1 Differential forms and integration

1.1 Smooth manifolds

The concept of an n-dimensional manifold is a simple extension of the calculus of several variables. It is a space constructed from open subsets of $\mathbb{R}^n$ by patching them together in a smooth way.

Let M be a topological space. We say that M has a countable base if there exists a collection $(U_i)_{i\in\mathbb{N}}$ of open subsets of M so that any open subset of M is given by the union of open subsets from the collection $(U_i)_{i\in\mathbb{N}}$. The space M is called Hausdorff if for any two distinct points $x, y \in M$ there exist open neighborhoods U_x and U_y of x and y with empty intersection.

1.1 Definition. *M is a topological manifold of dimension $n \geq 0$ if M is Hausdorff and has a countable base $(U_i)_{i\in\mathbb{N}}$ so that for any $i \geq 1, U_i$ is homeomorphic to an open subset of $\mathbb{R}^n$.*

It can be shown that any topological manifold M is *paracompact* – see e.g. [By]. It means that any open covering $(U_i)_{i\in I}$ of M has a locally finite refinement, i.e. an open covering $(V_j)_{j\in J}$ of M with the following two properties: (i) for any $j \in J$ there exists $i \in I$ with $V_j \subseteq U_i$ (refinement); (ii) for any $x \in M$ there exists a neighborhood W of x which intersects only a finite number of sets in $(V_j)_{j\in J}$ (locally finite).

A *chart* of an n-dimensional topological manifold is a pair (U, φ) consisting of an open subset $U \subseteq M$ and a homeomorphism $\varphi : U \to U'$ onto an open subset $U' \subseteq \mathbb{R}^n$.

We say that $\varphi^{-1} : U' \to U$ is a *local parametrization* of X. By a slight abuse of notation, in the sequel, we will sometimes refer to a local parametrization of M as a chart as well. An *atlas* A of a topological manifold is a collection of charts $(U_i, \varphi_i)_{i\in I}$ so that $(U_i)_{i\in I}$ is an open covering of M. One says that A is a *smooth atlas* if for any $i, j \in I$ the transition map

$$\varphi_{ji} := \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is a C^∞-diffeomorphism between open subsets of $\mathbb{R}^n$. Assume that A_1 and A_2 are two smooth atlases of M. We say that A_1 and A_2 are *equivalent* if $A_1 \cup A_2$ is a smooth atlas of M. One easily checks that this is an equivalence relation on the set of smooth atlases of M. An equivalence class of smooth atlases of M is called a *smooth structure* of M.

1.2 Definition. *A smooth manifold (of dimension n) is a pair $(M, \mathcal{A})$ consisting of a topological manifold M (of dimension n) and a smooth structure $\mathcal{A}$ of M.*

In the sequel we simply write M for a smooth manifold instead of $(M, \mathcal{A})$.

Assume that $f : M_1 \to M_2$ is a continuous map between two smooth manifolds M_1 and M_2. One says that f is *smooth (or C^∞) at $x \in M_1$* if there exist charts $\varphi_i : U_i \to U_i'$ of M_i $(i = 1, 2)$ with $x \in U_1$ and $f(x) \in U_2$ so that the following map between open subsets of Euclidean spaces,

$$\varphi_2 \circ f \circ \varphi_1^{-1} : \varphi_1(f^{-1}(U_2) \cap U_1) \to U_2'$$

is smooth at $\varphi_1(x)$. The map f is said to be *smooth* (or C^∞) if f is smooth at any point of M_1. It is a *diffeomorphism* (or more precisely, a C^∞-diffeomorphism) if f is a homeomorphism and both, f and f^{-1}, are smooth maps.

A maximal atlas A_{max} associated with a given smooth structure of M is defined as the set of all possible charts,

$$A_{\text{max:}} = \{\varphi : U \to U' \,\big|\, U \subseteq M \text{ open }; U' \subseteq \mathbb{R}^n \text{ open }; \varphi \text{ diffeomorphism } \}.$$

From now on, a chart $\varphi : U \to U'$ of a smooth manifold will be a chart in the maximal atlas. If not stated otherwise, U will be assumed to be contractible.

A very useful tool in the study of smooth manifolds is the notion of a partition of unity of M. It is a collection $(\eta_i)_{i \in I}$ of smooth functions $\eta_i : M \to \mathbb{R}_{\geq 0}$ so that for each $x \in M$ there exists an open neighborhood $U_x \subseteq M$ with the property that $U_x \cap \text{supp}(\eta_i) \neq \emptyset$ only for finitely many $i \in I$ and $\sum_{i \in I} \eta_i(x) = 1$. Here $\text{supp}(\eta_i)$ denotes the support of η_i, i.e. the closure (in M) of the set of all points in M where η_i does not vanish.

A partition of unity $(\eta_i)_{i \in I}$ is said to be *subordinate* to the (open) covering $(U_j)_{j \in J}$ of M if for each $i \in I$ there exists $j \in J$ so that $\text{supp}(\eta_i) \subseteq U_j$. Mostly we will use partitions of unity $(\eta_i)_{i \in I}$ subordinate to an open covering $(U_j)_{j \in J}$ of M with $J = I$ and $\text{supp}(\eta_i) \subseteq U_i$ $\forall i \in I$. The following result on the existence of partitions of unity is well known – see e.g. [By], [Wa].

1.3 Theorem. *Assume that M is a smooth manifold and $(U_j)_{j \in J}$ is an open covering of M. Then there exists a countable partition of unity $(\eta_i)_{i \in \mathbb{N}}$ subordinate to the cover $(U_j)_{j \in J}$ with* $\operatorname{supp}(\eta_i)$ *compact for any i in $\mathbb{N}$. If M is compact then there exists a partition of unity $(\eta_j)_{j \in J}$ with* $\operatorname{supp}(\eta_j) \subseteq U_j \ \forall j \in J$ *so that all except finitely many of the $(\eta_j)_{j \in J}$ vanish identically.*

Next we discuss the notion of a smooth submanifold. Let M be a smooth n-dimensional manifold. One says that $N \subseteq M$ is a *smooth submanifold of* M (of dimension k) if for any $x \in N$, there exists a chart $\varphi : U \to U' \subseteq \mathbb{R}^n$ of M such that $x \in U$ and $\varphi(U \cap N) = U' \cap (\mathbb{R}^k \times \{0\}^{n-k})$. (Then, in a canonical way, N is a smooth k-dimensional manifold.) A smooth map $f : M_1 \to M_2$ between smooth manifolds M_1 and M_2 is a *smooth embedding* if $f(M_1)$ is a smooth submanifold of M, and $f : M_1 \to f(M_1)$ is a diffeomorphism. The following well-known result says in particular that any smooth manifold can be viewed as a smooth submanifold of an Euclidean space.

