k+1,k})$. By Lemma 4.1, $P_k \times (c_{k+1}, c_k)$ is a smooth manifold with boundary

$$\partial(P_k \times (c_{k+1}, c_k)) = S_k \times (c_{k+1}, c_k).$$

Hence the claimed statement is established in this case.
$\underline{Q_k \cap (M_k^+ \times M_{k,k-1})}$: In this case, $T_k \cap (M_k^+ \times M_{k,k-1}) = \emptyset$ and

$$Q_k \cap (M_k^+ \times M_{k,k-1}) = Q_k' \cap (M_k^+ \times M_{k,k-1})$$

is a smooth manifold.
$\underline{Q_k \cap \left(M_k^+ \times h^{-1}((c_k - \varepsilon, c_k + \varepsilon))\right)}$: In this case we argue similarly as in the proof of Lemma 4.1 and first establish the claimed result for the canonical model where M is given by $\mathbb{R}^{n-\ell} \times \mathbb{R}^\ell$, $0 \le \ell \le n$, endowed with the Euclidean metric, and h by $h_\ell(y) = \frac{1}{2}\left(\|y^+\|^2 - \|y^-\|^2\right)$. Then 0 is the only critical point of h_ℓ and its index is ℓ. Let $S^+ := \{(y^+, 0) \mid \|y^+\|^2 = 2\varepsilon\}$ and define

$$\begin{aligned} Q &:= \{(x^+, x) \in \mathbb{R}^n \times \mathbb{R}^n \mid h_\ell(x^+) = \varepsilon;\ \|x\|^2 < 2\varepsilon;\ x^+ \sim x\} \\ Q' &:= \{(x^+, x) \in Q \mid x = (y^+, y^-) \text{ with } \|x\|^2 < 2\varepsilon \text{ and } y^+ \neq 0\} \\ T &:= S^+ \times \{(0, y^-) \mid \|y^-\|^2 < 2\varepsilon\}. \end{aligned}$$

Define the map

$$\theta : T \times [0, 1/2) \to \mathbb{R}^n \times \mathbb{R}^n, \left((y^+, 0), (0, y^-), s\right) \mapsto (x^+, x)$$

with $x^+(s) = x^+(s; y^+, y^-)$ and $x(s) \equiv x(s; y^+, y^-)$ given by

$$x^+(s) := f(s)(y^+, sy^-);\ x(s) := f(s)(sy^+, y^-)$$

and $f(s) := (1 - s^2\|y^-\|^2/2\varepsilon)^{-1/2}$. As $\|y^-\|^2 < 2\varepsilon$, and $0 \le s < 1/2, f(s)$ is well defined and satisfies $f(s) \le \sqrt{4/3}$. Note that $(x^+(s), x(s)) \in T$ only for $s = 0$ where $(x^+(s), x(s))$ is given by $((y^+, 0), (0, y^-))$. The point $x^+(s)$ is defined in such a way that $h_\ell(x^+(s)) = \varepsilon$ whereas $x(s)$ is defined so that $x(s) \sim x^+(s)$ for any $0 \le s < 1/2$, i.e. $x(s)$ lies on the (possibly broken) trajectory of the gradient vector field $-\text{grad}_g h_\ell$ going through $x^+(s)$. This shows that the range of θ is contained in Q and the one of the restriction $\theta\big|_{T\times(0,1/2)}$ in Q'. Clearly θ is a smooth embedding and the restriction $\theta\big|_{T\times\{0\}}$ is the standard inclusion. Hence for the standard model, the case under consideration is proved. To prove the considered case in the general situation we provide for any $v \in \text{Crit}(h)$ with $h(v) = c_k$ and $\text{index}(v) = \ell$ a smooth embedding

$$\theta_v : T_v \cap (M_k^+ \times U_3) \times [0, 1/2) \to M_k^+ \times U_3$$

where $U_3 = h^{-1}((c_k - \varepsilon, c_k + \varepsilon))$ such that

(i) $\theta_v\big|_{T_v \cap (M_k^+ \times U_3) \times \{0\}}$ is the standard inclusion,

(ii) $\theta_v\left(T_v \cap (M_k^+ \times U_3) \times [0, 1/2)\right) \subseteq Q_k$,

(iii) $\theta_v\left(T_v \cap (M_k^+ \times U_3) \times (0,1/2)\right) \subseteq Q_k'$.

As $T_v \cap (M_k^+ \times U_3)$ is contained in the standard chart U_v, the map θ_v can be expressed in terms of the standard coordinate map $\varphi_v : B_r \to U_v$. Consider the standard model with $\ell = \text{index}(v)$. Note that $T \subseteq B_r, \varphi_v(S^+) = S_v^+, \varphi_v((0,y^-)) \in W_v^- \cap U_v$ for any $y^- \in \mathbb{R}^k$ with $\|y^-\|^2 < 2\varepsilon$, and hence $\varphi_v(T) = T_v \cap (M_k^+ \times U_3)$. Further, $x^+(s)$ and $x(s)$ as defined above, are elements in B_r as for any $((y^+,0),(0,y^-)) \in T$ and $0 \le s < 1/2$.

$$\|x^+(s)\|^2 = f(s)^2\left(\|y^+\|^2 + s^2\|y^-\|^2\right) < 4\varepsilon/3$$

and

$$\|x(s)\|^2 = f(s)^2\left(s^2\|y^+\|^2 + \|y^-\|^2\right) < 4\varepsilon$$

and $4\varepsilon < r^2$ for ε sufficiently small. Hence for y^+, y^- and s as above one can define

$$\theta_v\left(\varphi_v(y^+,0), \varphi_v(0,y^-), s\right) := \left(\varphi_v(x^+(s)), \varphi_v(x(s))\right).$$

The map θ_v then satisfies the claimed properties (i) - (iii) as, by construction, θ satisfies the corresponding ones. The statements on the maps p_k and q_k are verified in a straight forward way. □

4.2. **Spaces of trajectories.** In this subsection we prove that for any $v, w \in \text{Crit}(h)$, the topological spaces $\mathcal{B}(v,w)$ (Theorem 4.3) and $\hat{W}_v^-$ (Theorem 4.4) have a canonical structure of a smooth manifold with corners with interior $\mathcal{T}(v,w)$ and W_v^-, respectively – see Section 3 for the notion of a manifold with corners M and the smooth submanifolds $\partial_k M$ of M of codimension k introduced there. Further we show that $\hat{i}_v : \hat{W}_v^- \to M$ is a smooth extension of the inclusion $i_v : W_v^- \to M$. Versions of Theorem 4.3 and Theorem 4.4 can be found in [18].

Theorem 4.3. *Assume that M is a smooth manifold, (h,X) a Morse-Smale pair and v, w any critical points of h with $w < v$. Then*

(i) *$\mathcal{B}(v,w)$ is compact and has a canonical structure of a smooth manifold with corners.*

(ii) *$\mathcal{B}(v,w)$ is of dimension $i(v) - i(w) - 1$ and for any $0 \le k \le \dim \mathcal{B}(v,w)$,*

$$\partial_k \mathcal{B}(v,w) = \bigsqcup_{w < v_k < \ldots < v_1 < v} \mathcal{T}(v,v_1) \times \ldots \times \mathcal{T}(v_k,w).$$

In particular,

$$\partial_0 \mathcal{B}(v,w) = \mathcal{T}(v,w).$$

Remark 4.1. Note that for $v, w \in \text{Crit}(h)$ not satisfying $w \le v, \mathcal{B}(v,w) = \emptyset$ whereas for $w = v, \mathcal{B}(v,w) = \{v\}$.

Proof. (i) Let $\ell_0 - 1 \le \ell$ be the integers satisfying $h(v) = c_{\ell_0 - 1}$ and $h(w) = c_{\ell+1}$ respectively. If $\ell_0 - 1 = \ell$, then $h(w) = c_{\ell_0}$ and hence $\mathcal{B}(v,w) = \mathcal{T}(v,w)$ which is a smooth manifold – see (2.15). For $\ell \ge \ell_0$ we want to use Lemm 3.4 and Lemma 4.1 to obtain a canonical differentiable structure of a manifold with corners for $\mathcal{B}(v,w)$.

To this end introduce

$$\begin{aligned}
\mathcal{P} \equiv \mathcal{P}_{\ell_0\ell} &:= \prod_{j=\ell_0}^{\ell} P_j \\
\mathcal{M} \equiv \mathcal{M}_{\ell_0\ell} &:= \prod_{j=\ell_0}^{\ell} M_j^+ \times M_j^- \\
\mathcal{N}' \equiv \mathcal{N}'_{vw} &:= \left(W_v^- \cap M_{\ell_0}^+\right) \times \prod_{j=\ell_0}^{\ell-1} M_j^- \times \left(W_w^+ \cap M_\ell^-\right)
\end{aligned}$$

where we recall that

$$M_j^\pm = h^{-1}\left(\{c_j \pm \varepsilon\}\right); \quad P_j := \left\{(x_j^+, x_j^-) \in M_j^+ \times M_j^- \,\middle|\, \varphi_k^+(x_j^+) = \varphi_k^-(x_j^-)\right\}.$$

By Lemma 4.1, P_j is a $(n-1)$ dimensional manifold with boundary, hence, by Corollary 3.2, $\mathcal{P}$ a manifold with corners of dimension $(\ell - \ell_0 + 1)(n-1)$ with $(0 \leq k \leq \dim \mathcal{P})$

$$\partial_k \mathcal{P} = \bigsqcup_{|\sigma|=k} \prod_{j=\ell_0}^{\ell} \partial_{\sigma(j)} P_j \tag{4.3}$$

where $\sigma = (\sigma(j))_j$ is a sequence of elements $\sigma(j) \in \{0,1\}$ and $|\sigma| := \sum_j \sigma(j)$. Further, $\mathcal{M}$ is a smooth manifold of dimension $2(\ell - \ell_0 + 1)(n-1)$ and, with $f_j : P_j \hookrightarrow M_j^+ \times M_j^-$ denoting the inclusion of $P_j \subseteq M_j^+ \times M_j^-$, the map

$$\mathfrak{f} := \prod_{j=\ell_0}^{\ell} f_j : \mathcal{P} \to \mathcal{M}$$

is a smooth embedding. Finally $\mathcal{N}'$ is a smooth manifold of dimension $i(v) - i(w) - 1 + (\ell - \ell_0 + 1)(n-1)$ and can be canonically identified with a submanifold of $\mathcal{M}$ as follows. Introduce

$$\theta : \mathcal{N}'_{vw} \to \mathcal{M}_{\ell_0\ell}, \quad (x_v^+, (x_j^-)_j, x_w^-) \mapsto \left(x_v^+, \left(x_j^-, \psi_j(x_j^-)\right)_j, x_w^-\right).$$

As the maps $\psi_j : M_j^- \to M_{j+1}^+$, defined in terms of the flow Ψ_t (cf section 4.1), are diffeomorphisms it follows that θ is a smooth embedding, hence

$$\mathcal{N} \equiv \mathcal{N}_{vw} := \theta(\mathcal{N}')$$

is a submanifold of $\mathcal{M}$. As in Subsection 2.3 one sees that $\mathcal{B}(v,w)$ can be identified with the image in $\mathcal{P}$ of the following smooth embedding

$$J : \gamma \mapsto (x_j^+, x_j^-)_j$$

where $x_j^\pm$ denote the points of intersection of the (possibly broken) trajectory γ with the level sets $M_j^\pm$. Clearly, the image of J coincides with $\mathfrak{f}^{-1}(\mathcal{N})$. Therefore $\mathcal{B}(v,w)$ and $\mathfrak{f}^{-1}(\mathcal{N})$ are identified as topological spaces and a differentiable structure of $\mathfrak{f}^{-1}(\mathcal{N})$ provides a differentiable structure on $\mathcal{B}(v,w)$. Next we prove that $\mathfrak{f}^{-1}(\mathcal{N})$ is a manifold with corners. In view of Lemma 3.4 this is the case if $\mathfrak{f}$ is transversal to $\mathcal{N}$, i.e. for any $0 \leq k \leq \dim \mathcal{P}$ and $\mathfrak{x} \in \partial_k \mathcal{P}$ with $\mathfrak{f}(\mathfrak{x}) \in \mathcal{N}$

$$T_{\mathfrak{f}(\mathfrak{x})}\mathcal{M} = T_{\mathfrak{f}(\mathfrak{x})}\mathcal{N} + d_{\mathfrak{x}}\mathfrak{f}(T_{\mathfrak{x}}\partial_k \mathcal{P}). \tag{4.4}$$

Using that X satisfies the Morse-Smale condition the transversality condition (4.4) will be verified in the subsequent subsection. As in Subsection 2.3 one argues that the induced differentiable structure on $\mathcal{B}(v,w)$ is independent of ε, hence canonical.

