MORSE THEORY, GEOMETRIC COMPLEX, INTEGRATION FROM de-RHAM TO GEOMETRIC COMPLEX

(Summary)

**Morse functions** $f : M^n \to mathbbR$

- Critical points of index $k \leq n$, $Cr_k(f)$.
- Morse charts. $\varphi_x : M \supset U \to \varphi(U_x) \subset \mathbb{R}^n$
- Riemannian metric $f-$compatible, ($g_{i,j} = \delta_{i,j}$ in some Morse charts. Vector field $X := -grad_g f$
- the flow of $X$ when $M$ is closed $\Phi : \mathbb{R} \times M \to M$, with $\Phi(s+t, x) = \Phi(s, \varphi(t, x))$, and $\Phi(0, \mathrm{x}) = \mathrm{x}$.
- the set of **rest points** $\mathcal{R}(X) := \{x \in M \mid \varphi(t, x) = x, \text{any t} \in \mathbb{R}\} = \mathrm{Cr(f)}$
- for $x \in \mathcal{R}(x)$ define the **stable/ unstable set**.

$$W_x^{\pm} := \{y \in M \mid \lim_{t \to \pm\infty} \varphi(y, t) = x\}$$

**Proposition 0.1** *For $X$ as above $W_x^-$ resp. $W_x^+$ are smooth submanifolds diffeomorphic to $\mathbb{R}^{index\ x}$ resp. $\mathbb{R}^{n-index\ x}$ with a specified orientation (when Morse charts are specified).*

**Example (illustration)**

- $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x_1, x_2, \cdots x_n) = c - 1/2 \sum_{i \leq k} x_i^2 + 1/2 \sum_{i > k} x_i^2$
- $Cr(f) = \{0 \in \mathbb{R}^n\}$,

$$X = \sum_{i \leq k} x_i\, \partial/\partial x_i - \sum_{i > k} x_i\, \partial/\partial x_i$$

$$\Phi(t; y_1, y_2, \cdots y_n) = (y_1 e^t, y_2 e^t, \cdots y_k e^t, y_{k+1} e^{-t} \ldots y_n e^{-t}).$$

Hence $W_0^- = \mathbb{R}^k \times (0, 0, \cdots 0)$, $W_0^+ = (0, 0 \cdots 0) \times \mathbb{R}^{n-k}$.

- Mores Smale pair $(f, g)$ is a Morse pair s.t. for any $x, y \in Cr(f)$ one has $W_x^- \pitchfork W_y^+$. hence $\mathcal{M}(x, y)$ smooth manifold of dimension $index\ x - index\ y$ hence $\mathcal{T}(x, y) = \mathcal{M}(x, y)/\mathbb{R}$ orientable smooth manifold of dimension $index\ x - index\ y - 1$ which is oriented provided Morse charts are chosen in the neighborhood of any critical point.

One write $x > y$ iff $\mathcal{T}(x, y) \neq \emptyset$.

**Proposition 0.2** *Given a pair $(f, g)$, $f$ a Morse function and for each critical point a Morse chart one can find $g'$ arbitrary closed to $g$ in $C^r-$ topology, $r \geq 0$ such that $g'$ differs from $g$ only in arbitrary small neighborhood of the critical points.*

**Manifold with corners,** $M^n$ Is a topological Hausdorff space equipped with an smooth atlas based on $\mathbb{R}^n_{\geq 0}$ -charts, ($\mathbb{R}^n_{\geq 0} = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \cdots \times \mathbb{R}_{\geq 0}$)

- the set $\partial_k M$of $k-$corners , (i.e. points which in some and then any chart have exactly $k-$coordinates equal to zero.

- each connected component of $\partial_k M$ is called a $k-$face and is a smooth submanifold of dimension $n - k$.

**Theorem 0.3** *Given a Morse Smale pair $(f, g)$ and for each critical point $x$ a Morse chart the manifold $W_x^-$ and the canonical inclusion $\iota_x : W_x^- \to M$ can be compactified to the compact manifold with corners $\hat{W}_x^-$ and the smooth map $\hat{\iota}_x : \hat{W}_x^- \to M$ s.t.*

$$\partial_k(\hat{W}_x^-) := \bigsqcup_{x>y_1>y_2>\cdots>y_k} \mathcal{T}(x, y_1) \times \mathcal{T}(y_1, y_2) \times \cdots \mathcal{T}(y_{k-1}, y_k) \times W_{y_k}^-.$$

*and*

*the restriction of $\hat{\iota}_x$ to the each component of $\partial_k(\hat{W}_x^-)$, the composition of the projection on $W_{y_k}^-$ with the inclusion $\iota_{y_k}$. Moreover $\hat{W}_x^-$ are homeomorphic to $D^{index\ x}$*

Under the hypotheses of the previous theorem (in view of the orientations provided by the choice of Morse charts) the collection $(M, \hat{\iota}_x : \hat{W}_x^- \to M)$ defines a smooth CW structure on $M$ and in particular $Int : (\Omega^*(M), d_*) \to \mathcal{C}^*(M, f, g), \partial_*)$ with the second cochain complex the one associated to the CW complex structure as specified.

**The geometric complex**

1. $Cr_q(f), I_r : Cr_q(f) \times Cr_{q-1}(f) \to \mathbb{Z}$ counting trajectories from $x$ to $y$
2. $C^q(f) := Maps(Cr_q(f), \mathbb{R}),\ \partial_{r-1} : C^{q-1}(f) \to C^q(f)$ defined by

$$\partial u(x) = \sum I_r(x, y)u(y).$$

In particular we have the main result of the theory.

**Theorem 0.4** *Suppose $(f, g)$ a Morse Smale pair.*

1. *For any $\omega \in \Omega^k(M)$ and any $x \in Cr_k(f)$ $\int_{W_x^-} \omega$ is convergent and provides a morphism of cochain complexes*

$$Int : \Omega^*(M), d_*) \to (C^*(f), \partial_*(f, g))$$

2. *$Int$ induces an isomorphism in cohomology.*
3. *If $\beta_k(M)$ denotes the dimension of $k-$dimensional cohomology vector space with coefficients in a field of characteristic zero and $c_k := \sharp Cr_k(f)$ then one has the Morse inequalities.*
   - *(a)* $\beta_r \leq c_r$
   - *(b)*

$$\begin{aligned} \sum_{r\leq q} \beta_r \leq \sum_{r\leq q} c_r \text{ if q even} \\ \sum_{r\leq q} \beta_r \geq \sum_{r\leq q} c_r \text{ if q odd} \end{aligned} \tag{1}$$

4. *The geometric complex (up to a non canonical isomorphism is independent of the metric $g$ and $\partial d_q \neq 0$ implies existence of trajectories from critical points of index $(q + 1)$ to critical points of index $q$ for any vector field $Y$ which admit $f$ as a Lyapunov function.*

Details for this material can be found in the attachment " refinedmorse" A sketch of the proof that a compact manifold with corners whose interior is contractible is diffeomorphic to the (closed) unit disc is included in the attachment "CWstructure "