We remark that Whitney's embedding theorem improves on the above theorem. It states that there exists a smooth embedding f where the codimension k of the embedding satisfies $k \leq n + 1$.

Given a smooth n-dimensional manifold M, the tangent space T_xM of M at a point $x \in M$ is used to study smooth maps on M near x. Let us recall its definition. Consider a chart $\varphi : U \to U'$ with $x \in U$. We say that two smooth curves $\gamma_i : t \mapsto \gamma_i(t) \in M$ $(i = 1, 2)$ with $-1 < t < 1$ passing through x at $t = 0$, are equivalent, $\gamma_1 \sim \gamma_2$, if $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$ where $'$ denotes the derivative with respect to t. (As $\gamma_i(t) \in U$ and hence $\varphi \circ \gamma_i(t)$ is defined for $|t|$ sufficiently small, so is $(\varphi \circ \gamma_i)'(0)$.) One easily verifies that $\sim$ is an equivalence relation and that this relation does not depend on the choice of the chart. The tangent space T_xM of M at x is then defined as the set of equivalence classes $[\gamma]$ of smooth curves $\gamma : (-1, 1) \to M$ with $\gamma(0) = x$. In a natural way, T_xM has the structure of an n-dimensional $\mathbb{R}$-vector space: It is defined in such a way that the map

$$T_xM \to \mathbb{R}^n, \ [\gamma] \mapsto (\varphi \circ \gamma)'(0)$$

is a linear isomorphism.

(One easily checks that the vector space structure is well-defined, i.e. does not depend on the choice of the chart.) Assume that $f : M \to N$ is a smooth map between two smooth manifolds M_1 and M_2. Then it induces for any

$x \in M$ a linear map $d_x f : T_x M_1 \to T_{f(x)} M_2, [\gamma] \mapsto [f \circ \gamma]$, referred to as tangent map or differential of f at x. Given charts $\varphi_1 : U_1 \to U_1' \subseteq \mathbb{R}^{n_1}$ of M_1 around x and $\varphi_2 : U_2 \to U_2' \subseteq \mathbb{R}^{n_2}$ of M_2 around $f(x), d_x f$, when expressed in local coordinates, is the linear map $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ given by the $n_2 \times n_1$ matrix $(\partial_{x_i}(\varphi_2 \circ f \circ \varphi_1^{-1})_j)_{ji}$ where $1 \leq j \leq n_2$ and $1 \leq i \leq n_1$ and where $\partial_{x_i}(\varphi_2 \circ f \circ \varphi_1^{-1})_j$ denotes the partial derivative of the j'th component $(\varphi_2 \circ f \circ \varphi_1^{-1})_j$ of $\varphi_2 \circ f \circ \varphi_1^{-1}$ with respect to x_i at $\varphi_1(x)$. Often it is convenient to choose in $T_x M$ the following basis associated to a chart $\varphi : U \to U'$ of M around x. Denote by $e_1, \ldots, e_n$ the standard basis of $\mathbb{R}^n$. Then

$$\left(\frac{\partial}{\partial x_i}\right)_x := (d_{\varphi(x)} \varphi^{-1})(e_i) \quad 1 \leq i \leq n$$

is a basis of $T_x M$ for any $x \in U$. Often we will drop the subscript x and simply write $\frac{\partial}{\partial x_i}$.

By $T_x^* M$ we denote the dual of $T_x M$. It is called the cotangent space of M at x. Using the dual pairing between $T_x M$ and $T_x^* M$, the basis $(\frac{\partial}{\partial x_i})_{1 \leq i \leq n}$ defines a basis of $T_x^* M$, dual to $(\frac{\partial}{\partial x_i})_{1 \leq i \leq n}$ which we denote by $(dx_i)_{1 \leq i \leq n}$.

1.2 Differential forms

Let V be a (real or complex) vector space of dimension n and denote by V^* the dual of V. By $\Lambda^k(V^*)$ with $k \geq 0$ we denote the set of all multilinear, anti-symmetric functions $\omega(v_1, \ldots, v_k)$ of k variables $v_j \in V$ with values in another vector space; it means that ω is linear in each of its arguments and that for any $1 \leq i < j \leq k$

$$\omega(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\omega(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

If not specified otherwise we will assume that ω takes complex values. Notice that $\Lambda^k(V^*)$ is a linear space. Further $\Lambda^k(V^*) = \{0\}$ for any $k \geq n+1$ and $\Lambda^1(V^*) = V^*$. The space $\Lambda^0(V^*)$ is set to be $\mathbb{C}$. Let $e_1, \ldots, e_n$ be a basis in V. Then an element $\omega \in \Lambda^k(V^*)$ is completely determined by the values $\omega(e_{i_1}, \ldots, e_{i_k})$ where $i_1 < i_2 < \ldots < i_k$. In fact, one uses multilinearity to express for $v_1, \ldots, v_k \in V$ arbitrary, $\omega(v_1, \ldots, v_k)$ in terms of $\omega(e_{i_1}, \ldots, e_{i_k})$ with $i_1, \ldots, i_k$ being arbitrary numbers between 1 and n. Because of the anti-symmetric property of ω, $\omega(e_{i_1}, \ldots, e_{i_k}) = 0$ if two of the numbers among $i_1, \ldots, i_k$ coincide and one can always permute $e_{i_1}, \ldots, e_{i_k}$ so that $i_1, \ldots, i_k$

appear in increasing order; that would result in multiplying ω by the sign of the permutation.