(ii) In view of Lemma 3.4,

$$\begin{aligned}\dim \mathfrak{f}^{-1}(\mathcal{N}) &= \dim \mathcal{P} - \operatorname{codim} \mathcal{N}\\ &= \dim \mathcal{P} - \dim \mathcal{M} + \dim \mathcal{N}'\\ &= i(v) - i(w) - 1\end{aligned}$$

and for any $0 \leq k \leq \dim \mathfrak{f}^{-1}(\mathcal{N})$

$$\partial_k \mathfrak{f}^{-1}(\mathcal{N}) = \mathfrak{f}^{-1}(\mathcal{N}) \cap \partial_k \mathcal{P}$$

with $\partial_k \mathcal{P}$ given by (4.3). Using the identification of $\mathcal{B}(v,w)$ with $\mathfrak{f}^{-1}(\mathcal{N})$ one sees that

$$\partial_k \mathcal{B}(v,w) = \bigsqcup_{w<v_k<\ldots<v_1<v} \mathcal{T}(v,v_1) \times \ldots \times \mathcal{T}(v_k,w).$$

□

The smooth structure on $\hat{W}_v^-$ is more elaborate. Recall that according to the definition in Subsection 2.3 $\hat{W}_v^- = \bigsqcup_{w\leq v} \mathcal{B}(v,w) \times W_w^-$. For ℓ_0 with $h(v) = c_{\ell_0 - 1}$, introduce the open covering $\left(\hat{W}_{v,\ell}^-\right)_{\ell \geq \ell_0 - 1}$ of $\hat{W}_v^-$ given by

$$\hat{W}_{v,\ell}^- := \{(\gamma, x) \in \hat{W}_v^- \,\big|\, c_{\ell+1} < \hat{h}_v(\gamma, x) < c_{\ell-1}\}$$

and $\hat{h}_v = h \circ \hat{i}_v$. The differentiable structure of $\hat{W}_v^-$ is defined by providing for any $\ell \geq \ell_0 - 1$ a differentiable structure on $\hat{W}_{v,\ell}^-$ so that these structures are compatible on the intersections. We note that $\hat{W}_{v,\ell}^- \cap \hat{W}_{v,\ell'}^- \neq \emptyset$ iff $|\ell - \ell'| \leq 1$ and that $\hat{W}_{v,\ell_0-1}^-$ is an open subset of W_v^-, hence a manifold.

For any $\ell \geq \ell_0$, $\hat{W}_{v,\ell}^-$ consists of (possibly broken) trajectories from v to a point $x \in M$ satisfying $c_{\ell+1} < h(x) < c_{\ell-1}$. Denote by γ_x the canonical parametrization (2.22) of the trajectory from v to x. To describe the differentiable structure of $\hat{W}_{v,\ell}^-$ we have to use a more complicated identification of γ_x than the one introduced in Subsection 2.3.

For an arbitrary element γ_x in $\hat{W}_{v,\ell}^-$ let

$$\hat{J}_\ell(\gamma_x) := \big((x_j^+, x_j^-)_j, (x_\ell^+, x)\big) \in \mathcal{M}_{\ell_0 \ell}$$

where

$$\mathcal{M}_{\ell_0 \ell} := \left(\prod_{j=\ell_0}^{\ell-1} M_j^+ \times M_j^-\right) \times M_\ell^+ \times h^{-1}\left((c_{\ell+1}, c_{\ell-1})\right)$$

and $x_j^\pm$ are the points of intersection of γ_x with the level sets

$$M_j^\pm := h^{-1}\left(\{c_j \pm \varepsilon\}\right)$$

with $\varepsilon > 0$ being chosen sufficiently small. Clearly, $\hat{J}_\ell : \hat{W}_{v,\ell}^- \to \mathcal{M}_{v,\ell}$ is an embedding. The component (x_j^+, x_j^-) of $\hat{J}_\ell(\gamma_x)$ is an element of P_j and the component (x_ℓ^+, x) is in Q_ℓ.

Theorem 4.4. *Assume that M is a smooth manifold, (h, X) a Morse-Smale pair and $v \in \mathrm{Crit}(h)$. Then,*

(i) *$\hat{W}_v^-$ has a canonical structure of a smooth manifold with corners.*

(ii) *$\hat{W}_v^-$ is of dimension $i(v)$ and for any $1 \leq k \leq \dim \hat{W}_v^-$*

$$\partial_k \hat{W}_v^- = \bigsqcup_{w<v} \partial_{k-1} \mathcal{B}(v,w) \times W_w^- .$$

whereas $\partial_0 \hat{W}_v^- = W_v^-$.

(iii) *The extension $\hat{i}_v : \hat{W}_v^- \to M$ of the inclusion $i_v : W_v^- \to M$ is smooth where $\hat{i}_v$ is given on $\mathcal{B}(v,w) \times W_w^-$ for any $w < v$ by the composition of the projection $\mathcal{B}(v,w) \times W_w^- \to W_w^-$ with the inclusion $W_w^- \hookrightarrow M$.*

Remark 4.2. Combined with Theorem 2.14, Theorem 4.4 implies that $\hat{i}_v$ and $\hat{h}_v := h \circ \hat{i}_v$ are smooth, proper maps.

Proof. As outlined above we consider the open covering $(\hat{W}_\ell^-)_{\ell \geq \ell_0 - 1}$ of $\hat{W}_v^-$ given by

$$\hat{W}_\ell^- := \left\{ \gamma_x \in \hat{W}_v^- \,\middle|\, c_{\ell+1} < h(x) < c_{\ell-1} \right\}$$

where ℓ_0 is the integer with $h(v) = c_{\ell_0 - 1}$. First we define a differentiable structure of a manifold with corners for each of the open sets $\hat{W}_\ell^-$ so that the restrictions of $\hat{i}_v$ to $\hat{W}_\ell^-$ is a smooth map. In a second step we then check that for any $\ell, \ell' \geq \ell_0 - 1$, $\hat{W}_\ell^-$ and $\hat{W}_{\ell'}^-$ induce the same differentiable structure on the intersection $\hat{W}_\ell^- \cap \hat{W}_{\ell'}^-$. This then proves that $\hat{W}_v^-$ has a structure of a smooth manifold with corners and that $\hat{i}_v$ is smooth. To define a differentiable structure on $\hat{W}_\ell^-$ we proceed in a similar way as for $\mathcal{B}(v,w)$ (cf proof of Theorem 4.3).
First note that $\hat{W}_{\ell_0-1}^-$ is an open subset of W_v^-, hence a smooth manifold. For $\ell \geq \ell_0$ introduce - with a view towards an application of Lemma 3.4 - the following spaces

$$\mathcal{P} \equiv \mathcal{P}_{\ell_0 \ell} := \left(\prod_{j=\ell_0}^{\ell-1} P_j \right) \times Q_\ell$$

$$\mathcal{M} \equiv \mathcal{M}_{\ell_0 \ell} := \prod_{j=\ell_0}^{\ell-1} \left(M_j^+ \times M_j^- \right) \times \left(M_\ell^+ \times M_{\ell+1,\ell-1} \right)$$

$$\mathcal{N}' \equiv \mathcal{N}'_{v,\ell} := \left(W_v^- \cap M_{\ell_0}^+ \right) \times \left(\prod_{j=\ell_0}^{\ell-1} M_j^- \right) \times M_{\ell+1,\ell-1}$$

where we recall that

$$M_{\ell+1,\ell-1} = \left\{ x \in M \,\middle|\, c_{\ell+1} < h(x) < c_{\ell-1} \right\}$$

and

$$Q_\ell = \left\{ (x^+, x) \in M_\ell^+ \times M_{\ell+1,\ell-1} \,\middle|\, x^+ \sim x \right\} .$$

By Lemma 4.1, Lemma 4.2, and Corollary 3.2, $\mathcal{P}$ is a manifold with corners of dimension $(\ell-\ell_0+1)(n-1)+1$ with $(0 \leq k \leq \dim \mathcal{P})$

$$\partial_k \mathcal{P} = \bigsqcup_{|\sigma|=k} \left(\prod_{j=\ell_0}^{\ell-1} \partial_{\sigma(j)} P_j \right) \times \partial_{\sigma(\ell)} Q_\ell$$

where $\sigma = (\sigma(j))_{\ell_0 \leq j \leq \ell}, \sigma(j) \in \{0,1\}$ and $|\sigma| = \sum_{j=\ell_0}^{\ell} \sigma(j)$. Further, $\mathcal{M}$ is a smooth manifold of dimension $2(\ell-\ell_0+1)(n-1)+1$ and

$$\mathfrak{f} := \left(\prod_{j=\ell_0}^{\ell-1} f_j \right) \times g_\ell : \mathcal{P} \to \mathcal{M}$$

is a smooth embedding where $f_j : P_j \to M_j^+ \times M_j^-$ $(\ell_0 \leq j \leq \ell-1)$ and $g_\ell : Q_\ell \to M_\ell^+ \times M_{\ell+1,\ell-1}$ denote the natural inclusions.
Finally, $\mathcal{N}'$ is a smooth manifold of dimension $i(v)+(\ell-\ell_0+1)(n-1)$ and can be canonically identified with a submanifold of $\mathcal{M}$ as follows: introduce

$$\theta : \mathcal{N}' \to \mathcal{M}, \ \left(x_v^+, (x_j^-)_j, x\right) \mapsto \left(x_v^+, \left(x_j^-, \psi_j(x_j^-)\right)_j, x\right).$$

As $\psi_j : M_j^- \to M_{j+1}^+$ are diffeomorphisms, θ is a smooth embedding and thus

$$\mathcal{N}_\ell \equiv \mathcal{N}_{v,\ell} := \theta(\mathcal{N}'_{v,\ell})$$

is a submanifold of $\mathcal{M}$. $\mathfrak{f}^{-1}(\mathcal{N}_\ell)$ can be canonically identified with $\hat{W}_\ell^-$, being the image of the embedding $\hat{J}_\ell : \hat{W}_\ell^- \to \mathcal{P}$ defined by

$$\hat{J}_\ell(\gamma_x) = \left((x_j^+, x_j^-)_j, (x_\ell^+, x)\right)$$

where $x_j^\pm$ are the points of intersection of γ_x with the level sets

$$M_j^\pm := h^{-1}\left(\{c_j \pm \varepsilon\}\right).$$

Hence we have to prove that $\mathfrak{f}^{-1}(\mathcal{N}_\ell)$ is a manifold with corners. In view of Lemma 3.4 this is the case if $\mathfrak{f}$ is transversal to $\mathcal{N}_\ell$, i.e. for any $0 \leq k \leq \dim \mathcal{P}$ and $\mathfrak{x} \in \partial_k \mathcal{P}$

$$T_{\mathfrak{f}(\mathfrak{x})}\mathcal{M} = T_{\mathfrak{f}(\mathfrak{x})}\mathcal{N}_\ell + d_{\mathfrak{x}}\mathfrak{f}(T_{\mathfrak{x}}\partial_k\mathcal{P}). \tag{4.5}$$

Again, these transversality conditions will be verified in the subsequent subsection using the assumption that the vector field X satisfies the Morse-Smale condition.
As in Subsection 2.3 one argues that the induced differentiable structure on $\hat{W}_\ell^-$ is independent of ε. By Lemma 3.4, for any $\ell \geq \ell_0$

$$\begin{aligned}\dim \mathfrak{f}^{-1}(\mathcal{N}_\ell) &= \dim \mathcal{P} - \text{codim } \mathcal{N}_\ell \\ &= \dim \mathcal{P} - \dim \mathcal{M} + \dim \mathcal{N}'_\ell \\ &= \iota(v)\end{aligned}$$

and for any $0 \leq k \leq \dim \mathfrak{f}^{-1}(\mathcal{N}_\ell)$,

$$\partial_k \mathfrak{f}^{-1}(\mathcal{N}_\ell) = \mathfrak{f}^{-1}(\mathcal{N}_\ell) \cap \partial_k \mathcal{P}.$$

Using the identification $\hat{J}_\ell : \hat{W}_\ell \to \mathcal{P}$ one sees that $\mathfrak{f}^{-1}(\mathcal{N}_\ell) \cap \partial_k \mathcal{P}$ corresponds to

$$\partial_k \hat{W}_\ell^- = \bigsqcup_{\substack{w \leq v \\ h(w) \geq c_\ell}} \partial_k \mathcal{B}(v,w) \times (W_w^- \cap M_{\ell+1,\ell-1}).$$

In particular, for $k = 0$, the interior $\partial_0 \hat{W}_\ell^-$ of $\hat{W}_\ell^-$ is given by $W_v^- \cap M_{\ell+1,\ell-1}$. Further, note that

$$\begin{tikzcd} \hat{W}_\ell^- \arrow[rr, "\hat{J}_\ell"] \arrow[dr, "\hat{i}_v|_{\hat{W}_\ell^-}"'] & & \mathfrak{f}^{-1}(\mathcal{N}_\ell) \arrow[dl, "\pi|_{\mathfrak{f}^{-1}(\mathcal{N}_\ell)}"] \\ & M_{\ell+1,\ell-1} & \end{tikzcd}$$

is commutative where

$$\pi : \mathcal{P} \to M, \ \big((x_j^+, x_j^-)_{\ell_0 \le j \le \ell-1}, x_\ell^+, x\big) \mapsto x$$

denotes the projection onto the last component of $\mathcal{P}$. Hence $\hat{i}_v \big|_{\hat{W}_\ell^-}$ is a composition of smooth maps, hence smooth.

