ALGEBRAIC TOPOLOGY- CELL COMPLEXES- COHOMOLOGY

(summary of lecture 1)

- Cell complex (Combinatorics)
- CW complexes
- Cochain complex associated to a CW complex and cohomology
- Relevant numbers
- Hodge decomposition in a cochain complex of f.d. hermitian vector spaces.

From a combinatirial point of view a **cell complex** consists of :

- 1. a collection of finite sets $\mathcal{X}_0, \mathcal{X}_1, \cdots \mathcal{X}_N \cdots$, of cardinality $n_0, n_1, \cdots n_N, \cdots$ referred to as $0, 1, 2, \cdots, N, \cdots -$cells and

- 2. a collection of matrices with entries integer numbers I_r, i.e. maps $I_r : \mathcal{X}_{r+1} \times \mathcal{X}_r \to \mathbb{Z}$, referred to as the **incidency matrix between** $r-$**cels and** $(r-1)-$ **cells** , which satisfy

$$I_{r-1} \cdot I_r = 0.$$

To these data and to any ring or field, for example $\kappa = \mathbb{Z}, \mathbb{R}, \mathbb{C}$, one associates the sequence $\mathcal{C}$ of $\kappa-$ modules C^r and $\kappa-$linear maps $d_r : C^r \to C^{r+1}$

$$\mathcal{C} : \quad \cdots \xrightarrow{d_{r-1}} C^r \xrightarrow{d_r} C^{r+1} \xrightarrow{d_{r+1}} C^{r+2} \xrightarrow{d_{r+2}} \cdots$$

with

$$C^r := Maps(\mathcal{X}_r, \kappa)$$

and

$$d_r(f)(y) := \sum_{x \in \mathcal{X}_r} I_{r+1}(y, x) f(x),$$

for $y \in \mathcal{X}_{r+1}, x \in \mathcal{X}_r$. Clearly $I_r \cdot I_{r+1}$ implies $d_{r+1} \cdot d_r = 0$.

Define

$$H^r(\mathcal{C}) := \ker d_r / img d_{r-1}$$

the cohomology of the cochain complex $\mathcal{C}$.

If $\kappa = \mathbb{Z}$ the finiteness of $\mathcal{X}_r'$s make $H^r(\mathcal{C})$ f.g. abelian groups with $Tor_r := \{g \in H_r \mid g \text{ of finite order}\}$ a finite group. Denote by

$$\beta_r := \text{rank} H^r$$

and by

$$t_r := \sharp T_r$$

and then , when $\mathcal{X}_r = \emptyset$ for N large enough the **Euler Poincaré** characteristic $\chi := \sum_{0 \leq r} (-1)^r b_r$ and the **Torsion** caracteristic $\tau := \prod_{0 \leq r} t_r^{(-1)^r}$.

An **open** $n-$**cell** for the space M is a continuous map $\overset{\circ}{\varphi} : \overset{\circ}{D^n} \to M$, homeomorphism on the image and an $n-$**cell** is a continuous map $\varphi : D^n \to M$ whose restriction to $\overset{\circ}{D^n}$ is an open cell. Here $\overset{\circ}{D^n}$ denotes the interior of D^n, the unit disc in R^n, viewed as an oriented manifold with boundary. If $n = 0$ make the convention that $\overset{\circ}{D^0} = D^0 = \text{one point}$.

Instead of D^n one can consider compact smooth manifold with corners W^n homeomorphic to the unit disc and in case M is a smooth manifold the CW complex structure is called a *smooth CW complex structure* if the maps φ are smooth maps with the restriction to the interior a diffeomorphism [1]

Denote by $\overset{\circ}{e} = \varphi(\overset{\circ}{D^n}) \subset X, e := \varphi(D^n) \subset X, \partial e = \varphi(\partial D^n) \subset X$. Clearly $D^n/\partial D^n = S^n$ with D^n and then S^n regarded as oriented manifolds whose orientations are induced from the orientation of $\mathbb{R}^n$ with $D^n := \{(x_1, \cdots, x_n) \in R^n \mid \sum x_i^2 \leq 1\}$.

Clearly the map φ induces the homeomorphism $\overline{\varphi} : S^n \to e/\partial e$

A CW complex structure on a compact space M consists of a finite collection of cells $\{\varphi_\alpha : D^{n_\alpha} \to X\}$ with the following properties.

1. $\varphi_\alpha(\overset{\circ}{D^{n_\alpha}}) \cap \varphi_\beta(\overset{\circ}{D^{n_\beta}}) = \emptyset$ if $\alpha \neq \beta$ and $X = \cup_\alpha \varphi_\alpha(\overset{\circ}{D}^{n_\alpha})$
2. $\varphi_\alpha(\partial D^{n_\alpha}) \subset X(n_\alpha - 1) = \cup_{n_\beta < n_\alpha} \varphi_\beta(\overset{\circ}{D^{n_\beta}})$ [2]
3. $K \subset X$ is closed iff $K \cap e_\beta$is closed for any β.