Let $1 \le i_1 < i_2 < \ldots < i_k \le n$. By $e^*_{i_1} \wedge \ldots \wedge e^*_{i_k}$ we denote the element from $\Lambda^k(V^*)$ such that

$$e^*_{i_1} \wedge \ldots \wedge e^*_{i_k}(e_{i_1}, \ldots, e_{i_k}) = 1$$

and

$$e^*_{i_1} \wedge \ldots \wedge e^*_{i_k}(e_{j_1}, \ldots, e_{j_k}) = 0$$

if $\{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\}$. Then the elements $e^*_{i_1} \wedge \ldots \wedge e^*_{i_k}$ form a basis in $\Lambda^k(V^*)$. In particular, $\{e^*_i\}$ is the basis in V^* that is dual to $\{e_i\}$. To make notations simpler, we sometimes will use multi-index notations $I = \{i_1 < \ldots < i_k\}$. Then $e^*_I = e^*_{i_1} \wedge \ldots \wedge e^*_{i_k}$. The dimension of $\Lambda^k(V^*)$ equals the number of multi-indices $I = \{i_1 < \ldots < i_k\}$ which is equal to $\binom{n}{k}$. Every element $\omega \in \Lambda^k(V^*)$ can be decomposed into the sum

$$\omega = \sum_{|I|=k} \omega_I e^*_I = \sum_{i_1 < \ldots < i_k} \omega_{i_1 \ldots i_k} e^*_{i_1} \wedge \ldots \wedge e^*_{i_k}.$$

The *exterior product* $\omega_1 \wedge \omega_2$ of $\omega_1 \in \Lambda^k(V^*)$ with $\omega_2 \in \Lambda^\ell(V^*)$ is defined as the multilinear map $(V^*)^{k+\ell} \to \mathbb{C}$ such that $\omega_1 \wedge \omega_2(v_1, \ldots, v_{k+\ell})$ is given by

$$\frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (-1)^{sgn\sigma} \omega_1(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot \omega_2(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}).$$

Here $S_{k+\ell}$ is the symmetric group of $k+\ell$ elements and $\mathrm{sgn}\sigma$ denotes the sign of the permutation $\sigma \in S_{k+\ell}$. One checks easily that $\omega_1 \wedge \omega_2$ is antisymmetric, i.e. $\omega_1 \wedge \omega_2 \in \Lambda^{k+\ell}(V^*)$.

To illustrate the notion of the exterior product let us consider the following example. Assume that $n \ge 2$ and let $\omega_1 = e^*_1 \in \Lambda^1(V^*) = V^*$ and $\omega_2 = e^*_2 \in \Lambda^1(V^*)$. Then

$$\begin{aligned}(\omega_1 \wedge \omega_2)(e_1, e_2) &= \omega_1(e_1)\omega_2(e_2) - \omega_1(e_2)\omega_2(e_1) \\ &= \omega_1(e_1)\omega_2(e_2) = 1,\end{aligned}$$

and, for any other pair of vectors, e_i and e_j, such that $\{i, j\} \neq \{1, 2\}$ one has

$$(\omega_1 \wedge \omega_2)(e_i, e_j) = 0.$$

It follows that $\omega_1 \wedge \omega_2 = e_1^* \wedge e_2^*$. More generally, one checks that the element $e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* \in \Lambda^k(V^*)$ introduced above is indeed the exterior product of the linear forms $e_{i_1}^*, \ldots, e_{i_k}^*$, thus justifying the notation.

1.4 Remark. The following properties of the exterior product can be verified in a straight forward way.

(i) For any elements $\omega_i \in \Lambda^{k_i}(V^*), 1 \leq i \leq 3$, one has

$$(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3).$$

(ii) Let $\omega_1 \in \Lambda^k(V^*)$ and $\omega_2 \in \Lambda^\ell(V^*)$. Then

$$\omega_1 \wedge \omega_2 = (-1)^{k\ell} \omega_2 \wedge \omega_1. \tag{1.1}$$

Let M be a smooth manifold of dimension n. For every point $x \in M, T_xM$ is the tangent space to M at x, and $T_x^*M = (T_xM)^*$ is the cotangent space. A differential k-form on M is a function on M, the value of which at a point $x \in M$ is an element of $\Lambda^k(T_x^*M)$. Let U be a coordinate neighborhood in M, and let $x_1, \ldots, x_n$ be local coordinates. At any point $x \in U$, the vectors $\partial/\partial x_1, \ldots, \partial/\partial x_n$ form a basis in T_xM; the dual basis in T_x^*M is denoted by $dx_1, \ldots, dx_n$. Then, on U, a differential k-form ω can be written as

$$\sum_{|I|=k} \omega_I(x) dx_I = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \omega_{i_1 \ldots i_k}(x) dx_{i_1} \wedge \ldots \wedge dx_{i_k}. \tag{1.2}$$

If for any coordinate neighborhood U of M, all the coefficients $\omega_I(x)$ are smooth functions of x then the differential form ω is called smooth. If not stated otherwise, we assume differential forms to be smooth, and the word "smooth" will be usually suppressed. Sometimes, we will be dealing with differential forms that are not smooth; then, we will specify explicitly, to which particular function space the coefficients $\omega_I(x)$ belong.

If $y_1, \ldots, y_n$ is another set of coordinates in U then the form (1.2) can be represented as

$$\omega(x) = \sum_{j_1, \ldots, j_k = 1}^{n} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \omega_{i_1 \ldots i_k}(x) \frac{\partial x_{i_1}}{\partial y_{j_1}} \cdots \frac{\partial x_{i_k}}{\partial y_{j_k}} \, dy_{j_1} \wedge \ldots \wedge dy_{j_k}. \tag{1.3}$$

In the representation (1.3), the indices $j_1, \ldots, j_k$ are arbitrary; they are not necessarily in increasing order. Notice that $dy_{j_1} \wedge \ldots \wedge dy_{j_k} = 0$ if two of the indices $j_1, \ldots, j_k$ coincide, and, if all of them are different, then one can put them in increasing order by a permutation and $dy_{j_1} \wedge \ldots \wedge dy_{j_k}$ is multiplied by the sign of the permutation.