In a second step we now prove that $\hat{W}_\ell^-$ and $\hat{W}_{\ell'}^-$ induce the same differentiable structure on the intersection $\hat{W}_\ell^- \cap \hat{W}_{\ell'}^-$. Arguing as in the proof of Proposition 2.13 first note that $\hat{W}_\ell^- \cap \hat{W}_{\ell'}^- = \emptyset$ for $|\ell - \ell'| \ge 2$. Hence it remains to consider the case where $\ell \ge \ell_0 - 1$ and $\ell' := \ell + 1$. Then $D_\ell := \hat{W}_\ell^- \cap \hat{W}_{\ell+1}^-$ is the set of elements $\gamma_x \in \hat{W}_v^-$ with $c_{\ell+1} < h(x) < c_\ell$. First let us treat the case $\ell \ge \ell_0$. Then

$$\hat{J}_{\ell+1}(\gamma_x) = \big((x_j^+, x_j^-)_{\ell_0 \le j \le \ell-1}, \ x_\ell^+, x_\ell^-, x_{\ell+1}^+, x\big)$$

and

$$\hat{J}_\ell(\gamma_x) = \big((x_j^+, x_j^-)_{\ell_0 \le j \le \ell-1}, x_\ell^+, x\big) \, .$$

Note that the points $x_\ell^-, x_{\ell+1}^+$ and x are on the trajectory γ_x and contained in $M_{\ell+1,\ell}$, hence

$$\gamma_x(c_{\ell+1} + \varepsilon) = \Psi_{c_{\ell+1}+\varepsilon-h(x)}(x)$$
$$\gamma_x(c_\ell - \varepsilon) = \Psi_{c_\ell-\varepsilon-h(x)}(x).$$

From the properties of the flow $\Psi_t(x)$ in the region $M_{\ell+1,\ell}$ one concludes that

$$\hat{J}_\ell(\gamma_x) \mapsto \hat{J}_{\ell+1}(\gamma_x) = \big((x_j^+, x_j^-)_{j \le \ell-1}, \ x_\ell^+, \Psi_{c_\ell-\varepsilon-h(x)}(x), \Psi_{c_{\ell+1}+\varepsilon-h(x)}(x), x\big)$$

is a diffeomorphism from $\hat{J}_\ell(D_\ell)$ onto $\hat{J}_{\ell+1}(D_\ell)$. This shows that for $\ell \ge \ell_0$, $\hat{W}_\ell^-$ and $\hat{W}_{\ell+1}^-$ induce the same differentiable structure on the intersection $\hat{W}_\ell^- \cap \hat{W}_{\ell+1}^-$. The case $\ell = \ell_0 - 1$ is treated in a similar fashion and thus (i) is proved. Statements (ii) and (iii) follow easily from the considerations above. $\square$

4.3. **Transversality properties.** In this subsection we verify the transversality conditions (4.4) and (4.5) stated in Subsection 4.2 which allow to apply Lemma 3.4 and hence to conclude that $\mathcal{B}(v, w)$ and, respectively, $\hat{W}_\ell^-(v)$ are manifolds with corners. Without further explanations we use the notation from the previous sections.

Transversality condition (4.4): To illustrate our arguments let us first verify (4.4) for $\mathfrak{x} = (x_j^+, x_j^-)_{\ell_0 \le j \le \ell} \in \partial_0 \mathcal{P}$. For such a point the image $d_\mathfrak{x}\mathfrak{f}(T_\mathfrak{x}\mathcal{P})$ of the tangent space $T_\mathfrak{x}\mathcal{P} = T_\mathfrak{x}\partial_0\mathcal{P}$ by the differential $d_\mathfrak{x}\mathfrak{f} : T_\mathfrak{x}\mathcal{P} \to T_{\mathfrak{f}(\mathfrak{x})}\mathcal{M}$ consists of vectors of the form

$$(\xi_j, d\varphi_j \cdot \xi_j)_{\ell_0 \le j \le \ell} \tag{4.6}$$

where $\xi_j \in T_{x_j^+} M_j^+$ and φ_j is given by (4.1) and $d\varphi_j \equiv d_{x_j^+}\varphi_j$. One computes

$$\dim d_{\mathfrak{r}}\mathfrak{f}(T_{\mathfrak{r}}\mathcal{P}) = \sum_{j=\ell_0}^{\ell} \dim M_j^+ = (n-1)(\ell - \ell_0 + 1).$$

The tangent space $T_{\mathfrak{f}(\mathfrak{r})}\mathcal{N}$ consists of all vectors of the form

$$\Big(\xi, (\zeta_j, d\psi_j(\zeta_j))_{\ell_0 \le j \le \ell-1}, \zeta\Big) \tag{4.7}$$

where $\xi \in T_{x_{\ell_0}^+}(W_v^- \cap M_{\ell_0}^+)$, $\zeta_j \in T_{x_j^-} M_j^-$, $d\psi_j \equiv d_{x_j^-}\psi_j$ and $\zeta \in T_{x_\ell^-}(W_w^+ \cap M_\ell^-)$. It is of dimension

$$\begin{aligned} \dim T_{\mathfrak{f}(\mathfrak{r})}\mathcal{N} &= i(v) - 1 + \sum_{j=\ell_0}^{\ell-1} \dim M_j^- + n - i(w) - 1 \\ &= i(v) - i(w) + (n-1)(\ell - \ell_0 + 1). \end{aligned}$$

As $\dim \mathcal{M} = 2(n-1)(\ell - \ell_0 + 1)$ it then follows that

$$\dim d_{\mathfrak{r}}\mathfrak{f}(T_{\mathfrak{r}}\mathcal{P}) + \dim T_{\mathfrak{f}(\mathfrak{r})}\mathcal{N} - \dim \mathcal{M} = i(v) - i(w) - 1.$$

To show the claimed transversality at the point $\mathfrak{r}$, it remains to verify that

$$\dim\big(d_{\mathfrak{r}}\mathfrak{f}(T_{\mathfrak{r}}\mathcal{P}) \cap T_{\mathfrak{f}(\mathfrak{r})}\mathcal{N}\big) = i(v) - i(w) - 1.$$

In view of (4.6) - (4.7) and of the fact that $d_{x_j^-}\psi_j$ and $d_{x_j^+}\varphi_j$ are isomorphisms (cf Lemma 4.1), the linear space $d_{\mathfrak{r}}\mathfrak{f}(T_{\mathfrak{r}}\mathcal{P}) \cap T_{\mathfrak{f}(\mathfrak{r})}\mathcal{N}$ is linearly isomorphic to the space of all elements (ξ, ζ) in $T_{x_{\ell_0}^+}(W_v^- \cap M_{\ell_0}^+) \times T_{x_\ell^-}(W_w^+ \cap M_\ell^-)$ satisfying

$$\zeta = d\varphi_\ell \circ d\psi_{\ell-1} \circ \ldots \circ d\varphi_{\ell_0}\xi.$$

Hence $d_{\mathfrak{r}}\mathfrak{f}(T_{\mathfrak{r}}\mathcal{P}) \cap T_{\mathfrak{f}(\mathfrak{r})}\mathcal{N}$ is linearly isomorphic to the graph of

$$d\varphi_\ell \circ d\psi_{\ell-1} \circ \ldots \circ d\varphi_{\ell_0} : T_{x_{\ell_0}^+}(W_w^+ \cap W_v^- \cap M_{\ell_0}^+) \to T_{x_\ell^-}(W_w^+ \cap W_v^- \cap M_\ell^-).$$

As, by assumption, X is Morse-Smale, it follows that in the case where $W_w^+ \cap W_v^- \neq \emptyset$

$$\dim\big(d_{\mathfrak{r}}\mathfrak{f}(T_{\mathfrak{r}}\mathcal{P}) \cap T_{\mathfrak{f}(\mathfrak{r})}\mathcal{N}'\big) = i(v) - i(w) - 1.$$

To prove the transversality condition (4.4) for $\mathfrak{r}$ in $\partial_k\mathcal{P}$ with $1 \le k \le \dim \mathfrak{f}^{-1}(\mathcal{N})$, let us first introduce some more notation. For any $1 \le k \le \dim \mathfrak{f}^{-1}(\mathcal{N})$ and any $\sigma = (\sigma(j))_{\ell_0 \le j \le \ell}$ with $\sigma(j) \in \{0, 1\}$ and $|\sigma| = \sum_j \sigma(j) = k$, choose any element

$$\mathfrak{r} = (x_j^+, x_j^-)_j \in \prod_{j=\ell_0}^{\ell} \partial_{\sigma(j)} P_j \subseteq \partial_k\mathcal{P}.$$

As

$$\partial_1 P_j = \bigsqcup_{h(u)=c_j} S_u^+ \times S_u^-$$

for any $\ell_0 \le j \le \ell$ such that $\sigma(j) = 1$ there exists a critical point $u_j \in M_j$ with $x_j^\pm \in S_{u_j}^\pm$. Hence the tangent space $T_{\mathfrak{r}}\partial_k\mathcal{P}$ is of the form $\prod_{j=\ell_0}^{\ell} E_j$ where

$$E_j = \begin{cases} T_{(x_j^+, x_j^-)}P_j = (Id \times d\varphi_j) \cdot T_{x_j^+}M_j^+ & \text{if } \sigma(j) = 0 \\ T_{x_j^+}S_{u_j}^+ \times T_{x_j^-}S_{u_j}^- & \text{if } \sigma(j) = 1. \end{cases}$$

Further, write $T_{\mathfrak{x}}\mathcal{M} = \prod_{j=\ell_0}^{\ell} F_j$ where $F_j := F_j^+ \times F_j^-$ with

$$F_j^{\pm} := T_{x_j^{\pm}} M_j^{\pm},$$

and let $\alpha := d_{\mathfrak{x}}\mathfrak{f} = \prod_{j=\ell_0}^{\ell} \alpha_j$ where $\alpha_j = \alpha_j^+ \times \alpha_j^-$ is given by the canonical projections,

$$\alpha_j^{\pm} : T_{(x_j^+, x_j^-)} P_j \longrightarrow T_{x_j^{\pm}} M_j^{\pm}$$

when $\sigma(j) = 0$ whereas $\alpha_j = \mathrm{diag}(\alpha_j^+, \alpha_j^-)$ with $\alpha_j^{\pm}$ denoting now the natural inclusions

$$\alpha_j^{\pm} : T_{x_j^{\pm}} S_{u_j}^{\pm} \to T_{x_j^{\pm}} M_j^{\pm}$$

when $\sigma(j) = 1$. In the sequel we will not distinguish between

$$\alpha_j^{\pm} : T_{x_j^{\pm}} S_{u_j}^{\pm} \to T_{x_j^{\pm}} M_j^{\pm}$$

and its trivial extension

$$\alpha_j^{\pm} : T_{x_j^+} S_{u_j}^+ \times T_{x_j^-} S_{u_j}^+ \to T_{x_j^{\pm}} M_j^{\pm}.$$

Finally, $T_{\mathfrak{f}(\mathfrak{x})}\mathcal{N}$ is isomorphic to $\prod_{j=\ell_0-1}^{\ell} G_j$ where

$$G_j := \begin{cases} T_{x_{\ell_0}^+}(W_v^- \cap M_{\ell_0}^+) & j = \ell_0 - 1 \\ T_{x_j^-}(M_j^-) & \ell_0 \leq j \leq \ell - 1 \\ T_{x_\ell^-}(W_w^+ \cap M_\ell^-) & j = \ell. \end{cases}$$

The linear map $\beta : \prod_{j=\ell_0-1}^{\ell} G_j \to T_{\mathfrak{f}(\mathfrak{x})}\mathcal{M}$ identifying $\prod_{j=\ell_0-1}^{\ell} G_j$ with $T_{\mathfrak{f}(\mathfrak{x})}\mathcal{N}$ is given by $\beta = \beta_{\ell_0-1} \times \prod_{j=\ell_0}^{\ell-1} \overline{\beta}_j \times \beta_\ell$ where $\beta_{\ell_0-1} : G_{\ell_0-1} \hookrightarrow T_{x_{\ell_0}^+} M_{\ell_0}^+$ and $\beta_\ell : G_\ell \hookrightarrow T_{x_\ell^-} M_\ell^-$ are the natural inclusions; for $\ell_0 \leq j \leq \ell - 1$

$$\overline{\beta}_j : G_j \to T_{x_j^-} M_j^- \times T_{x_{j+1}^+} M_{j+1}^+$$

is given by

$$\overline{\beta}_j := Id \times \beta_j \ ; \quad \beta_j := d_{x_j^-}\psi_j.$$

The situation at hand can best be described with the following diagram

$$\begin{tikzcd}
& E_{\ell_0} \arrow[dl, "\alpha^+_{\ell_0}"'] \arrow[dr, "\alpha^-_{\ell_0}"] & & & & E_{\ell_0+1} \arrow[dl, "\alpha^+_{\ell_0+1}"'] \arrow[dr, "\alpha^-_{\ell_0+1}"] & & \ldots\ldots \\
F^+_{\ell_0} & & F^-_{\ell_0} & & F^+_{\ell_0+1} & & {} & \ldots\ldots \\
G_{\ell_0-1} \arrow[u, "\beta_{\ell_0-1}"] & & & G_{\ell_0} \arrow[ul, "Id"] \arrow[ur, "\beta_{\ell_0}"'] & & & & \ldots\ldots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots\cdots & & & & E_\ell \arrow[dl, "\alpha_\ell^+"'] \arrow[dr, "\alpha_\ell^-"] & \\
\cdots\cdots & F_{\ell-1}^- & & F_\ell^+ & & F_\ell^- \\
\cdots\cdots & & G_{\ell-1} \arrow[ul, "Id"] \arrow[ur, "\beta_{\ell-1}"'] & & & G_\ell \arrow[u, "\beta_\ell"]
\end{tikzcd}$$

Diagram 1

To prove (4.4) in the case $1 \leq k \leq \dim \mathfrak{f}^{-1}(\mathcal{N})$ it is to show that Diagram 1 satisfies the transversality condition

$$\alpha\left(\prod_{j=\ell_0}^{\ell} E_j\right) + \beta\left(\prod_{j=\ell_0-1}^{\ell} G_j\right) = \prod_{j=\ell_0}^{\ell} F_j. \tag{4.8}$$

