CALCULUS ON SMOOTH MANIFOLDS (what you should know- a summary)

- Differential structure
  - chart
  - atlas
  - maximal atlas = equivalent atlases
  - partition of unity
- Orientability, Orientation
- Differential forms , exterior differential,
  - $\mathcal{C}(M)$ the algebra of $\kappa-$valued smooth functions , $\kappa = \mathbb{R}$or $\mathbb{C}$
    $\Omega^0(M) = \mathbb{C}(M)$,
    $\Omega^r(M) = \Omega^1(M) \wedge_{\Omega^0(M)} \Omega^1(M) \wedge_{\Omega^0(M)} \cdots \wedge_{\Omega^0(M)} \Omega^1(M)$ $r-$times
  - $\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3$
  - $\omega' \wedge \omega'' = (-1)^{\deg \omega' \cdot \deg \omega"} \omega" \wedge \omega'$
  - $d_r : \Omega^r(M) \to \Omega^{r+1}(M)$ first order differential opereator with $d(\omega \wedge \omega') = d(\omega) \wedge \omega' + (-1)^{deg\omega} \omega \wedge \omega'$
  - $d_{r+1} \cdot d_r = 0$
  - deRham complex: $\cdots \longrightarrow \Omega^{r-1}(M) \xrightarrow{d_{r-1}} \Omega^r(M) \xrightarrow{d_r} \Omega^{r+1}(M) \xrightarrow{d_{r+1}} \cdots$ with deRham cohomology $H^r_{DM} := \ker d_r / imgd_{r-1}$.
- Vector fields, contraction of forms along a vector field X, Lie derivative along the vector field X ,
  - $\mathcal{X}(M) := \{X : \mathcal{C}(M) \to (M) \mid X(f \cdot g) = X(f) \cdot g + f \cdot X(g)\}$
  - $\iota_X : \Omega^r(M) \to \Omega^{r-1}$, $\mathcal{C}(M)-$linear, with $(\iota_X \omega)(X_1, X_2, \cdots, X_{r-1}) = \omega(X, X_1, X_2, \cdots, X_{r-1})$
  - $L_X : \Omega^r(M) \to \Omega^r(M)$, $L_X = d_{r-1} \cdot \iota_X - \iota_X \cdot d_r$
  - $[X, Y] := X \cdot Y - Y \cdot X$, the braket.
- Integration of top degree forms on an oriented manifold

Important Review section on integration of the attached material

- description in coordinates $X = \sum \alpha_i(x_1, \cdots x_n) \partial / \partial x_i$
  $\omega = \sum 1_1 < i_2 <, \cdots i_r \omega_{i_1, i_2, \cdots i_r}(x_1, x_2, \cdots x_n) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_r}$ an $r-$ differential form,
  $d\omega := \sum (-1)^i \partial / \partial x_i (\omega_{i_1, i_2, \cdots i_r}(x_1, x_2, \cdots x_n)) dx_i \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_r}$
  ...
- coordinate free description. [1]

[1]

- $\Omega^1 := I/I^2,\ withI = \{F \in \mathcal{C}(M \times M) \mid F(,x,x) = 0\}$, I is an ideal in the algebra $\mathcal{C}(M \times M)$
- The $\mathcal{C}(M) = \Omega^0(M)-$ module structure on $\Omega^1(M)$ is given by $(f, F) \to G$ with $G(x,y) = f(x) \cdot F(x,y)$, equivalently $G(x,y) = F(x,y) \cdot f(y)$ since $F(x,y) \cdot (f(x) - f(y)) \in I^2$.
- $d_0 : \Omega^0 \to \Omega^1(M)$ is given by $f \to F$ with $F(x,y) = f(x) - f(y)$ and $d_1 : \Omega^1 \to \Omega^2(M)$ is induced by $d_1(f \cdot dg) \to d_0(f) \wedge d_0(g)$
- the remaining d_r are derived inductively using the formula for $d(\omega \wedge \omega')$

- for M compact with respect to the countable collection of norms $||\cdot||_k$, $k = 1, 2, 3, \cdots$, $\Omega^r(M)$ is a Frechet space with d_r continuous operator.