1.5 Example. Let $\omega = f(x)dx_1 \wedge \ldots \wedge dx_n$. Then

$$\begin{aligned}\omega(x) &= f(x) \sum_{\sigma \in S_n} (-1)^{sgn\sigma} \frac{\partial x_1}{\partial y_{\sigma(1)}} \cdots \frac{\partial x_n}{\partial y_{\sigma(n)}}\; dy_1 \wedge \ldots \wedge dy_n \\ &= f(x) \cdot \det\left(\frac{\partial x}{\partial y}\right) dy_1 \wedge \ldots \wedge dy_n.\end{aligned}$$

The space of all (smooth) differential forms on M of degree k ($k \geq 0$) will be denoted by $\Omega^k(M)$. Note that $\Omega^k(M)$ is a vector space and $\Omega^k(M) = \{0\}$ for $k \geq n+1$.

An important operation is taking the differential of a form. The operator d maps $\Omega^k(M)$ into $\Omega^{k+1}(M)$, and if a form ω is given in local coordinates by (1.2) then

$$d\omega = \sum_{j=1}^{n} \sum_{|I|=k} \partial_{x_j}(\omega_I(x))dx_j \wedge dx_I \tag{1.4}$$

where $\partial_{x_j} = \frac{\partial}{\partial x_j}$. On an open subset $U' \subseteq \mathbb{R}^n$, for any $1 \leq j \leq n$, one can define the two operations

$$\frac{\partial}{\partial x_j} : \Omega^k(U') \to \Omega^k(U') \text{ and } \Lambda_j : \Omega^k(U') \to \Omega^{k+1}(U')$$

by

$$\partial_{x_j}\omega = \sum_{|I|=k} \partial_{x_j}(\omega_I(x))dx_I$$

and

$$\Lambda_j\omega = dx_j \wedge \omega.$$

Notice that both $\frac{\partial}{\partial x_j}$ and Λ_j depend on the choice of coordinates. Formula (1.4) can be rewritten as

$$d = \sum_{j=1}^{n} \Lambda_j \circ \frac{\partial}{\partial x_j}. \tag{1.5}$$

To see that (1.4) defines a well defined operator $\Omega^k(M) \to \Omega^{k+1}(M)$ we have to see how $\sum_{j=1}^n \Lambda_j \circ \frac{\partial}{\partial x_j}$ changes under a coordinate transformation

$$\varphi : V' \to U',\ (y_1, \ldots, y_n) \mapsto (x_1, \ldots, x_n) = \varphi(y_1, \ldots, y_n)$$

between open subsets V' and U' of $\mathbb{R}^n$. Let $\omega \in \Omega^k(U')$. To simplify our exposition assume that ω is of the form $\omega = \omega_I(x)dx_I$ with $I = \{1 \leq i_1 < \ldots < i_k \leq n\}$. Then

$$\varphi^*\left(\left(\sum_{i=1}^n \Lambda_i \circ \frac{\partial}{\partial x_i}\right)\omega\right)(y) = \sum_{i=1}^n (\partial_{x_i}\omega_I)(\varphi(y))\varphi^*(dx_i \wedge dx_I)(y).$$

From the definition of the exterior product one sees that $\varphi^*(dx_i \wedge dx_I) = \varphi^*(dx_i) \wedge \varphi^*(dx_I)$. Moreover, it is easy to see that

$$\varphi^*(dx_i)(y) = \sum_{k=1}^n (\partial_{y_k}\varphi_i)(y)dy_k.$$

Hence we get

$$\begin{aligned}
&\varphi^*\left(\left(\sum_{i=1}^n \Lambda_i \circ \frac{\partial}{\partial x_i}\right)\omega\right)(y) \\
&= \sum_{k=1}^n \partial_{y_k}(\varphi^*\omega_I)(y)(dy_k \wedge \varphi^* dx_I)(y) \\
&= \sum_{k=1}^n \left(\tilde{\Lambda}_k \circ \frac{\partial}{\partial y_k}\right)(\varphi^*\omega)(y).
\end{aligned}$$

This shows that the operator d, defined in local coordinates by $\sum_{i=1}^n \Lambda_i \circ \frac{\partial}{\partial x_i}$ is a well defined operator from $\Omega^k(M)$ into $\Omega^{k+1}(M)$. In the sequel, we will often suppress the sign for composing operators in (1.5) and simply write

$$d = \sum_{j=1}^n \Lambda_j \frac{\partial}{\partial x_j}.$$

1.6 Remark. One can define the differential d in an invariant way, without using coordinates. Let $X_1, \ldots, X_{k+1}$ be vector fields on M. It is a good exercise to show that

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) &= \sum_{\ell=1}^{k+1} (-1)^{\ell-1} X_\ell \left(\omega(X_1, \ldots, \hat{X}_\ell, \ldots, X_{k+1}) \right) \\ &+ \sum_{1 \leq \ell < i \leq k+1} (-1)^{i+\ell} \omega([X_\ell, X_i], X_1, \ldots, \hat{X}_\ell, \ldots, \hat{X}_i, \ldots, X_{k+1}). \end{aligned} \tag{1.6}$$

Here $\hat{X}_\ell$ means that the argument X_ℓ is missing and $[X_\ell, X_i]$ is the commutator of the vector fields X_ℓ and X_i. Formula (1.6) can be taken as a definition of d.

In the following proposition we formulate important properties of the differential.

1.7 Proposition. (i) $d \circ d = 0$.

(ii) *Let* $\omega_1 \in \Omega^k(M), \omega_2 = \Omega^\ell(M)$. *Then*

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2.$$

This identity is referred to as Leibniz rule.

Proof. To prove both statements, we use local coordinates. First,

$$\begin{aligned} d \circ d &= \sum_{j=1}^{n} \sum_{k=1}^{n} \Lambda_j \frac{\partial}{\partial x_j} \circ \Lambda_k \frac{\partial}{\partial x_k} \\ &= \sum_{j,k=1}^{n} \Lambda_j \Lambda_k \frac{\partial^2}{\partial x_j \partial x_k} \\ &= \sum_{1 \leq j < k \leq n} (\Lambda_j \Lambda_k + \Lambda_k \Lambda_j) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} \Lambda_j^2 \frac{\partial^2}{\partial x_j^2}. \end{aligned}$$

Notice that $\Lambda_j^2 \omega = dx_j \wedge dx_j \wedge \omega = 0$ and

$$(\Lambda_j \Lambda_k + \Lambda_k \Lambda_j)\omega = dx_j \wedge dx_k \wedge \omega + dx_k \wedge dx_j \wedge \omega = 0$$

for any form ω. Therefore, $\Lambda_j^2 = 0$ and $\Lambda_j\Lambda_k + \Lambda_k\Lambda_j = 0$, so $d \circ d = 0$.