From the definition of α_j one sees that Diagram 1 *splits* at any E_j with $\sigma(j) = 1$. As we treat the case $|\sigma| = k \geq 1$ this implies that Diagram 1 splits up into Diagram 2 (beginning), $|\sigma| - 1$ diagrams of the type of Diagram 3 (middle pieces) and Diagram 4 (end).

$$\begin{tikzcd}
 & E_{\ell_0} \arrow[dl, "\alpha_{\ell_0}^+"'] \arrow[dr, "\alpha_{\ell_0}^-"] & & & & \cdots\cdots & & & T_{x_j^+} S_{u_j}^+ \arrow[d, "\alpha_j^+"'] \\
F_{\ell_0}^+ & & F_{\ell_0}^- & & F_{\ell_0+1}^+ & \cdots\cdots & F_{j-1}^- & & F_j^+ \\
G_{\ell_0-1} \arrow[u, "\beta_{\ell_0-1}"] & & & G_{\ell_0} \arrow[ul, "Id"] \arrow[ur, "\beta_{\ell_0}"'] & & \cdots\cdots & & G_{j-1} \arrow[ul, "Id"] \arrow[ur, "\beta_{j-1}"'] &
\end{tikzcd}$$

Diagram 2

$$\begin{tikzcd}
T_{x_j^-} S_{u_j}^- \arrow[dr, "\alpha_j^-"] & & & & \cdots\cdots & & & & T_{x_i^+} S_{u_i}^+ \arrow[dl, "\alpha_i^+"] \\
 & F_j^- & & F_{j+1}^+ & \cdots\cdots & F_{i-1}^- & & F_i^+ & \\
 & & G_j \arrow[ul, "Id"] \arrow[ur, "\beta_j"'] & & \cdots\cdots & & G_{i-1} \arrow[ul, "Id"] \arrow[ur, "\beta_{i-1}"'] & &
\end{tikzcd}$$

Diagram 3

$$
\begin{tikzcd}
T_{x^-_{j'}} S^-_{u_{j'}} \arrow[dr, "\alpha^-_{j'}"] & & & & \cdots\cdots & & & & E_\ell \arrow[dl, "\alpha^+_\ell"'] \arrow[dr, "\alpha^-_\ell"] & \\
& F^-_{j'} & & F^+_{j'+1} & \cdots\cdots & F^-_{\ell-1} & & F^+_\ell & & F^-_\ell \\
& & G_{j'} \arrow[ul, "Id"] \arrow[ur, "\beta_{j'}"'] & & \cdots\cdots & & G_{\ell-1} \arrow[ul, "Id"] \arrow[ur, "\beta_{\ell-1}"'] & & & G_\ell \arrow[u, "\beta_\ell"]
\end{tikzcd}
$$

Diagram 4

In each of the latter three diagrams all maps are linear isomorphisms (cf Lemma 4.1) except for β_{ℓ_0} and β_ℓ which are both $1-1$. As in the case $k = 0$ treated above, the transversality of these diagrams then all follow from the assumption that X is Morse-Smale, i.e. that for any $u, u' \in \mathrm{Crit}(h)$ and j,

$$\left(W^-_u \cap M^\pm_j\right) \pitchfork \left(W^+_{u'} \cap M^\pm_j\right).$$

Hence (4.4) is proved for any $0 \le k \le \dim f^{-1}(\mathcal{N})$.

Transversality condition (4.5): The proof of (4.5) is very similar to the one for (4.4). The only difference is that the last component of $\mathcal{P}, \mathcal{M}$, and $\mathcal{N}$ get changed from P_ℓ, M^-_ℓ and $W^+_w \cap M^-_\ell$ to $Q_\ell, M_{\ell+1,\ell-1}$ and $M_{\ell+1,\ell-1}$, respectively.

For the purpose of illustration let us again first verify (4.5) for

$$\mathfrak{r} = \left((x^+_j, x^-_j)_{\ell_0 \le j \le \ell-1}, x^+_\ell, x\right) \in \partial_0 \mathcal{P}.$$

Note that the image $d_\mathfrak{r}\mathfrak{f}(T_\mathfrak{r}\mathcal{P})$ of the tangent space $T_\mathfrak{r}\mathcal{P} = T_\mathfrak{r}\partial_0\mathcal{P}$ by the differential $d_\mathfrak{r}\mathfrak{f}$ consists of vectors of the form

$$\left((\xi_j, d\varphi_j \cdot \xi_j)_{\ell_0 \le j \le \ell-1},\ (d\eta^+_\ell \cdot \xi_\ell, \xi_\ell)\right) \tag{4.9}$$

where $\xi_j \in T_{x^+_j} M^+_j (\ell_0 \le j \le \ell - 1), \xi_\ell \in T_x M$, and $d\eta^+_\ell \equiv d_x \eta^+_\ell$ with

$$\eta^+_\ell : M_{\ell+1,\ell-1} \backslash W^-_\ell \to M^+_\ell, x \mapsto x^+_\ell$$

defined in terms of the flow Ψ_t – see (4.2) in Subsection 4.1. One computes

$$
\begin{aligned}
\dim d_\mathfrak{r}\mathfrak{f}(T_\mathfrak{r}\mathcal{P}) &= \sum_{j=\ell_0}^{\ell-1} \dim M^+_j + \dim Q_\ell \\
&= (n-1)(\ell - \ell_0 + 1) + 1.
\end{aligned}
$$

The space $T_{\mathfrak{f}(\mathfrak{r})}\mathcal{N}$ consists of all vectors of the form

$$(\xi, (\zeta_j, d\psi_j \cdot \zeta_j)_{\ell_0 \le j \le \ell-1}, \zeta) \tag{4.10}$$

where $\xi \in T_{x^+_{\ell_0}}(W^-_v \cap M^+_{\ell_0}),\ \zeta_j \in T_{x^-_j} M^-_j$, and $\zeta \in T_x M_{\ell+1,\ell-1}$. It is of dimension

$$
\begin{aligned}
\dim T_{\mathfrak{f}(\mathfrak{r})}\mathcal{N} &= i(v) - 1 + \sum_{j=\ell_0}^{\ell-1} \dim M^-_j + n \\
&= i(v) + (n-1)(\ell - \ell_0 + 1).
\end{aligned}
$$

As $\dim \mathcal{M} = 2(n-1)(\ell - \ell_0 + 1) + 1$ it then follows that

$$\dim d_\mathfrak{r}\mathfrak{f}(T_\mathfrak{r}\mathcal{P}) + \dim T_{\mathfrak{f}(\mathfrak{r})}\mathcal{N} - \dim \mathcal{M} = i(v).$$

Hence to show the claimed transversality at $\mathfrak{x}$ it remains to verify that

$$\dim\left(d_{\mathfrak{x}}\mathfrak{f}(T_{\mathfrak{x}}\mathcal{P})\cap T_{\mathfrak{f}(\mathfrak{x})}\mathcal{N}\right)=i(v).$$

In view of (4.9) - (4.10) and the fact that $d_{x_j^-}\psi_j$ and $d_{x_j^+}\varphi_j$ are isomorphisms (cf Lemma 4.1) and $d_x\eta_\ell^+$ is onto (cf Lemma 4.2) the linear space $d_{\mathfrak{x}}\mathfrak{f}(T_{\mathfrak{x}}\mathcal{P})\cap T_{\mathfrak{f}(\mathfrak{x})}\mathcal{N}$ is linearly isomorphic to the subspace of

$$T_{x^+_{\ell_0}}(W_v^-\cap M^+_{\ell_0})\times T_xM_{\ell-1,\ell+1}$$

consisting of elements (ξ,ζ) satisfying

$$d\psi_{\ell-1}\circ\cdots\circ d\varphi_{\ell_0}\xi=d_x\eta_\ell^+\zeta.$$

As $\dim\left(T_{x^+_{\ell_0}}(W_v^-\cap M^+_{\ell_0})\right)=i(v)-1$ and $d_x\eta_\ell^+$ has a one dimensional null space it follows that

$$\dim\left(d_{\mathfrak{x}}\mathfrak{f}(T_{\mathfrak{x}}\mathcal{P})\cap T_{\mathfrak{f}(\mathfrak{x})}\mathcal{N}\right)=i(v).$$

To prove the transversality condition (4.5) for $\mathfrak{x}=((x_j^+,x_j^-)_{\ell_0\le j\le\ell-1},x_\ell^+,x)$ in $\partial_k\mathcal{P}$ with $1\le k\le\dim\mathfrak{f}^{-1}(\mathcal{N})$ we introduce first some more notation. Recall that Q_ℓ is a manifold with boundary and that the boundary ∂_1Q_ℓ is given by

$$\partial_1Q_\ell=\bigsqcup_{\substack{w\in\mathrm{Crit}(h)\\ h(w)=c_\ell}}S_w^+\times(W_w^-\cap M_{\ell+1,\ell-1}).$$

The tangent space $T_{\mathfrak{x}}\partial_k\mathcal{P}$ is again of the form $\prod_{j=\ell_0}^{\ell}E_j$ as defined above except that the last component E_ℓ is now given by

$$E_\ell=\begin{cases}T_{(x_\ell^+,x)}Q_\ell=(d_x\eta_\ell^+\times Id)T_xM_{\ell+1,\ell-1} & \text{if } \sigma(\ell)=0\\ T_{x_\ell^+}S_w^+\times T_x(W_w^-\cap M_{\ell+1,\ell-1}) & \text{if } \sigma(\ell)=1\end{cases}$$

where $w\in\mathrm{Crit}(h)$ is the critical point so that

$$(x_\ell^+,x)\in S_w^+\times(W_w^-\cap M_{\ell+1,\ell-1}).$$

Similarly, $T_{\mathfrak{x}}\mathcal{M}=\prod_{j=\ell_0}^{\ell}F_j$ except that F_ℓ^- in $F_\ell=F_\ell^+\times F_\ell^-$ is now given by

$$F_\ell^-=T_xM_{\ell+1,\ell-1}$$

and $\alpha:=d_{\mathfrak{x}}\mathfrak{f}=\prod_{j=\ell_0}^{\ell}\alpha_j$ with the exception that α_ℓ^- in $\alpha_\ell=\alpha_\ell^+\times\alpha_\ell^-$ in the case $\sigma(\ell)=0$ is given by the canonical projection on the last component

$$\alpha_\ell^-:T_{(x_\ell^+,x)}Q_\ell\to T_xM_{\ell+1,\ell-1}$$

whereas when $\sigma(\ell)=1$, α_ℓ^- in $\alpha_\ell=diag(\alpha_\ell^+,\alpha_\ell^-)$ is given by the natural inclusion

$$\alpha_\ell^-:T_xW_w^-\to T_xM_{\ell+1,\ell-1}.$$

Furthermore, $T_x\mathfrak{f}^{-1}(\mathcal{N})=\prod_{j=\ell_0-1}^{\ell}G_j$ where G_ℓ is now given by

$$G_\ell:=T_xM_{\ell+1,\ell-1}.$$

Finally the map $\beta=\beta_{\ell_0-1}\times\prod_{j=\ell_0}^{\ell-1}\overline{\beta_j}\times\beta_\ell$ is the same as above with the exception that now $\beta_\ell:G_\ell\to T_xM_{\ell+1,\ell-1}$ is the identity map. With these changes made one then argues as above to prove the transversality conditions (4.5).

5. Geometric complex and integration map

In this section we introduce the geometric complex associated to a Morse-Smale pair (h, X). Using that the unstable manifolds of the vector field X admit a compactification with the structure of a smooth manifold with corners we then define a morphism between the de Rham complex and the geometric complex by integrating forms on unstable manifolds. This map can be proven to induce an isomorphism in cohomology.

5.1. **Coherent orientations and coverings.** In this subsection we discuss some additional notions and results needed for defining the geometric complex.
Orientation of a manifold with corners: Let M be a smooth manifold with corners of dimension n. Denote by $\det(M) \to M$ the vector bundle of rank 1 whose fibre at $x \in M$ is the n'th exterior product $\Lambda^n T_xM$ of the tangent space T_xM. As already mentioned in Subsection 3.2 an orientation $\mathcal{O}$ of M is an equivalence class of a nowhere vanishing sections where two smooth sections $\sigma_j : M \to \det(M)$ $(j = 1, 2)$ are said to be equivalent if there exists a positive smooth function $\lambda : M \to \mathbb{R}_{>0}$ so that $\sigma_1(x) = \lambda(x)\sigma_2(x)$ for any $x \in M$. Moreover, at the end of Subsection **??** we have seen that the Cartesian product $M_1 \times M_2$ of smooth manifolds with corners M_j with orientation $\mathcal{O}_j$ $(j = 1, 2)$ admits a canonical orientation $\mathcal{O}_1 \otimes \mathcal{O}_2$, referred to as the product orientation of $\mathcal{O}_1$ and $\mathcal{O}_2$.
Coherent orientations: Let (h, X) be a Morse-Smale pair. Recall that for any $v, w \in \mathrm{Crit}(h)$ with $W_v^- \cap W_w^+ \neq \emptyset$, $\mathcal{B}(v, w)$ is defined as the space of (broken and unbroken) trajectories from v to w and is endowed with a canonical structure of a smooth manifold with corners – see Theorem 4.3. Its interior $\partial_0\mathcal{B}(v, w)$ is given by the space $\mathcal{T}(v, w)$ of unbroken trajectories from v to w.