To a CW complex structure on X one associate the cell structure with $\mathcal{X}_r := \{\alpha \mid n_\alpha = r\}$ and with $I_r(\alpha, \beta), \alpha \in]\mathcal{X}_r, b\eta \in \mathcal{X}_{r-1}$ defined by the degree of the following composition

$$\begin{array}{ccccc}
S^{r-1} = \partial D^{n_\alpha} & & \bigvee_{n_\beta = r-1} S^{n_\beta} & \longrightarrow & S^{n_\beta} \\
\Big\downarrow{\scriptstyle \varphi_\alpha} & & \Big\downarrow{\scriptstyle \bigvee \overline{\varphi}_\beta} & & \\
X(r) & \longrightarrow & X(r)/X(r-1) = \bigvee_{n_\beta = r-1} e_\beta/\partial e_\beta & &
\end{array}$$

Note that $\bigvee \overline{\varphi}_\beta$ is a homeomorphism hence invertible.

The cohomology derived from this CW complex structure is canonically isomorphic to the singular cohomology of the underlying space X hence independent on the CW complex structure.

Hodge decomposition

For a a cochain complex of finite dimensional complex vector spaces,

$$\cdots \xrightarrow{d_{r-1}} C^r \xrightarrow{d_r} C^{r+1} \xrightarrow{d_{r+1}} C^{r+2} \xrightarrow{d_{r+2}} \cdots$$

with C^r equipped with hermitian scalar products $(\ ,\ )_r$ consider $\delta_r : C^{r+1} \to C^r$ the adjoint of d_r , i.e. the unique linear map which satisfies $(d_r(x), y)_{r+1} = (x, \delta_r(y))_r$ for $x \in C^r, y \in C^{r+1}$. Observe that $\delta_{r-1} \cdot \delta_r = 0$.

Define

1. $C^r_+ = \mathrm{i}mg\ d_{r-1}$
2. $C^r_- = \mathrm{i}mg\ \delta_r$
3. $\underline{d}_r : C^r_- \to C^r_+$ the restriction of d_r which is an isomorphism
4. $\Delta_r = \delta_r \cdot d_r + d_{r-1} \cdot \delta_{r-1}$

[1] recall that an $n-$smooth manifold with corners is locally diffeomorphic to open sets in $\mathbb{R}^n_+ := \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_i \geq 0\}$

[2] $X(r) = \cup_{n_\alpha \leq r} \varphi_\alpha(\overset{\circ}{D}^{n\alpha}) = \cup_{n_\alpha \leq r} \varphi_\alpha(D^{n_\alpha})$

5. $\mathcal{H}^r := \ker \Delta_r \ker d_r \cap \ker \delta_{r-1}$

The proof is an easy exercise.

The folowing definitions might be useful.

1. $\Delta_r^+ := \delta_r \cdot d_r : C^r \to C^r_+ \subset C^r$,
2. $\Delta_r^- := d_{r-1} \cdot \delta_{r-1} : C^r \to C^r_- \subset C^r$, hence
3. $\Delta_r = \Delta_r^+ + \Delta_r^-$ and $d_r \cdot \Delta_r^- = \Delta_{r+1}^+ \cdot d_r$ and $\Delta_r^- \cdot \delta_r = \delta_r \cdot \Delta_{r+1}^+$,

Observe that the nonzero eigenvalues of Δ_r^- are the same as the nonzero eigenvalues of Δ_{r+1}^+ and te Consider the eigenvalues of Δ_r, $\lambda_1^r, \lambda_2^r, \cdots, \lambda_{n_r}^r$ and define the *modified determinant* $\det' \Delta_r := \prod_{\lambda_i \neq 0} \lambda_i$.

For an injective linear map $f : V_1 \to V_2$ between two f.d. hermitian vector spaces denote by $Vol(f) := \sqrt{\det(f^* f)^{1/2}}$.

Proposition 0.1

1. $\mathcal{H}^r = \ker d_r \cap \ker \delta_{r-1}$,

2. the subspace $C^r_-, C^r_+, \mathcal{H}^r$ *are mutually orthogonal and* $C^r = C^r_- \oplus C^r_+ \oplus \mathcal{H}^r$ *with.* $d_r = \begin{bmatrix} 0 & 0 & 0 \\ \underline{d}_r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

3. $H^r(C^*) = \mathcal{H}^r$,

4. $\prod (Vol(\underline{d}_r))^{(-1)^r} = \prod (\det' \Delta_r)^{(-1)^{r+1} r}$.

Again a rather easy exercise. provided that one note that $Vol(\underline{d}_r)$ is exactly the product of the nonzero eigenvalues of of Δ_r^-.

Note: A fancy way to define the modified determinant, of a hermitian matrix which might work for self adjoint operators with infinite many eigenvalues λ_i

$$\log det' A := -d/ds_{s=0} \sum_{\lambda_i \neq 0} \lambda_i^{-s}$$