Secondly,

$$\begin{aligned} d(\omega_1 \wedge \omega_2) &= \sum_{j=1}^{n} dx_j \wedge \partial_{x_j}(\omega_1 \wedge \omega_2) \\ &= \sum_{j=1}^{n} dx_j \wedge \left(\partial_{x_j}\omega_1 \wedge \omega_2 + \omega_1 \wedge \partial_{x_j}\omega_2\right) \\ &= \left(\sum_{j=1}^{n} dx_j \wedge \partial_{x_j}\omega_1\right) \wedge \omega_2 + (-1)^k \omega_1 \wedge \left(\sum_{j=1}^{n} dx_j \wedge \partial_{x_j}\omega_2\right) \\ &= d\omega_1 \wedge \omega_2 + (-1)^k \omega_k \wedge d\omega_2. \end{aligned}$$

■

Let N and M be manifolds, and let $F : N \to M$ be a smooth mapping. For a point $x \in N$, the differential d_xF of F at x is a linear map from T_xN to $T_{F(x)}M$. Suppose that $\omega \in \Omega^k(M)$. Then, the pull-back $F^*\omega \in \Omega^k(N)$ by F is defined by the formula

$$(F^*\omega)(X_1, \ldots, X_k) = \omega\left(d_xF(X_1), \ldots, d_xF(X_k)\right) \tag{1.7}$$

for $X_1, \ldots, X_k \in T_xN$.

It easily follows from the formula (1.6) that

$$d(F^*\omega) = F^*(d\omega). \tag{1.8}$$

1.3 Integration

First, we define the integral of a differential form over an Euclidean domain. Let V be a connected bounded domain in $\mathbb{R}^n$, and let ω be a smooth n-form in $\overline{V}$. This means that ω is the restriction to $\overline{V}$ of a smooth n-form that is defined in a neighborhood of $\overline{V}$. We denote by $(x_1, \ldots, x_n)$ the coordinates in $\mathbb{R}^n$. With respect to these coordinates, the form ω can be written as

$\omega = f(x)dx_1 \wedge \ldots \wedge dx_n$ where $f : \mathbb{R}^n \to \mathbb{C}$ is a smooth function. Then, the integral of ω is defined as

$$\int_V \omega = \int_V f(x)dx. \tag{1.9}$$

Suppose that $(y_1, \ldots, y_n)$ is another coordinate system in V. It is induced by a diffeomorphism $\varphi : V' \to V$ where V' is also a domain in $\mathbb{R}^n$; the coordinates in V' are denoted by $(y_1, \ldots, y_n)$. The form ω can be represented in the coordinate system y – see (1.3); one can treat the resulting form as a form on V'. This form is denoted by $\varphi^*\omega$, and it is called the pull-back of ω with respect to φ. We have seen in Example 1.5 that

$$\varphi^*\omega = f(\varphi(y))\det(d_y\varphi)dy_1 \wedge \ldots \wedge dy_n. \tag{1.10}$$

Then

$$\begin{aligned}\int_{V'} \varphi^*\omega &= \int_{V'} f(\varphi(y))\det(d_y\varphi)dy_1 \ldots dy_n \\ &= \int_V f(x)\det\left(d_{\varphi^{-1}(x)}\varphi\right) \left|\det(d_x\varphi^{-1})\right| dx_1 \ldots dx_n.\end{aligned} \tag{1.11}$$

Notice that

$$\det(d_{\varphi^{-1}(x)}\varphi) \cdot \det(d_x\varphi^{-1}) = 1,$$

so

$$\det(d_{\varphi^{-1}(x)}\varphi) \left|\det\, d_x\varphi^{-1}\right| = \text{sgn}\,\det(d_y\varphi)$$

where $y = \varphi^{-1}(x)$. The determinant $\det(d_y\varphi)$ is of constant sign: it is a continuous, non-vanishing function on a connected domain. We call a diffeomorphism φ positive ($\text{sgn}\,\varphi = 1$) if $\text{sgn}\,\det(d_y\varphi) = 1$, and we call it negative ($\text{sgn}\,\varphi = -1$) if $\text{sgn}\,\det(d_y\varphi) = -1$. Now, we can re-write (1.11) as

$$\int_{V'} \varphi^*\omega = \text{sgn}\,\varphi \int_V \omega. \tag{1.12}$$

Formula (1.12) shows that the definition of the integral of a differential form is invariant under coordinate changes only up to a sign; it is invariant under positive coordinate changes.

We now turn to the definition of the integral of a differential form over a manifold. To make our exposition simpler, we assume the manifolds considered to be compact; one can easily make generalizations to the case of non-compact manifolds. First we need to introduce the notion of a manifold with boundary. Let M be a compact manifold with boundary. It means that one can cover M by a finite number of coordinate charts U_j, $j = 1, \dots, J$. Each chart is the image of a diffeomorphism $\varphi_j : U'_j \to U_j$ where here U'_j is an open bounded (contractible) subset of $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}$. In addition, for any two different charts U_j and U_ℓ, such that $U_j \cap U_\ell$ is non-empty, the transition map

$$\varphi_\ell^{-1} \circ \varphi_j : \varphi_j^{-1}(U_j \cap U_\ell) \to \varphi_\ell^{-1}(U_j \cap U_\ell)$$

is a diffeomorphism. M is called *orientable* if one can find an atlas consisting of coordinate charts $\varphi_j : U'_j \to U_j$ of the type above such that all transition maps $\varphi_\ell^{-1} \circ \varphi_j$ are positive. Not all manifolds are orientable. The Möbius strip, the Klein bottle, and even-dimensional real projective spaces are examples of non-orientable manifolds.

Let M be an orientable manifold. We say that an atlas is consistent if all the transition maps $\varphi_\ell^{-1} \circ \varphi_j$ are positive. Let $\{\varphi_j\}$ and $\{\tilde{\varphi}_\ell\}$ be two consistent atlases. Then the maps $\tilde{\varphi}_\ell^{-1} \circ \varphi_j$ are either all positive or all negative. In the first case, we call these two atlases equivalent. For an orientable manifold, there are two equivalence classes of consistent atlases. The choice of one of them is called an *orientation* of M. An orientable manifold, together with an orientation, is called an *oriented manifold*. For an oriented manifold, an atlas is *compatible with orientations* if it belongs to the equivalence class of the given orientation.