Lemma 5.1. *In the above set-up, $\mathcal{T}(v, w)$ and hence $\mathcal{B}(v, w)$ are orientable.*

Proof. Recall that the unstable manifold W_v^- is diffeomorphic to $\mathbb{R}^{i(v)}$, and hence orientable. Further recall that W_v^- is the interior of $\hat{W}_v^-, \partial_0\hat{W}_v^- = W_v^-$. By Lemma 3.5 it then follows that $\hat{W}_v^-$ as well as $\partial_k\hat{W}_v^-$ $(k \geq 1)$ are orientable. As $\mathcal{T}(v, w) \times W_w^-$ is contained in $\partial_1\hat{W}_v^-$ and $\mathcal{T}(v, w)$ is the interior of $\mathcal{B}(v, w)$ it follows that $\mathcal{T}(v, w)$ and hence $\mathcal{B}(v, w)$ are orientable as well. □

The following concept of coherent orientations will be important in Subsection 5.2 for constructing the geometric complex.

Definition 5.1. *A collection $\{\mathcal{O}_{uw}\}$ of orientations $\mathcal{O}_{uw}$ of $\mathcal{T}(u, w)$ (or equivalently of $\mathcal{B}(u, w)$) for u, w in* $\mathrm{Crit}(h)$ *with $\mathcal{T}(u, w) \neq \emptyset$ is said to be a collection of* coherent orientations [1] *if for any three critical points u, v, w of h with $\mathcal{T}(u, v), \mathcal{T}(v, w)$, and $\mathcal{T}(u, w)$ nonempty, the product orientation $\mathcal{O}_{uv} \otimes \mathcal{O}_{vw}$ on $\mathcal{T}(u, v) \times \mathcal{T}(v, w)$ is the opposite of the one canonically induced by the orientation $\mathcal{O}_{uw}$ on $\mathcal{B}(u, w)$ when viewing $\mathcal{T}(u, v) \times \mathcal{T}(v, w)$ as a subset of $\partial_1\mathcal{B}(u, w)$.*

Choose for any unstable manifold W_u^- an orientation $\mathcal{O}_u^-$. By the procedure explained in the proof of Lemma 5.1, $\mathcal{O}_u^-$ induces in a canonical way an orientation on $\mathcal{T}(u, w)$ for any $w \in \mathrm{Crit}(h)$ with $\mathcal{T}(u, w) \neq \emptyset$. In the sequel we denote this

[1] The concept of coherent orientations has been used in the framework of Floer theory by Floer and Hofer [13].

orientation by $\mathcal{O}_{uw} \equiv \mathcal{O}_{uw}(\mathcal{O}_u^-)$ to indicate that it is derived from the orientation $\mathcal{O}_u^-$of W_u^-.

Proposition 5.2. *Assume that (h, X) is a Morse-Smale pair and choose for any $u \in \mathrm{Crit}(h)$ an orientation $\mathcal{O}_u^-$ of W_u^-. Then $\{\mathcal{O}_{uw} \equiv \mathcal{O}_{uw}(\mathcal{O}_u^-)\}$ is a collection of coherent orientations.*

Proof. Let $u, v, w \in \mathrm{Crit}(h)$ so that $\mathcal{T}(u,v), \mathcal{T}(v,w)$ and $\mathcal{T}(u,w)$ are not empty. Denote by $\mathcal{O}_{uvw}$ the orientation on $\mathcal{T}(u,v) \times \mathcal{T}(v,w) \subseteq \partial_1 \mathcal{B}(u,w)$ induced from $\mathcal{B}(u,w)$ in a canonical way as explained in the proof of Lemma 3.5 (iii) by viewing $\mathcal{T}(u,v) \times \mathcal{T}(v,w)$ as a subset of $\partial_1 \mathcal{B}(u,w)$. It is to prove that $\mathcal{O}_{uvw} = -\mathcal{O}_{uv} \otimes \mathcal{O}_{vw}$. The manifold $\mathcal{T}(u,v) \times \mathcal{T}(v,w) \times W_w^-$, being a subset of $\partial_2 \hat{W}_u^-$, is contained in $\partial_1 (\mathcal{B}(u,w)) \times W_w^-$ as well as in $\mathcal{T}(u,v) \times \partial_1 \hat{W}_v^-$. Denote by $\mathcal{O}^{(1)}$ and $\mathcal{O}^{(2)}$ the orientations on $\mathcal{T}(u,v) \times \mathcal{T}(v,w) \times W_w^-$ induced from the orientations on $\partial_1(\mathcal{B}(u,w)) \times W_w^-$ and $\mathcal{T}(u,v) \times \partial_1 \hat{W}_v^-$, respectively. Following the procedure explained in the proof of Lemma 3.5 (iii) one sees, that $\mathcal{O}^{(2)} = \mathcal{O}_{uv} \otimes \mathcal{O}_{vw} \otimes \mathcal{O}_w^-$ whereas $\mathcal{O}^{(1)} = \mathcal{O}_{uvw} \otimes \mathcal{O}_w^-$. Hence $\mathcal{O}_{uvw} = -\mathcal{O}_{uv} \otimes \mathcal{O}_{vw}$ if and only if $\mathcal{O}^{(1)} = -\mathcal{O}^{(2)}$. To prove the latter identity, choose $\tau^{(j)} \in \mathcal{O}^{(j)}$ $(j = 1, 2)$ and let $\mathfrak{x} \in \mathcal{T}(u,v) \times \mathcal{T}(v,w) \times W_w^-$ be an arbitrary point. Then there exist

$$a \in T_{\mathfrak{x}}(\mathcal{B}(u,w) \times W_w^-) \cap \mathcal{C}(\mathfrak{x}) \subseteq T_{\mathfrak{x}}(\hat{W}_u^-)$$

and

$$b \in T_{\mathfrak{x}}(\mathcal{T}(u,v) \times \hat{W}_v^-) \cap \mathcal{C}(\mathfrak{x}) \subseteq T_{\mathfrak{x}}(\hat{W}_u^-)$$

so that both, a and b, are transversal to $T_{\mathfrak{x}}(\mathcal{T}(u,v) \times \mathcal{T}(v,w) \times W_w^-)$. Here $\mathcal{C}(\mathfrak{x})$ denotes the cone of directions tangent to $\hat{W}_u^-$ at $\mathfrak{x}$ – see Subsection 3.1. As $T_{\mathfrak{x}}(\mathcal{B}(u,w) \times W_w^-) \neq T_{\mathfrak{x}}(\mathcal{T}(u,v) \times \hat{W}_v^-)$, a and b are linearly independent. Hence, by the definition of $\mathcal{O}^{(j)}$, there exist $t_j > 0$ so that $\sigma(\mathfrak{x}) = t_1 \tau^{(1)}(\mathfrak{x}) \wedge a \wedge b$ and $\sigma(\mathfrak{x}) = t_2 \tau^{(2)}(\mathfrak{x}) \wedge b \wedge a$ where $\sigma \in \mathcal{O}_u^-$. Thus $\tau^{(1)}(\mathfrak{x}) = -\tau^{(2)}(\mathfrak{x})$. As $\mathfrak{x} \in \mathcal{T}(u,v) \times \mathcal{T}(v,w) \times W_w^-$ is arbitrary we have shown that $\tau^{(1)} = -\tau^{(2)}$. ∎

Using Proposition 5.2, Stokes' theorem as stated in Theorem 3.6 leads to a formula which we will use below. Recall that for any given $v, w \in \mathrm{Crit}(h)$ with $i(v) = q$ and $i(w) = q - 1$, $\mathcal{T}(v,w)$, if not empty, is a smooth compact manifold of dimension $i(v) - i(w) - 1 = 0$. Hence it consists of finitely many elements and the determinant bundle $\det(\mathcal{T}(v,w)) \to \mathcal{T}(v,w)$ is canonically isomorphic to the trivial line bundle $\mathcal{T}(v,w) \times \mathbb{R} \to \mathcal{T}(v,w)$. In this case an orientation of $\mathcal{T}(v,w)$ is represented by a function $\mathcal{T}(v,w) \to \{\pm 1\}$. Denote by $\varepsilon(\gamma) \in \{\pm 1\}$ the sign representing the orientation $\mathcal{O}_{vw}$ at $\gamma \in \mathcal{T}(v,w)$ as given by Proposition 5.2.

Proposition 5.3. *Assume that M is a smooth manifold and (h, X) a Morse-Smale pair. Let $v \in \mathrm{Crit}(h)$ be a critical point of index q and let ω be a smooth $(q-1)$-form on M. Then*

$$\int_{W_v^-} i_v^*(d\omega) = \sum_{\substack{w < v \\ i(w) = q-1}} \sum_{\gamma \in \mathcal{T}(v,w)} \varepsilon(\gamma) \int_{W_w^-} i_w^* \omega \tag{5.1}$$

where $i_v : W_v \to M$ and $i_w : W_w \to M$ are the natural inclusions.

Proof. As $i_v^* d\omega = d i_v^* \omega$ one has

$$\int_{W_v^-} i_v^* d\omega = \int_{W_v^-} d(i_v^*\omega) = \int_{\hat{W}_v^-} d(\hat{i}_v^*\omega)$$

where $\hat{i}_v : \hat{W}_v^- \to M$ is the smooth extension of the embedding $i_v : W_v^- \hookrightarrow M$ - see Theorem 4.4 (iii). By Theorem 4.3 (ii) and Theorem 4.4 (ii), $\partial_1 \hat{W}_v^-$ is given by the disjoint union $\sqcup_{w<v} \mathcal{T}(v,w) \times W_w^-$. Hence by Theorem 3.6 (Stokes' theorem)

$$\begin{aligned} \int_{W_v^-} i_v^* d\omega &= \int_{\partial_1 \hat{W}_v^-} i^*_{\partial_1 \hat{W}_v^-} (\hat{i}_v^* \omega) \\ &= \sum_{w<v} \int_{\mathcal{T}(v,w)\times W_w^-} i^*_{\mathcal{T}(v,w)\times W_w^-} (\hat{i}_v^*\omega). \end{aligned} \tag{5.2}$$

By Theorem 4.4 (iii) one has

$$i^*_{\mathcal{T}(v,w)\times W_w^-} \circ \hat{i}_v^* = p_{vw}^* \circ i_w^* \tag{5.3}$$

where $p_{vw} : \mathcal{T}(v,w) \times W_w^- \to W_w^-$ denotes the projection onto the second component of the product $\mathcal{T}(v,w) \times W_w^-$. Hence for any critical point $w < v$ with $\dim(W_w^-) \leq q-2$, one has $i^*_{\mathcal{T}(v,w)\times W_w^-} \circ \hat{i}_v^* \omega = 0$ as $i_w^*\omega = 0$, being a $(q-1)$-form on a manifold of dimension strictly smaller than $q-1$. As a consequence, we need only to take the sum in (5.2) over all critical points $w < v$ with $i(w) = q-1$. As noted above it then follows that $\mathcal{T}(v,w)$ is a 0-dimensional compact manifold, hence a finite set. By the definition of the orientation of $\mathcal{O}_{vw}$ on $\mathcal{T}(v,w)$ it follows that $\mathcal{O}_{vw} \otimes \mathcal{O}_w^-$ coincides with the orientation induced from $\mathcal{O}_v^-$ on W_v^-. Using (5.3) one then obtains

$$\begin{aligned} &\int_{\mathcal{T}(v,w)\times W_w^-} i^*_{\mathcal{T}(v,w)\times W_w^-} (\hat{i}_v^*\omega) \\ &= \sum_{\gamma \in \mathcal{T}(v,w)} \varepsilon(\gamma) \int_{W_w^-} i_w^* \omega \end{aligned}$$

where $\varepsilon(\gamma) \in \{\pm 1\}$ defines the orientation $\mathcal{O}_{vw}$ at $\gamma \in \mathcal{T}(v,w)$. Combining this with (5.2), the claimed formula follows. □

We remark that the manifolds W_v^- and $\mathcal{T}(v,w)$ as well as their orientations are the same for equivalent Morse-Smale pairs. In particular they do not depend on the Morse function but only on the vector field.

Coverings: Throughout this paragraph, let $\tilde{M}$ be a smooth manifold and let G be a discrete group, i.e. a group with countably many elements, endowed with the discrete topology. Assume that G acts on $\tilde{M}$ by diffeomorphisms and that this action, denoted by μ,

$$\mu : G \times \tilde{M} \to \tilde{M}, \ (g,x) \mapsto \mu(g,x) \equiv g \cdot x$$

is free and properly discontinuous. It means that for any $x, y \in \tilde{M}$ with $y \notin G \cdot x$ there exist neighborhoods U_x of x and V_y of y in $\tilde{M}$ so that $U_x \cap G \cdot V_y = \emptyset$ and $U_x \cap g \cdot U_x = \emptyset$ for any $g \neq e$ where e is the neutral element of G. It then follows that $\tilde{M}/G$ is a smooth manifold and the canonical projection $p : \tilde{M} \to \tilde{M}/G$ is a local diffeomorphism.

Definition 5.2. *$\pi : \tilde{M} \to M$ is the* principal G-covering *of a smooth manifold M, associated to μ, if there exists a diffeomorphism $\theta : \tilde{M}/G \to M$ so that $\pi = \theta \circ p$.*

Throughout the remainder of this paragraph assume that $\pi : \tilde{M} \to M$ is a principal G-covering. We note that for any $x \in M$, there are an open connected neighborhood U of x and an open connected set $\tilde{U}$ in $\tilde{M}$ so that $\pi^{-1}(U) = \bigcup_{g \in G} g \cdot \tilde{U}$ is a decomposition of $\pi^{-1}(U)$ into its (open) connected components and $\pi : g \cdot \tilde{U} \to U$ is a diffeomorphism for any $g \in G$.

