LINEAR ALGEBRA
(summary)

Let $A : V \to V$ be a linear endomorphism of complex vector spaces (=vector space over the field $\mathbb{C}$) of dimension n. With respect to a base $\{e_1, e_2 \cdots e_n\}$ A can be viewed as an $n \times n$ matrix $A = ||a_{i,k}||, i, k = 1, 2, \cdots, n$ with complex coefficients $a_{i,k} \in \mathbb{C}$.

- $\mathrm{Tr}A$ and $\det A$ are the two basic invariants (up to conjugation).
- The set $SpecA := \{\lambda \in \mathbb{C} \mid \det(A - \lambda I) = 0\}$consists of the eigenvalues of A and $\sharp SpecA = n$, when counting the eigenvalues with their multiplicity.
  The subspace $H_\lambda := \{v \in V \mid A(v) = \lambda v\} = \ker(A - \lambda I)$ defines the eigenspace of λ.
  $P^A(\lambda) := \det(A - \lambda I)$, denotes the characteristic polynomial of degree A which is of degree n.
- Note that $TrA = \sum \lambda_i$, $\det A = \prod \lambda_i$ and denote $\det' A = \prod_{\lambda_i \neq 0} \lambda_i$.
- Γ denotes a simple closed curve in the complex plane, counter-clockwise oriented, containing $SpecA$ and Γ_λ a simple closed curve containing only one eigenvalue $\lambda \in SpecA$.

For $f : U \to \mathbb{C}$ a holomorphic function on a 1-connected domain U, a domain containing the set $SpecA$ and the curve Γ resp. Γ_λ define

$$f(A) := 1/2\pi i \int_\Gamma f(z)(z - A)^{-1} dz.$$

in particullar one has

$$Id := 1/2\pi i \int_\Gamma (z - A)^{-1} dz$$

$$A := 1/2\pi i \int_\Gamma z(z - A)^{-1} dz$$

and

$$E^A_\lambda := 1/2\pi i \int_{\Gamma_\lambda} (z - A)^{-1} dz$$

the projection on H_λ.

Note that for each $z \in U \setminus SpecA$, the linear map $(z - A)$ is invertible, hence the formulae above make sense.

Based on Cauchy theorem the results are independent on the Γ resp. Γ_λ.

For a complex vector space V a Hermitian scalar product is a complex valued function on two variables $(,) : V \times V \to \mathbb{C}$, linear in the first variable, conjugate linear in the second which satisfies the following: $(x, y) = \overline{(y, x)}$ $(x, x) = ||x|| \geq 0 and ||x|| = 0$ implies $x = 0$.

Let $A : V_1 \to V_2$ be a linear map between two complex vector spaces of dimension n_1 resp. n_2, equipped with Hermitian scalar products $(,)_1$ and $(,)_2$. Define

$$||A|| := \sup_{||u||_1 = 1} ||A(u)||_2.$$

Denote by $A^* : V_2 \to V_1$ the unique linear map defined by

$$(A(x), y)_2 = (x, A^* y)_1,\ x \in V_1,\ y \in V_2.$$

If $e^1_1, e^1_2, \cdots e^1_{n_1}$ and $e^2_1, e^2_2, \cdots e^2_{n_2}$ are orthonormal bases in V_1 and V_2 and A is represented by the matrix with components $a_{i,j}, 1 \leq i \leq n_2, 1 \leq j \leq n_1$, then A^* is represented by the matrix with entries $b_{i,j}, 1 \leq i \leq n_1, 1 \leq j \leq n_2$ with $b_{i,j} = \overline{a}_{j,i}$.

Suppose $V_1 = V_2 = V$.

1. The linear map $A : V \to V$ is **selfadjoint = hermitian symmetric** if $A = A^*$ i.e. $a_{i,j} = \overline{a}_{j,i}$.
   Note that $A^* \cdot A,\ A \cdot A^*,\ A + A^*$ are all selfadjoint with the first two nonnegative definite ([1].
2. The linear map $U : V \to V$ is unitary or bijective isometry iff $U^* \cdot U = Id$.

One can prove:

Proposition 0.1
Let $A : V \to V$ be a selfadjoint map.