Let M be an oriented compact manifold of dimension n, and let $\{\varphi_j : U'_j \to U_j\}$ $(1 \leq j \leq J)$ be an atlas compatible with the orientation. Let $\{\eta_j\}$ be a partition of unity subordinated to the covering $\{U_j\}$ of M. This means that η_j are smooth functions with $0 \leq \eta_j(x) \leq 1$, such that $\operatorname{supp} \eta_j \subset U_j$ and $\sum_j \eta_j(x) = 1$. For a form $\omega \in \Omega^n(M)$, one defines

$$\int_M \omega = \sum_j \int_{U'_j} \varphi_j^*(\eta_j \omega). \tag{1.13}$$

We have to check that $\int_M \omega$ is independent of the choice of $\{\varphi_j\}$ and $\{\eta_j\}$. Let $\{\tilde{\varphi}_\ell : \tilde{U}'_\ell \to \tilde{U}_\ell\}$ $(1 \leq \ell \leq L)$ be another atlas compatible with the orientation of M, and let $\{\tilde{\eta}_\ell(x)\}$ be a partition of unity subordinated to $\tilde{U}_\ell$.

First,

$$\sum_j \int_{U'_j} \varphi_j^*(\eta_j \omega) = \sum_j \sum_\ell \int_{U'_j} \varphi_j^*(\eta_j \tilde{\eta}_\ell \omega).$$

The form $\eta_j \tilde{\eta}_\ell \omega$ is supported in $U_j \cap \tilde{U}_\ell$, so

$$\int_{U'_j} \varphi_j^*(\eta_j \tilde{\eta}_\ell \omega) = \int_{\varphi_j^{-1}(U_j \cap \tilde{U}_\ell)} \varphi_j^*(\eta_j \tilde{\eta}_\ell \omega).$$

The map $\varphi_j^{-1} \circ \tilde{\varphi}_\ell$ is a positive diffeomorphism $\tilde{\varphi}_\ell^{-1}(U_j \cap \tilde{U}_\ell) \to \varphi_j^{-1}(U_j \cap \tilde{U}_\ell)$, and

$$(\varphi_j^{-1} \circ \tilde{\varphi}_\ell)^*(\varphi_j^*(\eta_j \tilde{\eta}_\ell \omega)) = \tilde{\varphi}_\ell^*(\eta_j \tilde{\eta}_\ell \omega),$$

so, by (1.12),

$$\int_{\varphi_j^{-1}(U_j \cap \tilde{U}_\ell)} \varphi_j^*(\eta_j \tilde{\eta}_\ell \omega) = \int_{\tilde{\varphi}_\ell^{-1}(U_j \cap \tilde{U}_\ell)} \tilde{\varphi}_\ell^*(\eta_j \tilde{\eta}_\ell \omega).$$

Finally,

$$\begin{aligned} \sum_j \int_{U'_j} \varphi_j^*(\eta_j \omega) &= \sum_{j,\ell} \int_{U'_j} \varphi_j^*(\eta_j \tilde{\eta}_\ell \omega) \\ &= \sum_{j,\ell} \int_{\tilde{U}'_\ell} \tilde{\varphi}_\ell^*(\eta_j \tilde{\eta}_\ell \omega) \\ &= \sum_\ell \int_{\tilde{U}'_\ell} \tilde{\varphi}_\ell^*(\tilde{\eta}_\ell \omega). \end{aligned}$$

One of the basic theorems of the calculus of differential forms is Stokes's theorem. Stokes's theorem is the version of the Fundamental Theorem of calculus of real valued functions in one variable in the calculus of differential forms, and as special cases it contains Green's theorem, the Divergence theorem and the classical Stokes's theorem from vector calculus.

To formulate Stokes's theorem we first have to define the orientation of the boundary induced by the orientation of an orientable manifold with boundary. Let M be an oriented manifold with boundary. Let $\{\varphi_j : U'_j \to U_j\}, 1 \leq j \leq J$, be an atlas compatible with the orientation of M. Each domain U'_j is an open set in $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1} = \{x = (x_1, \ldots, x_n) : x_1 \geq 0\}$. Suppose that a domain U'_j has non-empty intersection with $\{0\} \times \mathbb{R}^{n-1} = \{x \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1} :$

$x_1 = 0\}$. We denote by B'_j the intersection of U'_j and $\{0\} \times \mathbb{R}^{n-1}$. Let ψ_j be the restriction of φ_j to B'_j. Then $\psi_j : B'_j \to B_j$ where $B_j = U_j \cap \partial M$ is an open subset of the boundary ∂M of M. Let $p \in \partial M$, and let U_j and U_ℓ be two coordinate neighborhoods in M that contain p. The Jacobi matrix of the diffeomorphism $\varphi_\ell^{-1} \circ \varphi_j$ at the point $p' = \varphi_j^{-1}(p)$ is of the form

$$\left(\begin{array}{c|ccc} c & 0 & \dots & 0 \\ \hline * & & & \\ \vdots & & (\psi_\ell^{-1} \circ \psi_j)^*(p') & \\ * & & & \end{array}\right)$$

where $c > 0$. Therefore,

$$\text{sgn}\ \det(d_{p'}(\psi_\ell^{-1} \circ \psi_j)) = \text{sgn}\ \det(d_{p'}(\varphi_\ell^{-1} \circ \varphi_j)) = 1.$$

We conclude that $\{\psi_j : B'_j \to B_j\}$ is a consistent atlas of ∂M. In particular, ∂M is an orientable manifold. The orientation of ∂M induced by the orientation of M is the orientation *opposite* to the orientation of the equivalence class of the atlas $\{\psi_j\}_{1 \le j \le J}$. Though this definition may not look natural at first glance, it turns out to be convenient as it leads to a positive sign in Stokes's theorem.

Now, we are in a position to formulate Stokes's theorem.