Given a Morse-Smale pair (h, X) on M, let $\tilde{h} := h \circ \pi$ be the pullback of the Morse function h to $\tilde{M}$ and denote by $\tilde{X} := \pi^* X$ the pullback of the vector field X to $\tilde{M}$. Then $\tilde{h}$ is a smooth Morse function, albeit not necessarily proper, with $\pi\left(\mathrm{Crit}(\tilde{h})\right) = \mathrm{Crit}(h)$ and $i(\tilde{v}) = i\left(\pi(\tilde{v})\right)$ for any $\tilde{v} \in \mathrm{Crit}(\tilde{h})$. In addition, $\tilde{X}(\tilde{h}) = X(h) \circ \pi$. In particular, $\tilde{X}(\tilde{h})(x) < 0$ for any x in $\tilde{M} \backslash \mathrm{Crit}(\tilde{h})$. Hence $(\tilde{h}, \tilde{X})$ satisfies condition (MS1) of Definition 2.3. Denote by $\tilde{\Phi}_t(\tilde{x})$ the lift of the solution $\Phi_t(x)$ $(t \in \mathbb{R}, x \in M)$ of

$$\frac{d}{dt}\Phi_t(x) = X\left(\Phi_t(x)\right); \;\Phi_0(x) = x$$

with the property that $\tilde{\Phi}_0(\tilde{x}) = \tilde{x}$. Then $\tilde{\Phi}_t(\tilde{x})$ is defined for all $t \in \mathbb{R}$ and solves $\frac{d}{dt}\tilde{\Phi}_t(\tilde{x}) = \tilde{X}\left(\tilde{\Phi}_t(\tilde{x})\right)$. Hence we may introduce the stable and unstable manifolds, $W^+_{\tilde{v}}$ and $W^-_{\tilde{v}}$, of any critical point $\tilde{v} \in \mathrm{Crit}(\tilde{h})$. We claim that $(\tilde{h}, \tilde{X})$ satisfies the Morse-Smale condition (MS2) of Definition 2.3. To see it first note that for any $\tilde{v} \in \mathrm{Crit}(\tilde{h}), \pi\big|_{W^{\pm}_{\tilde{v}}} : W^{\pm}_{\tilde{v}} \to W^{\pm}_{\pi(\tilde{v})}$ is a diffeomorphism as paths on M originating from $\pi(\tilde{v})$ can be lifted to paths originating from $\tilde{v}$ in a unique way. Further, for any $w \in \mathrm{Crit}(h)$, π maps the disjoint union $\sqcup_{\tilde{w} \in \pi^{-1}(w)} W^-_{\tilde{v}} \times W^+_{\tilde{w}}$ bijectively onto $W^-_{\pi(\tilde{v})} \cap W^+_w$. Hence $\{\pi(W^-_{\tilde{v}} \cap W^+_{\tilde{w}}) | \tilde{w} \in \pi^{-1}(w)\}$ are disjoint components of $W^-_v \cap W^+_w$. (Note that for some $\tilde{w} \in \pi^{-1}(w), W^-_{\tilde{v}} \cap W^+_{\tilde{w}}$ might be empty.) As (h, X) is assumed to be a Morse-Smale pair, $W^-_{\pi(\tilde{v})} \cap W^+_w$, if not empty, is a smooth manifold of dimension $i(\pi(\tilde{v})) - i(w)$. Therefore it follows that for any $\tilde{w} \in \pi^{-1}(w)$ with $W^-_{\tilde{v}} \cap W^+_{\tilde{w}} \neq \emptyset, W^-_{\tilde{v}} \cap W^+_{\tilde{w}}$ is a smooth manifold of dimension $i(\pi(\tilde{v})) - i(w) = i(\tilde{v}) - i(\tilde{w})$ and we conclude that $W^-_{\tilde{v}}$ and $W^+_{\tilde{w}}$ intersect transversally. Hence $(\tilde{h}, \tilde{X})$ satisfies (MS2). Together with the considerations above we conclude that $(\tilde{h}, \tilde{X})$ is a Morse-Smale pair except for the fact that $\tilde{h}$ might not be proper. Further we conclude that $\mathcal{T}(\tilde{v}, \tilde{w}) := (W^-_{\tilde{v}} \cap W^+_{\tilde{w}})/\mathbb{R}$ is a smooth manifold of dimension $i(\tilde{v}) - i(\tilde{w}) - 1$ and

$$\Pi : \bigsqcup_{\tilde{w} \in \pi^{-1}(w)} \mathcal{T}(\tilde{v}, \tilde{w}) \to \mathcal{T}(\pi(\tilde{v}), w), \quad [\gamma] \to [\pi \circ \gamma]$$

is a diffeomorphism as well. Finally we introduce the set of (possibly broken) trajectories $\mathcal{B}(\tilde{v}, \tilde{w})$ from $\tilde{v}$ to $\tilde{w}$ where $\tilde{v}, \tilde{w} \in \mathrm{Crit}(\tilde{h})$

$$\mathcal{B}(\tilde{v}, \tilde{w}) := \bigsqcup_{\tilde{w} < \tilde{v}_\ell < \ldots < \hat{v}_1 < \hat{v}} \mathcal{T}(\tilde{v}, \tilde{v}_1) \times \ldots \times \mathcal{T}(\tilde{v}_\ell, \tilde{w})$$

and the set of (possibly broken) trajectories originating from $\tilde{v}$,

$$\hat{W}^-_{\tilde{v}} := \bigsqcup_{\tilde{w} \leq \tilde{v}} \mathcal{B}(\tilde{v}, \tilde{w}) \times W^-_{\tilde{w}}$$

where for any $\tilde{v}, \tilde{w} \in \mathrm{Crit}(\tilde{h})$ the relation $\tilde{w} < \tilde{v}$ means that $i(\tilde{w}) < i(\tilde{v})$ and $\tilde{h}(\tilde{w}) < \tilde{h}(\tilde{v})$ whereas $\tilde{w} \leq \tilde{v}$ says that $\tilde{w} < \tilde{v}$ or $\tilde{w} = \tilde{v}$ hold. For any given

$\tilde{v} \in \mathrm{Crit}(\tilde{h})$ and $w \in \mathrm{Crit}(h)$, the map Π defined above can then be extended in an obvious way to the bijective maps

$$\Pi_{\tilde{v}w} : \bigsqcup_{\tilde{w} \in \pi^{-1}(w)} \mathcal{B}(\tilde{v}, \tilde{w}) \to \mathcal{B}(v, w)$$

and

$$\Pi_{\tilde{v}} : \hat{W}^-_{\tilde{v}} \to \hat{W}^-_v$$

where as above, $v = \pi(\tilde{v})$. In this way $\mathcal{B}(\tilde{v}, \tilde{w})$ and $\hat{W}^-_{\tilde{v}}$ become compact smooth manifolds with corners and the extension $\hat{i}_{\tilde{v}} : \hat{W}^-_{\tilde{v}} \to \tilde{M}$ of the inclusion $W^-_{\tilde{v}} \hookrightarrow \tilde{M}$ is smooth.
Let us summarize our results in the following proposition.

Proposition 5.4. *Assume that $\pi : \tilde{M} \to M$ is the principal G-covering of a smooth manifold M of a discrete group G acting on $\tilde{M}$ by diffeomorphisms. Further let (h, X) be a Morse-Smale pair, $\tilde{h} := h \circ \pi$ the pullback of h by π and $\tilde{v}, \tilde{w}$ any critical points of $\tilde{h}$ with $\tilde{w} < \tilde{v}$. Then*

(i) *$\mathcal{B}(\tilde{v}, \tilde{w})$, unless empty, is compact and has a canonical structure of a smooth manifold with corners.*

(ii) *$\mathcal{B}(\tilde{v}, \tilde{w})$, unless empty, is of dimension $i(\tilde{v}) - i(\tilde{w}) - 1$ and for any $0 \leq k \leq \dim \mathcal{B}(\tilde{v}, \tilde{w})$ the k-boundary is given by*

$$\partial_k \mathcal{B}(\tilde{v}, \tilde{w}) = \bigsqcup_{\tilde{w} < \tilde{v}_k < \ldots < \tilde{v}_1 < \tilde{v}} \mathcal{T}(\tilde{v}, \tilde{v}_1) \times \ldots \times \mathcal{T}(\tilde{v}_k, \tilde{w}).$$

(iii) *For any critical points v, w with $w < v$ and any $\tilde{v} \in \pi^{-1}(v)$*

$$\Pi : \bigsqcup_{\tilde{w} \in \pi^{-1}(w)} \mathcal{T}(\tilde{v}, \tilde{w}) \to \mathcal{T}(v, w),$$

defined by associating to a solution $\tilde{\Phi}.(\tilde{x})$ of the vector field $\tilde{X}$ on $\tilde{M}$ its projection $\Phi.(\pi(\tilde{x}))$ on M, is a diffeomorphism.

(iv) *For any given $\tilde{v} \in \mathrm{Crit}(\tilde{h})$ and $w \in \mathrm{Crit}(h)$ with $w < v := \pi(\tilde{v})$, the above map Π can be extended in an obvious way to the bijective map*

$$\Pi \equiv \Pi_{\tilde{v}w} : \bigsqcup_{\tilde{w} \in \pi^{-1}(w)} \mathcal{B}(\tilde{v}, \tilde{w}) \to \mathcal{B}(v, w)$$

which is also a diffeomorphism.

(v) *The set $\hat{W}^-_{\tilde{v}} := \sqcup_{\tilde{w} \leq \tilde{v}} \mathcal{B}(\tilde{v}, \tilde{w}) \times W^-_{\tilde{w}}$ is a smooth compact manifold with corners, the natural extension of Π to $\hat{W}^-_{\tilde{v}}$, $\Pi_{\tilde{v}} : \hat{W}^-_{\tilde{v}} \to \hat{W}^-_v$, is a diffeomorphism and the extension $\hat{i}_{\tilde{v}} : \hat{W}^-_{\tilde{v}} \to \tilde{M}$ of the inclusion $W^-_{\tilde{v}} \hookrightarrow \tilde{M}$ is smooth. In particular there are at most finitely many critical points $\tilde{w}$ of $\tilde{h}$ for which there is a (possibly broken) trajectory from $\tilde{v}$ to $\tilde{w}$.*

5.2. **Geometric complex.** Let (h, X) be a Morse-Smale pair in the sense of Definition 2.3, consisting of a Morse function $h : M \to \mathbb{R}$ and a smooth vector field X on a *closed* manifold M of dimension n and let $\pi : \tilde{M} \to M$ be a G-principal covering where G is a discrete group. According to Definition 5.2, this means that there exists a smooth, free action of G on $\tilde{M}$ such that π can be identified with the projection $\tilde{M} \to \tilde{M}/G$.

In this subsection we define the geometric complex complex. Recall that a cochain complex $A^\bullet = (A^i, d^i)$

$$\cdots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \to \cdots,$$

consists of a sequence of vector spaces A^i (possibly of infinite dimension) and linear maps $d^i : A^i \to A^{i+1}$ satisfying $d^{i+1} \circ d^i = 0$. A morphism $f : A^\bullet \to B^\bullet$ between two chain complexes $A^\bullet = (A^i, d^i_A)$ and $B^\bullet = (B^i, d^i_B)$ consists of a family $f = \{f^i\}$ of linear maps $f^i : A^i \to B^i$ satisfying $d^i_B \circ f^i = f^{i+1} \circ d^i_A$

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

We denote by $\mathcal{X}_q := \mathrm{Crit}_q(h)$ the set of critical points of index q of the Morse function h. For any point $v \in \mathcal{X}_q$, one has the canonical embeddings $i^\pm_v : W^\pm_v \to M$ of the stable and unstable manifolds of v into M. By Theorem 4.4 (iii), these embeddings extend to smooth maps $\hat{i}^\pm_v : \hat{W}^\pm_v \to M$. In what follows we will often suppress the minus superscript, so e.g. we will write i_v for i^-_v and W_v instead of W^-_v.

Let $\tilde{h} := h \circ \pi$ be the lifting of the function h to $\tilde{M}$ and $\tilde{X} := \pi^* X$ the pullback of the vector field X to $\tilde{M}$. Let by $\tilde{\mathcal{X}}_q := \mathrm{Crit}_q(\tilde{h})$. Clearly $\tilde{\mathcal{X}}_q = \pi^{-1}(\mathrm{Crit}_q(h))$ and the projection π establishes a diffeomorphism between the unstable manifold $W_{\tilde{v}} \subseteq \tilde{M}$ of a critical point $\tilde{v}$ of $\tilde{h}$ and $W_v \subseteq M$ with $v = \pi(\tilde{v})$.