1. *$A : V \to V$ has all eigenvalues real and eigenspaces mutually orthogonal.*
2. *There exists unitary linear maps U s.t. $A = U \cdot D \cdot U^*$ with $D = \begin{bmatrix} \lambda_1, & 0, & \cdots & ,0 \\ 0, & \lambda_2, & \cdots & ,0 \\ \cdots \\ 0, & 0, & \cdots, & \lambda_n \end{bmatrix}$ with λ_i real numbers; in particular if the linear transformation A is selfadjoint and positive definite then A has a square root given by $U \cdot D' \cdot U^*$ with $D' = \begin{bmatrix} \lambda_1^{1/2}, & 0, & \cdots & ,0 \\ 0, & \lambda_2^{1/2}, & \cdots & ,0 \\ \cdots \\ 0, & 0, & \cdots, & \lambda_n^{1/2} \end{bmatrix}$.*
3. *(**"Gap in the spectrum" lemma**) If $A : V \to V$ is selfadjoint, V_1, V_2 two subspaces of V with $V = V_1 \oplus V_2$ which satisfy $||A(x)|| \leq a||x||$ for $x \in V_1$ and $||A(y) \geq b||y||$ for $y \in V_2$ with $a < b$, then $Spec A \cap (a, b) = \emptyset$.*

Proof: Item 1. $A(x) = \lambda x$ implies $\lambda||x||^2 = (A(x), x) = (x, Ax) = (x, \lambda x) = \overline{\lambda}||x||^2$, hence $\lambda = \overline{\lambda}$, hence $\lambda \in \mathbb{R}$. If $x \in H_\lambda, y \in H_\mu$ one has $A(x) = \lambda x$ and $A(y) = \mu y$ and then $\lambda(x, y) = (Ax, y) = (x, Ay) = \mu(x, y)$. In view of selfadjointess of A, which implies $\overline{\mu} = \mu$, one has $(\lambda - \mu)(x, y) = 0$. If $\lambda \neq \mu$ then $(x, y) = 0$. q.e.d.

Item 2 follows by induction on dimension of V once one observes that if A is selfadjoint with $v \in V$ eigenvector corresponding to the eigenvalue λ then A leaves invariant the orthogonal complement of v.

Item 3. Suppose $Spect A \ni \lambda \in (a, b)$ hence theexista $v = v_1 + v_2$ with $A(v) = \lambda v$. We want to show this is impossible. Indeed $A(v) = \lambda v_1 + \lambda v_2$. which implies

$$(A(v_1), v_1) + (A(v_2), v_1) = \lambda||v_1||^2 + \lambda(v_2, v_1) \tag{1}$$

and

$$(A(v_1), v_2) + A(v_2), v_2) = \lambda||v_2||^2 + \lambda(v_1, v_2). \tag{2}$$

In view of selfadjointness (1) is equivalent to

$$(A(v_1), v_1) - \lambda||v_1||^2 = -\overline{(A(v_1), v_2)} + \lambda\overline{(v_1, v_2)} \tag{3}$$

[1] A selfadjoint is called nonnegative definite if all eigenvalues of A are nonnegative real numbers

which because the left side is real so is rhe right side, hence

$$(A(v_1), v_1) - \lambda ||v_1||^2 = -(A(v_1), v_2) + \lambda(v_1, v_2) \tag{4}$$

Equality (2) is equivalent to

$$(A(v_2), v_2) - \lambda ||v_2||^2 = -(A(v_1), v_2) + \lambda(v_1, v_2). \tag{5}$$

One concludes that $(A(v_1), v_1) - \lambda ||v_1||^2 = (A(v_2), v_2) - \lambda ||v_2||^2$, clearly impossible, since the left side is strictly negative and the right side is strictly positive by hypothesis. ■

For A selfadjoint and $f : R \to R$ one also consider the self adjoint linear map $f(A) := UD'U^*$ with the diagonal matrix D' having the entries $f(\lambda_1), f(\lambda_2), \cdots, f(\lambda_n)$ when $A = UDU^*$, is as in Proposition 0.1 item 2. Note that when f is a restriction of holomorphic map and A is selfadjoint then this definition is consistent with the definitiion of $f(A)$ from the begining of these remarks.

A very Important result

Theorem 0.2 *(Relliich- Kato)*
Suppose $A(z) = A_0 + \sum_{i\geq 1} A_i z^i$ is a convergent power series whose coefficients A_i are selfadjoint matrices with complex coefficients (i.e. the matrix $A(z)$ has entries $a_{i,j}(z) = \sum_{k\geq 0} a^k_{i,j} z^k$ convergent power series with the properties that $\overline{a^k_{j,i}} = a^k_{i,j}$ [2]. Then there exists the convergent power series $\lambda_1(z), \cdots, \lambda_n(z),\ \lambda_i(z) \in \mathbb{C}$ and $v_1(z), \cdots, v_n(z),\ v_i(z) \in V$ such that $A(z)(v_i(z)) = \lambda_i(z) v_i(z)$ and $\lambda_i(t) \in \mathbb{R}$ and $v_i(t) = 1$ for $z = t, t \in \mathbb{R}$. [3]

In particular if $A(z) = A_0 + A_1 z + \cdots A_n z^n$ with A_i selfadjoint matrices, there exists the set of holomorphic maps $\lambda_1(z), \cdots \lambda_n(z),\ \lambda_i(z) \in \mathbb{C}$ and $v_1(z), \cdots v_n(z),\ v_i(z) \in V$ holomorphic in some small small neighborhood U of $\mathbb{R} \subset U \subset C$ with $A(z)(v_i(z)) = \lambda_i(z) v_i(z), z \in U$ and with $\lambda_i(t) \in \mathbb{R}$ $||v_i(t)|| = 1$ for $z = t \in \mathbb{R}$, exhausting the eigenvalues of $A(z)$ for $z \in U$.