1.8 Theorem. *(Stokes's theorem) Let M be an oriented manifold of dimension n with boundary. Suppose that the boundary ∂M of M is taken with the induced orientation. Then, for any $\omega = \Omega^{n-1}(M)$,*

$$\int_M d\omega = \int_{\partial M} \omega. \tag{1.14}$$

1.9 Remark. In fact, the expression $\int_{\partial M} \omega$ is a shorthand notation. To be more precise, let $\iota : \partial M \to M$ be the inclusion. Then we simply write $\int_{\partial M} \omega$ instead of $\int_{\partial M} \iota^* \omega$.

Proof. The case where $\dim M = 1$ is more elementary and left to the reader. So let us assume that $\dim M \ge 2$. Let $\{\varphi_j : U'_j \to U_j\}$ $(1 \le j \le J)$ be an atlas of M compatible with the orientation of M, and let $\{\xi_j(x)\}$ be a partition of unity subordinated to $\{U_j\}$. Then the integral of $d\omega = \sum_j d(\xi_j \omega)$ is computed using (1.8) as follows

$$\int_M d\omega = \sum_j \int_{U'_j} \varphi_j^* \left(d(\xi_j \omega)\right) = \sum_j \int_{U'_j} d\left(\varphi_j^*(\xi_j \omega)\right).$$

Let

$$\eta_j = \varphi_j^*(\xi_j \omega) = \sum_{p=1}^n a_p(x) dx_1 \wedge \ldots \wedge \widehat{dx_p} \wedge \ldots \wedge dx_n,$$

where $a_p(x)$ is a smooth function with compact support in U'_j. Then

$$d\eta_j = \left(\sum_{p=1}^n (-1)^{p-1} \frac{\partial a_p(x)}{\partial x_p} \right) dx_1 \wedge \ldots \wedge dx_n.$$

In the case when $U'_j \cap (\{0\} \times \mathbb{R}^{n-1}) = \emptyset$,

$$\int_{U'_j} d\eta_j = \sum_{p=1}^n (-1)^{p-1} \int_{U'_j} \frac{\partial a_p(x)}{\partial x_p} dx_1 \ldots dx_n = 0$$

by Fubini's theorem and the Fundamental Theorem of Calculus. If $B'_j := U'_j \cap (\{0\} \times \mathbb{R}^{n-1}) \neq \emptyset$ then, arguing in the same way, one has for $2 \leq p \leq n$, $\int_{U'_j} \frac{\partial a_p}{\partial x_p} dx_1 \ldots dx_n = 0$ where as for $p = 1$, $\int_{U'_j} \frac{\partial a_1}{\partial x_1} dx_1 \ldots dx_n = -\int_{B'_j} a_1(x) dx_2 \ldots dx_n$. Hence

$$\int_{U'_j} d\eta_j = -\int_{B'_j} a_1(x) dx_2 \ldots dx_n.$$

The intersection B'_j is the image of $B_j = U_j \cap \partial M$ under the mapping $\psi_j^{-1} = \varphi_j^{-1}|_{B_j}$. Notice that the restriction of η_j to B'_j is $a_1(x) dx_2 \wedge \ldots \wedge dx_n$ (see Remark 1.9), and

$$\int_{B'_j} \eta_j = -\int_{B'_j} a_1(x) dx_2 \ldots dx_n.$$

Here, the "-" sign comes from our choice of the orientation of the boundary. Hence $\int_{U'_j} d\eta_j = \int_{B'_j} \eta_j$ and

$$\int_M d\omega = \sum_j \int_{U'_j} d\eta_j = \sum_j \int_{B'_j} \eta_j = \sum_j \int_{B'_j} \psi_j^*(\xi_j \omega).$$

The restrictions of the functions $\xi_j(x)$ to ∂M form a partition of unity on ∂M subordinated to the covering $\{B_j\}$ and formula (1.14) follows. ■

1.4 De Rham complex and its cohomology

Let M be a closed manifold. In section 1.2 we defined the spaces $\Omega^k(M)$ of differential forms of degree k on M and the operator $w \mapsto d\omega$, mapping a differential form ω on M to the differential $d\omega$ of this form. We denote the restriction of d to $\Omega^k(M)$ by d_k instead of again d,

$$d_k : \Omega^k(M) \to \Omega^{k+1}(M),$$

to indicate the degree of a differential form to which the operator d is applied. The first statement of Proposition 1.7 is that

$$d_k \circ d_{k-1} = 0. \tag{1.15}$$

It means that the spaces $\Omega^k(M)$ form a complex

$$0 \to \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} \Omega^n(M) \to 0$$

where we have used that $\Omega^k(M) = 0$ for any $k \geq n+1$. This complex is referred to as the de Rham complex of differential forms on M (with values in $\mathbb{C}$). A differential form $\omega \in \Omega^k(M)$ is called *closed* if $d_k\omega = 0$. It is called *exact* if $\omega = d_{k-1}\eta$ for some $\eta \in \Omega^{k-1}(M)$. Let us denote the vector space of closed differential forms of degree k by $Z^k(M) = Z^k_{dR}(M)$ and the vector space of exact differential forms of degree k by $B^k(M) = B^k_{dR}(M)$. Then (1.15) says that

$$B^k(M) \subseteq Z^k(M).$$

The quotient space $Z^k(M)/B^k(M)$ is called the space of the k-th de Rham cohomologies of the manifold M, and it is denoted by $H^k(M) = H^k_{dR}(M)$. (The subscript dR is often suppressed.) Note that $H^k(M)$ is a vector space and, as $\Omega^k(M) = \{0\}$ $\forall k \geq n+1$ it follows that $H^k(M) = \{0\}$ $\forall k \geq n+1$. The de Rham cohomologies has a clear analytic meaning. Suppose that a form $\omega \in \Omega^k(M)$ is given, and one wants to solve the equation

$$d_{k-1}\eta = \omega \tag{1.16}$$

for η. If one writes both η and ω in local coordinates, then (1.16) becomes a linear system of first order differential equations for the coefficients of η. This system is not always solvable. In view of (1.15) the condition

$$d_k\omega = 0 \tag{1.17}$$

is a necessary condition for the solvability of (1.16). Moreover, Poincaré's Lemma (see [MT, Theorem 3.15]) says that (1.17) is sufficient for the *local* solvability of (1.16). However, there might be additional obstructions for the *global* solvability of (1.16). The space $H^k_{dR}(M)$ is the space of these global obstructions. More precisely, given a closed k-form ω, the equation (1.16) has a solution $\eta \in \Omega^{k-1}(M)$ iff $[\omega] = 0$ where $[\omega]$ denotes the class in $H^k(M)$, with representative ω.