To construct the geometric complex associated to a given Morse-Smale pair (h, X) we introduce for any $0 \leq q \leq n$ the incidence functions $I_q : \mathcal{X}_q \times \mathcal{X}_{q-1} \to \mathbb{Z}$ and $\tilde{I}_q : \tilde{\mathcal{X}}_q \times \tilde{\mathcal{X}}_{q-1} \to \mathbb{Z}$ as follows. According to Theorem 4.3, the space $\mathcal{T}(v, w)$ of trajectories from a critical point $v \in \mathcal{X}_q$ to a critical point $w \in \mathcal{X}_{q-1}$ is in case $\mathcal{T}(v, w) \neq \emptyset$, a manifold of dimension 0 and precompact. Hence $\mathcal{T}(v, w)$ consists of at most finitely many trajectories. Assume that $\{\mathcal{O}^-_v \mid v \in \mathrm{Crit}(h)\}$ are orientations of $\{W^-_v \mid v \in \mathrm{Crit}(h)\}$ and denote by $\mathcal{O}_{vw}$ the orientation on $\mathcal{T}(v, w)$ so that the product orientation on $\mathcal{T}(v, w) \times W^-_w$ coincides with the orientation induced from $\hat{W}^-_v$ by viewing $\mathcal{T}(v, w) \times W^-_w$ as a subset of the 1-boundary $\partial_1 \hat{W}^-_v$. By Proposition 5.2, $\{\mathcal{O}_{vw}\}$ is a collection of coherent orientations, i.e. the product orientation on $\mathcal{T}(u, v) \times \mathcal{T}(v, w)$ is the opposite to the one induced from the 1-boundary $\partial_1 \mathcal{B}(u, w)$. For any $(v, w) \in \mathcal{X}_q \times \mathcal{X}_{q-1}$ with $\mathcal{T}(v, w) \neq \emptyset$ and $\gamma \in \mathcal{T}(v, w)$ let $\mathcal{O}_\gamma$ be the orientation induced on the element γ by the direction of the flow Φ_t. We then define $\varepsilon(\gamma) \in \{1, -1\}$ by

$$\mathcal{O}_\gamma = \varepsilon(\gamma) \mathcal{O}_{vw}|_\gamma.$$

The incidence functions I_q and $\tilde{I}_q$ are then given by

$$I_q(v, w) := \sum_{\gamma \in \mathcal{T}(v,w)} \varepsilon(\gamma) \tag{5.4}$$

and for any $\tilde{v} \in \pi^{-1}(v), \tilde{w} \in \pi^{-1}(w)$

$$\tilde{I}_q(\tilde{v}, \tilde{w}) := \sum_{\tilde{\gamma} \in \mathcal{T}(\tilde{v},\tilde{w})} \varepsilon(\pi \circ \tilde{\gamma}). \tag{5.5}$$

The sums in (5.4) and (5.5) count the (finite) number of unbroken trajectories between v and w, respectively $\tilde{v}$ and $\tilde{w}$, in an algebraic way. Recall that by Proposition 5.4

$$\{\pi \circ \tilde{\gamma} \mid \tilde{\gamma} \in \mathcal{T}(\tilde{v}, \tilde{w})\} \subseteq \mathcal{T}(v, w).$$

The following proposition states the basic properties of I_q and $\tilde{I}_q$.

Proposition 5.5. (i) *For any* $(\tilde{v}, \tilde{w}) \in \tilde{\mathcal{X}}_q \times \tilde{\mathcal{X}}_{q-1}$ *and* $g \in G$,

$$\tilde{I}_q(g\tilde{v}, g\tilde{w}) = \tilde{I}_q(\tilde{v}, \tilde{w}). \tag{5.6}$$

(ii) *For any* $\tilde{v} \in \tilde{\mathcal{X}}_q$*, the set of critical points* $\tilde{w} \in \tilde{\mathcal{X}}_{q-1}$ *with* $\tilde{I}_q(\tilde{v}, \tilde{w}) \neq 0$ *is finite.*

(iii) *For any* $(v, w) \in \mathcal{X}_q \times \mathcal{X}_{q-1}$ *and* $(\tilde{v}, \tilde{w}) \in \pi^{-1}(v) \times \pi^{-1}(w)$

$$I_q(v, w) = \sum_{g \in G} \tilde{I}_q(\tilde{v}, g\tilde{w}) \tag{5.7}$$

and

$$I_q(v, w) = \sum_{g \in G} \tilde{I}_q(g\tilde{v}, \tilde{w}). \tag{5.8}$$

(iv) *For any* $(u, w) \in \mathcal{X}_q \times \mathcal{X}_{q-2}$

$$\sum_{v \in \mathcal{X}_{q-1}} I_q(u, v) I_{q-1}(v, w) = 0 \tag{5.9}$$

and for any $(\tilde{u}, \tilde{w}) \in \tilde{\mathcal{X}}_q \times \tilde{\mathcal{X}}_{q-2}$

$$\sum_{\tilde{v} \in \tilde{\mathcal{X}}_{q-1}} \tilde{I}_q(\tilde{u}, \tilde{v}) \tilde{I}_{q-1}(\tilde{v}, \tilde{w}) = 0. \tag{5.10}$$

Proof. (i) Any element $g \in G$ induces a bijection between $\mathcal{T}(\tilde{v}, \tilde{w})$ and $\mathcal{T}(g\tilde{v}, g\tilde{w})$. As $\pi \circ g\tilde{\gamma} = \pi\tilde{\gamma}$ it follows from the definition (5.5) of $\tilde{I}_q$ that $\tilde{I}_q(g\tilde{v}, g\tilde{w}) = \tilde{I}_q(\tilde{v}, \tilde{w})$.

(ii) By Proposition 5.4 (v) one has

$$\sharp\left\{\tilde{w} \in \tilde{\mathcal{X}}_{q-1} \mid \mathcal{T}(\tilde{v}, \tilde{w}) \neq \emptyset\right\} < \infty.$$

(iii) By Proposition 5.4 (v) the projection π induces a bijection between the disjoint union $\sqcup_{g \in G} \mathcal{T}(\tilde{v}, g\tilde{w})$ and $\mathcal{T}(v, w)$, the identity (5.7) follows from the definitions of I_q and $\tilde{I}_q$. Formula (5.8) is easily obtained from (5.6) and (5.7).

(iv) The identity (5.9) follows from (5.10) by substituting (5.7) and (5.8) into the left hand side of (5.9). Hence it remains to prove (5.10). Let $(\tilde{u}, \tilde{w}) \in \tilde{\mathcal{X}}_q \times \tilde{\mathcal{X}}_{q-2}$. According to Proposition 5.4, $\mathcal{B}(\tilde{u}, \tilde{w})$ is a smooth compact manifold with corners of dimension 1. Hence the (finitely many) connected components of $\mathcal{B}(\tilde{u}, \tilde{w})$ consist of circles and closed intervals. Denote the family of intervals in $\mathcal{B}(\tilde{u}, \tilde{w})$ by $[\xi_j^-, \xi_j^+]$, $j \in J$. As these intervals are pairwise disjoint, the broken trajectories ξ_j^+, ξ_j^-, $j \in J$, are all different. The 1-boundary $\partial_1 \mathcal{B}(\tilde{u}, \tilde{w})$ of $\mathcal{B}(\tilde{u}, \tilde{w})$, given by the (finite) set $\Xi = \{\xi_j^+, \xi_j^- | j \in J\}$, is thus in bijective correspondance to $\bigcup_{\substack{\mathcal{T}(\tilde{u}, \tilde{v}) \neq \emptyset \\ \mathcal{T}(\tilde{v}, \tilde{w}) \neq \emptyset}} \mathcal{T}(\tilde{u}, \tilde{v}) \times \mathcal{T}(\tilde{v}, \tilde{w})$.

(Note that for $\tilde{v} \in \mathrm{Crit}(\tilde{h})$, with $\mathcal{T}(\tilde{u}, \tilde{v}) \neq \emptyset$ and $\mathcal{T}(\tilde{v}, \tilde{w}) \neq 0$, it follows that $\tilde{v} \in \mathcal{X}_{q-1}$.) Hence

$$\begin{aligned}
&\sum_{\tilde{v} \in \mathcal{X}_{q-1}} \tilde{I}_q(\tilde{u}, \tilde{v}) \tilde{I}_{q-1}(\tilde{v}, \tilde{w}) = \\
&= \sum_{\substack{\mathcal{T}(\tilde{u},\tilde{v}) \neq \emptyset \\ \mathcal{T}(\tilde{v},\tilde{w}) \neq \emptyset}} \sum_{\substack{\gamma \in \mathcal{T}(\tilde{u},\tilde{v}) \\ \delta \in \mathcal{T}(\tilde{v},\tilde{w})}} \varepsilon(\pi \circ \gamma) \cdot \varepsilon(\pi \circ \delta) \\
&= \sum_{j \in J} (\varepsilon(\pi \circ \gamma_j^+) \cdot \varepsilon(\pi \circ \delta_j^+) + \varepsilon(\pi \circ \gamma_j^-) \cdot \varepsilon(\pi \circ \delta_j^-))
\end{aligned}$$

where for any $j \in J, (\gamma_j^\pm, \delta_j^\pm) := \xi_j^\pm$. We now prove the identity (5.10) by showing that for any $j \in J$,

$$\varepsilon(\pi \circ \gamma_j^+) \cdot \varepsilon(\pi \circ \delta_j^+) + \varepsilon(\pi \circ \gamma_j^-) \cdot \varepsilon(\pi \circ \delta_j^-)) = 0. \tag{5.11}$$

To make notation lighter we suppress the subscript j in the sequel. Then $(\gamma^\pm, \delta^\pm)$ is an element $\mathcal{T}(\tilde{u}, \tilde{v}^\pm) \times \mathcal{T}(\tilde{v}^\pm, \tilde{w})$. Viewing $\mathcal{T}(\tilde{u}, \tilde{w})$ as a subset of $\mathcal{T}(\pi(\tilde{u}), \pi(\tilde{w}))$ we denote by $\mathcal{O}_{\tilde{u}\tilde{w}}$ the restriction of the orientation on $\mathcal{T}(\pi(\tilde{u}), \pi(\tilde{w}))$ to $\mathcal{T}(\tilde{u}, \tilde{w})$. It induces in a canonical way an orientation on $\mathcal{T}(\tilde{u}, \tilde{v}^\pm) \times \mathcal{T}(\tilde{v}^\pm, \tilde{w}) \subseteq \partial_1 \mathcal{B}(\tilde{u}, \tilde{w})$ which we denote by $\mathcal{O}_{\tilde{u}\tilde{v}^\pm\tilde{w}}$. As $\{\mathcal{O}_{uw}\}$ is a collection of coherent orientations (cf Proposition 5.2) one has

$$\mathcal{O}_{\tilde{u}\tilde{v}^\pm\tilde{w}} = -\mathcal{O}_{\tilde{u}\tilde{v}^\pm} \otimes \mathcal{O}_{\tilde{v}^\pm\tilde{w}}.$$

Further, by definition, we have

$$\mathcal{O}_{\tilde{u}\tilde{v}^\pm} \big|_{\gamma^\pm} = \varepsilon(\gamma^\pm) \mathcal{O}_{\gamma^\pm} \quad \text{and} \quad \mathcal{O}_{\tilde{v}^\pm\tilde{w}} \big|_{\delta^\pm} = \varepsilon(\delta^\pm) \mathcal{O}_{\delta^\pm}$$

where $\mathcal{O}_{\gamma^\pm}$ denotes the orientation at $\gamma^\pm$ given by the flow $\tilde{\Phi}$. Hence

$$\mathcal{O}_{\tilde{u}\tilde{v}^\pm\tilde{w}} \big|_{(\gamma^\pm, \delta^\pm)} = -\varepsilon(\gamma^\pm) \cdot \varepsilon(\delta^\pm) \cdot \mathcal{O}_{\gamma^\pm} \otimes \mathcal{O}_{\delta^\pm}. \tag{5.12}$$

On the other hand, $\mathcal{O}_{\tilde{u}\tilde{v}^\pm\tilde{w}} \big|_{(\gamma^\pm, \delta^\pm)}$ is determined in a canonical way by $\mathcal{O}_{\tilde{u}\tilde{w}}$,

$$\mathcal{O}_{\tilde{u}\tilde{v}^\pm\tilde{w}} \big|_{(\gamma^\pm, \delta^\pm)} = \sigma^\pm \mathcal{O}_{\gamma^\pm} \otimes \mathcal{O}_{\delta^\pm} \tag{5.13}$$

where $\sigma^\pm \in \{\pm 1\}$. As the orientation $\mathcal{O}_{\tilde{u}\tilde{v}^\pm\tilde{w}} \big|_{(\gamma^\pm, \delta^\pm)}$ is defined by using directions at $\xi^\pm$ which are pointing inwards of the interval $[\xi^-, \xi^+]$ and as $\mathcal{O}_{\gamma^\pm}$ and $\mathcal{O}_{\delta^\pm}$ are defined by the flow $\tilde{\Phi}$ it follows that $\sigma^+ + \sigma^- = 1$. Thus, by combining (5.12) and (5.13) one obtains

$$\varepsilon(\gamma^+) \cdot \varepsilon(\delta^+) + \varepsilon(\gamma^-) \cdot \varepsilon(\delta^-) = 0$$

and identity (5.10) is established. □

Let E be a finite dimensional k-vector space, with k denoting either the field of real or complex numbers, and let $\rho : G \to GL(E)$ be a representation of the group G. For any $0 \leq q \leq n$ denote by $\tilde{\mathcal{C}}_E^q$ the k-vector space of maps from $\tilde{\mathcal{X}}_q$ to E. The group G acts on $\tilde{\mathcal{C}}_E^q$

$$\rho_* : G \times \tilde{\mathcal{C}}_E^q \to \tilde{\mathcal{C}}_E^q, \ (g, f) \mapsto g \cdot f$$

where for any $\tilde{v} \in \tilde{\mathcal{X}}_q$

$$(g \cdot f)(\tilde{v}) := \rho(g) f(g^{-1} \cdot \tilde{v}). \tag{5.14}$$

We denote by $\mathcal{C}_\rho^q$ the subspace of $\tilde{\mathcal{C}}_E^q$ consisting of all ρ_*-invariant functions, i.e. $g \cdot f = f$ for any $g \in G$. As $\mathcal{X}_q$ is finite, $\mathcal{C}_\rho^q$ is finite dimensional. In the case where

E is the 1-dimensional vector space k and ρ is the trivial representation, we write simply $\tilde{\mathcal{C}}^q$ and $\mathcal{C}^q$ instead of $\tilde{\mathcal{C}}^q_E$ and $\mathcal{C}^q_\rho$. Clearly, $\mathcal{C}^q$ can be interpreted as the (finite dimensional) vector space of all functions $f : \mathcal{X}_q \to k$.
Furthermore, introduce the linear map $\tilde{\delta}^q$. $\tilde{\mathcal{C}}^q_E \to \tilde{\mathcal{C}}^{q+1}_E$ defined for $f \in \tilde{\mathcal{C}}^q_E,\ \tilde{v} \in \tilde{\mathcal{X}}_{q+1}$ by

$$\tilde{\delta}^q(f)(\tilde{v}) := \sum_{\tilde{w} \in \tilde{\mathcal{X}}_q} \tilde{I}_{q+1}(\tilde{v}, \tilde{w}) f(\tilde{w}). \tag{5.15}$$

In a straightforward way it follows from (5.7) - (5.8) that the maps $\tilde{\delta}^q$ commute with the action ρ_* of the group G. Hence they induce linear maps $\delta^q : \mathcal{C}^q_\rho \to \mathcal{C}^{q+1}_\rho$ between these vector spaces of G-invariant functions. By formula (5.10) of Proposition 5.5,

$$\tilde{\delta}^{q+1} \circ \tilde{\delta}^q = 0. \tag{5.16}$$

We summarize the results obtained so far in this subsection in the following proposition.