Proof:

Sketch of proof (following Rellich)

The proof will be accomplished in four steps:

Step 1 : One writes the solution of the characteristic polynomial (equated to zero) as a Puiseux series, $\lambda(z) = \lambda + \sum_{n_0 \leq k} b_k z^{k/h},\ h$ a positive integer,

Step 2 : One establishes that $n_0 \geq 1$,

Step 3 : Using the fact that $A(z)$ is selfadjoint for $z \in \mathbb{R}$ real on concludes inductively that $b_k = 0$ unless h divides k which makes $\lambda(z)$ a convergent power series.

Step 4: One shows the following: Let $\gamma_{i,k}(z), i = 1, \cdots n$ are convergent power series in the neighborhood of $z = 0$ and suppose that $\det(\gamma_{i,k}(z)) = 0$. Then there exists power series $c_1(z), \cdots c_n(z)$, convergent in the neighborhood of $z = 0$ such that $\sum_{k=1}^n \gamma_{i,k}(z) c_k(z) = 0,\ i = 1, \cdots, n$ and for real z, $\sum_{k=1}^n |c_k(z)|^2 = 1$.

The result can be easily derived from these steps.

■

The above steps follow the arguments in Rellich' lecture notes on "Perturbation Theory of Eigenvalue Problems, F Rellich, New York University, Institut of Mathematical Sciences. Pages 36-45. These pages

[2] $A(z)$ is selfadjoint only for $z \in \mathbb{R}$

[3] Without the selfadjoint hypothesis the statement fails. Consider $A(z) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z$. One can not write $\lambda_1(z)$ and $\lambda_2(z)$ as two power series in z.

attached in the course directory as " Relleichbook.doc. The entire text of this book can be found on INTERNET. An Alternative proof can recovered from Kato 's book, T.Kato, Perturbation theory for Linear operators Springer =Verlag, second edition

Puiseux series

A Puiseux series is a formal Laurent power series in the variable $T^{1/n}$ denoted by T_n with coefficients in the field K of the form $f = \sum_{k_0 \leq k < \infty} c_k T^{k/n}$ for n fixed. The set of these powe series is denoted by $K((T_n))$. The union $\bigcup_n K((T_n))$ is refered to as the set of all Puiseux power series. If we denote by $T_n = T^{1/n}$ and for any $m|n$ one considers $T_m \to (T_n)^{n/m}$ which induces an injective map from $K((T_m)) \to K((T_n))$ The direct limit is equal to $\bigcup_{n>0} K((T_n))$ which is in an obvious way a field. When K is the field of complex numbers the element in $f \in K((T_n))$ is said to have the radius of convergence $\epsilon > 0$ if for any $z \in \mathbb{C}$ with $|z^n| < \epsilon$ the series obtained by replacing T_n by z is convergent. complex number with z^n in absolute value smaller than ϵ provide, when replacing by replacing T_n by z, a convergent series. The series is convergent if it has a positive radius of convergence.

It is a result which can be very well attributed to Newton but certainly present in the work to Puiseux, that both the field of all Puiseux as well as the subfield of convergent Puiseux series with complex coefficients is algebraically closed. Hence any equation $z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$ with a_i Puiseux series resp. convergent Puiseux series has exactly n (possibly with multiplicity) solutions each Puiseux resp. convergent Puiseux series. An power series in z is recognized among the Puiseux series with complex coefficients as having all c_k with k not divisilble by n equal to 0.

Wikipedia may be a good reference or most of the books in algebraic geometry like
Shafarevici, Igor Rostislavovici (1994) Basic Algebraic Geometry Springer Verlag or
Walker, R, J (1978) Algebraic curves Springer Verlag)

Spectral density function

For $A : V_1 \to V_2$ linear map between two vector spaces equipped norms $|| \, ||_1$ and $|| \, ||_2$ define the spectral density function by the formula

$$F^A(\lambda) := \sup\{\dim L \mid ||f_L|| \leq \lambda\}.$$

This is a step function continuous to the left. If the norms come from a scalar product one has

$$F^A(\lambda) = F^{(A^* \cdot A)^{1/2}}(\lambda)$$

and the jump values are exactly the eigenvalues of $(A^* \cdot A)^{1/2}$.