Proposition 5.6. *Assume that M is a closed manifold, $\pi : \tilde{M} \to M$ a G-principal covering where G is a discrete group, (h, X) a Morse-Smale pair and $\{\mathcal{O}^-_v | v \in \mathrm{Crit}(h)\}$ a collection of orientations of the unstable manifolds $\{W^-_v | v \in \mathrm{Crit}(h)\}$ of the vector field X. Further assume that E is a finite dimensional k-vector space ($k = \mathbb{R}$ or $\mathbb{C}$) and $\rho : G \to GL(E)$ a representation of G. Then $\tilde{\mathcal{C}}^\bullet = (\tilde{\mathcal{C}}^q_E, \tilde{\delta})$ is a cochain complex of G-representations and $\mathcal{C}^\bullet_\rho = (\mathcal{C}^q_\rho, \sigma)$ is a finite dimensional subcomplex.*

We refer to $\mathcal{C}^\bullet_\rho \equiv \mathcal{C}^\bullet_\rho((h, X), \mathcal{O})$ as the geometric complex associated to the data $(h, X), \mathcal{O} = \{\mathcal{O}^-_v\}, \rho : G \to GL(E)$.
De Rham: Let M be a smooth, but not necessarily closed manifold. We denote by $(\Omega^\bullet(M), d)$ the de Rham complex. Here $\Omega^q(M)$ is the space of smooth q-forms on M, and

$$d \equiv d^q : \Omega^q(M) \to \Omega^{q+1}(M)$$

is the exterior differential.
More generally, assume that $\pi : \tilde{M} \to M$ is a G-principal covering of a smooth, closed manifold M and $\rho : G \to GL(E)$ a representation of the group G on a finite dimensional k-vector space E. Let $\Omega^\bullet(\tilde{M}; E) := \Omega^\bullet(\tilde{M}) \otimes E$ denote the space of differential forms with values in E. Then the de Rham differential $\tilde{d}$ on $\Omega^\bullet(\tilde{M})$ can be extended to the differential $\tilde{d}_E$, mapping $\Omega^\bullet(\tilde{M}; E)$ to $\Omega^{\bullet+1}(\tilde{M}; E)$,

$$\tilde{d}_E = \tilde{d} \otimes Id_E.$$

To make notation lighter we will often suppress the subscript E. The action ρ_* of G on functions in $\tilde{\mathcal{C}}^\bullet_E$ defined in (5.14) extends to an action on forms with values in E and is again denoted by ρ_*. In particular for any $g \in G, e \in E$, and $\omega \in \Omega^q(\tilde{M})$,

$$g \cdot (\omega \otimes e) := \big((g^{-1})^* \cdot \omega\big) \otimes \rho(g)e$$

where for any $g \in G, g^* : \Omega^q(\tilde{M}) \to \Omega^q(\tilde{M})$ is the map induced by the map $g : \tilde{M} \to \tilde{M}, \tilde{x} \mapsto g\tilde{x}$, i.e. for any $\tilde{x} \in \tilde{M}, \omega \in \Omega^q(\tilde{M})$,

$$g^*\omega(\tilde{x})(\xi_1, \cdots, \xi_q) = \omega(g\tilde{x})(d_{\tilde{x}}g\xi_1, \cdots, d_{\tilde{x}}g\xi_q)$$

for any $\xi_1, \cdots, \xi_q \in T_{\tilde{x}}\tilde{M}$. This action commutes with the de Rham differential $\tilde{d}_E$.

Denote by $\Omega^\bullet(M,\rho)$ the subspace of $\Omega^\bullet(\tilde{M};E)$ consisting of G-invariant differential forms on $\tilde{M}$ with values in E. Let d_ρ be the restriction of the de Rham differential $\tilde{d}_E$ to $\Omega^\bullet(M,\rho)$. In this way we get the de Rham complex $(\Omega^\bullet(M,\rho),d_\rho)$ with coefficients in ρ.

5.3. **Integration map.** Let M be a smooth manifold of dimension n, W a compact oriented smooth manifold with corners of dimension $q \leq n$ and $i : W \to M$ a smooth map. Then one can define the integration map $Int \equiv Int_W : \Omega^q(M) \to k$ given by

$$Int(\omega) := \int_W i^*\omega.$$

This is applied to the following situation. Suppose (h,X) is a Morse-Smale pair on a closed manifold M, $\pi : \tilde{M} \to M$ a G-principal covering and $\rho : G \to GL(E)$ a representation of G. For any critical point $v \in \mathrm{Crit}(h)$, choose an orientation $\mathcal{O}_v$ of its unstable manifold W_v^-. As $\pi : W_{\tilde{v}}^- \to W_{\pi(\tilde{v})}^-$ is a diffeomorphism for any $\tilde{v} \in \mathrm{Crit}(\tilde{h})$, $\mathcal{O}_v$ lifts to an orientation $\mathcal{O}_{\tilde{v}}$ of the unstable manifold $W_{\tilde{v}}^-$ of any critical point $\tilde{v}$ of $\tilde{h}$ with $\pi(\tilde{v}) = v$. For any $0 \leq q \leq n$ we then define the map

$$\widetilde{Int} \equiv \widetilde{Int}^q : \Omega^q(\tilde{M};E) \to \tilde{\mathcal{C}}_E^q$$

as follows: for any $\tilde{\omega} \in \Omega^q(\tilde{M};E)$, the value of $\widetilde{Int}^q(\tilde{\omega})$ at a point $\tilde{v}$ in $\mathcal{X}_q := \{\tilde{v} \in \mathrm{Crit}(\tilde{h}) : i(\tilde{v}) = q\}$ is given by

$$\widetilde{Int}^q(\tilde{\omega})(\tilde{v}) = \int_{W_{\tilde{v}}^-} i_{\tilde{v}}^*\tilde{\omega} = \int_{\hat{W}_{\tilde{v}}^-} \hat{i}_{\tilde{v}}^*\tilde{\omega} \in E.$$

As $\hat{W}_{\tilde{v}}^-$ is a compact manifold with corners, both integrals are well defined.
By Proposition 5.3 (version of Stokes' theorem) one obtains the following identities.

Proposition 5.7. *For any* $0 \leq q \leq n$

$$\tilde{\delta}^q \circ \widetilde{Int}^q = \widetilde{Int}^{q+1} \circ \tilde{d}_E^q.$$

As a consequence, $\widetilde{\mathrm{Int}} : (\tilde{\Omega}^\bullet(\tilde{M},E),\tilde{d}_E) \to (\tilde{\mathcal{C}}_E^\bullet,\tilde{\delta})$ *is a morphism of cochain complexes. Since* $\widetilde{\mathrm{Int}}$ *commutes with the action of* G *its restriction to* $(\Omega^\bullet(M,\rho),d_\rho^\bullet)$, *denoted by* Int, *is also a morphism of cochain complexes,*

$$\mathrm{Int} : (\Omega^\bullet(M,\rho),d_\delta) \to (\mathcal{C}_\rho^\bullet,\delta).$$

Remark 5.1. It can be shown that both morphisms, $\widetilde{Int}$ and Int, induce an isomorphism in cohomology.

Proof. Let $\omega \in \Omega^q(\tilde{M};E)$ where $0 \leq q \leq n$. By the definition of $\widetilde{Int}^{q+1}$ one has, for any $\tilde{v} \in \tilde{\mathcal{X}}_{q+1}$,

$$\begin{aligned}\widetilde{Int}^{q+1}(\tilde{d}_E\omega)(\tilde{v}) &= \int_{W_{\tilde{v}}^-} i_{\tilde{v}}^*(\tilde{d}_E\omega) \\ &= \sum_{\substack{\tilde{w}\in\tilde{\mathcal{X}}_q \\ \tilde{w}<\tilde{v}}} \sum_{\gamma\in\mathcal{T}(\tilde{v},\tilde{w})} \varepsilon(\pi(\gamma)) \int_{W_{\tilde{w}}^-} i_{\tilde{w}}^*\omega\end{aligned}$$

where for the latter identity we applied Proposition 5.3. (Recall that $(\tilde{h},\tilde{X})$ is a Morse-Smale pair except for the fact that $\tilde{h}$ might not be proper. However $\hat{W}_{\tilde{v}}^-$ is compact and all arguments in the proof of Proposition 5.4 remain valid.) By the definition (5.5) of $\tilde{I}_{q+1}(\tilde{v},\tilde{w})$ one gets

$$\begin{aligned}\widetilde{Int}^{q+1}(\tilde{d}_E\omega)(\tilde{v}) &= \sum_{\substack{\tilde{w}\in\tilde{\mathfrak{X}}_q\\ \tilde{w}<\tilde{v}}} \tilde{I}_{q+1}(\tilde{v},\tilde{w})\widetilde{Int}^{q}(\omega)(\tilde{w})\\ &= \tilde{\delta}^q\left(\widetilde{Int}^{q}(\omega)\right)(\tilde{v})\end{aligned}$$

where for the latter identity we used the definition (5.15) of $\tilde{\delta}^q(f)(\tilde{v})$. This establishes the claimed identity. □

6. Epilogue

In his seminal paper [35], Witten proposed an analytic approach to Morse theory, inspired by quantum mechanics. Given a Morse function $h(x)$ on a closed Riemannian manifold, he introduced the deformed de Rham differential

$$d(t) = e^{-th}de^{th} = d + tdh \wedge .$$

As $d(t)2 = 0$, the space of forms on M together with this differential defines again a complex, referred to as the deformed de Rham complex. The deformed differential gives rise to deformed Laplacians

$$\Delta_q(t) = d_q^*(t)d_q(t) + d_{q-1}(t)d_{q-1}^*(t),$$

acting on q-forms on M; here $d_q(t)$ is the restriction of $d(t)$ to the space of q-forms. It turns out that for t sufficiently large, the spectrum of $\Delta_q(t)$ splits into two parts, one of which lies exponentially close to 0 and consists of finitely many eigenvalues, whereas the other one consists of infinitely many eigenvalues and is contained in the half line $[Ct,\infty)$ for some constant $C > 0$. For such a t, let $\Lambda^q_{\mathrm{sm}}(t)$ be the space of q-forms, spanned by the eigenforms of $\Delta_q(t)$ corresponding to exponentially small eigenvalues. Witten showed that the dimension of $\Lambda^q_{\mathrm{sm}}(t)$ equals the number of critical points of $h(x)$ of index q. As $d_q(t)$ maps $\Lambda^q_{\mathrm{sm}}(t)$ into $\Lambda^{q+1}_{\mathrm{sm}}(t)$, it follows that $\Lambda^\bullet_{\mathrm{sm}}(t)$ is a subcomplex of the deformed de Rham complex, sometimes referred to as the *small complex*. Suppose now that the gradient vector field $X = -\mathrm{grad}h$ satisfies the Morse-Smale condition. As explained in Section 5, the cell decomposition provided by the unstable manifolds, W_v^-, $v \in \mathrm{Crit}(h)$, leads to a complex of finite dimensional vector spaces. The grading of the complex is provided by the index of the critical points and the chain maps are defined in terms of the trajectories (instantons) between critical points whose indices differ by 1 and a coherent orientation on spaces of trajectories between two critical points of h. The corresponding cochain complex is called the *geometric complex*. Actually, according to [19], or more recently [37], it can be shown to be a CW complex. Witten conjectured that this complex is isomorphic to the small complex. His conjecture was first proved by Helffer and Sjöstrand [15]. Using methods of semiclassical analysis, they analysed in detail the restriction of the deformed de Rham differential to the small complex. Later on, Bismut and Zhang [2] discovered that the integration map provides an isomorphism of complexes between the small complex and the geometric complex. In this way, they could simplify the arguments of Helffer and Sjöstrand and provide a new proof of de Rham's theorem which says that the integration map induces an isomorphism between cohomologies. The present paper provides important elements of the topological part of the so called Witten-Helffer-Sjöstrand theory, which will be treated in our book [7] in preparation, together with some of the applications of this theory in topology and geometric analysis.

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Department of Mathematics, The Ohio State University, USA
E-mail address: burghele@mps.ohio-state.edu

Department of Mathematics, University of Arizona, USA.
E-mail address: friedlan@math.arizona.edu

Department of Mathematics, University of Zurich, Switzerland.
E-mail address: tk@math.unizh.ch