Let us illustrate the nature of global obstructions for the solvability of (1.17) with the following example. Let M be the two-dimensional torus, $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$. In the sequel we will identify differential forms on $\mathbb{T}^2$ with differential forms on $\mathbb{R}^2$ with 2π-doubly periodic coefficients. Consider the differential 1-form

$$\omega = f(x,y)dx + g(x,y)dy \in \Omega^1(\mathbb{R}^2),$$

where f and g are smooth 2π-periodic functions in both variables x and y. Suppose that one wants to find a smooth 2π-doubly periodic function $h(x,y)$ such that $\omega = dh$. It means that one wants to solve the system of equations

$$\frac{\partial h}{\partial x} = f, \qquad \frac{\partial h}{\partial y} = g. \tag{1.18}$$

One has

$$d\omega = \left(\tfrac{\partial g}{\partial x} - \tfrac{\partial f}{\partial y}\right) dx \wedge dy,$$

so $d\omega = 0$ means

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}. \tag{1.19}$$

The last condition is clearly necessary for the solvability of (1.18). However, to be able to find *periodic* solutions of (1.18), one needs, in addition, to require that both f and g have zero average over the torus (or, equivalently, in the representation of these functions as Fourier series, the constant term must vanish). The two averages

$$f_0 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x,y)dxdy \quad \text{and} \quad g_0 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x,y)dxdy$$

are the only global obstructions to the solvability of (1.18) on the torus $\mathbb{T}^2$. The space $H^1_{dR}(\mathbb{T}^2)$ is two-dimensional.

The exterior product of differential forms induces a product

$$\cdot : H^k(M) \times H^\ell(M) \to H^{k+\ell}(M)$$

defined as follows. Let $\alpha \in H^k(M)$ and $\beta \in H^\ell(M)$. Take forms $\omega_\alpha \in Z^k(M)$ and $\omega_\beta \in Z^\ell(M)$ that represent the classes α and β. Note that $\omega_\alpha \wedge \omega_\beta$ is closed as $d(\omega_\alpha \wedge \omega_\beta) = d\omega_\alpha \wedge \omega_\beta + (-1)^k \omega_\alpha \wedge d\omega_\beta$ and both ω_α and ω_β are closed. Hence $\omega_\alpha \wedge \omega_\beta \in Z^{k+\ell}(M)$. The cohomology class of $\omega_\alpha \wedge \omega_\beta$ does not depend on the choice of ω_α and ω_β. Indeed, let $\omega'_\alpha \in Z^k(M)$ and $\omega'_\beta \in Z^\ell(M)$ be other forms representing the same classes α and β, respectively. Then

$$\omega'_\alpha = \omega_\alpha + d\eta_\alpha \text{ and } \omega'_\beta = \omega_\beta + d\eta_\beta$$

where $\eta_\alpha \in \Omega^{k-1}(M)$ and $\eta_\beta \in \Omega^{\ell-1}(M)$. Using that ω_α and ω_β are closed one has

$$\begin{aligned} \omega'_\alpha \wedge \omega'_\beta - \omega_\alpha \wedge \omega_\beta &= d\eta_\alpha \wedge \omega_\beta + \omega_\alpha \wedge d\eta_\beta + d\eta_\alpha \wedge d\eta_\beta \\ &= d(\eta_\alpha \wedge \omega_\beta + (-1)^k \omega_\alpha \wedge \eta_\beta + \eta_\alpha \wedge d\eta_\beta) \in B^{k+\ell}(M). \end{aligned}$$

This means that the cohomology class of $\omega'_\alpha \wedge \omega'_\beta - \omega_\alpha \wedge \omega_\beta$ is zero. By definition, $\alpha \cdot \beta$ is the cohomology class of $\omega_\alpha \wedge \omega_\beta$. In this way the total space of cohomologies $H^\bullet(M) = \oplus_{k=0}^n H^k(M)$ becomes a *commutative graded algebra*. By (1.1) the commutativity property reads

$$\alpha \cdot \beta = (-1)^{k\ell} \beta \cdot \alpha \quad \forall \alpha \in H^k(M),\ \forall \beta \in H^\ell(M).$$

It descends from the similar property of the exterior product (1.1).

There are many cohomology theories. It turns out that for (closed) smooth manifolds the corresponding cohomology spaces coincide. As an example which will be reconsidered in Chapter 5, we discuss briefly the simplicial cohomology.

First let us briefly review the concept of a smooth triangulation of a closed manifold. [1] By σ^k we denote the standard k-dimensional simplex in $\mathbb{R}^{k+1}$,

$$\sigma^k = \{(x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} : x_i \geq 0 \ \forall i;\ x_1 + \ldots + x_{k+1} = 1\}.$$

Its interior is

$$\mathring{\sigma}^k = \{x \in \sigma^k : x_i > 0 \quad \forall i\}.$$

[1] In section **??** we will consider a special class of smooth triangulations

The simplex σ^k lies in the hyperplane

$$P_k = \{x = (x_1, \dots, x_{k+1}) \in \mathbb{R}^{k+1} : x_1 + \dots + x_{k+1} = 1\},$$

which is the boundary of the half-space

$$H_{k+1} = \{x \in \mathbb{R}^{k+1} : x_1 + \dots + x_{k+1} \le 1\}.$$

Note that H_{k+1} has a natural orientation, coming from the standard orientation from $\mathbb{R}^{k+1}$. Remark that the orientation of an affine subspace of $\mathbb{R}^{k+1}$ can also be characterized by a (equivalence class of) basis of the corresponding linear subspace. In this way the standard basis of $\mathbb{R}^{k+1}$ represents the standard orientation of $\mathbb{R}^{k+1}$. As orientation on $P_k = \partial H_{k+1}$ (and hence on σ^k) we choose the one represented by the basis $e_2 - e_1, \dots, e_{k+1} - e_1$ (or, equivalently, for $2 \le i \le k+1$, by the basis $e_i - e_1, \dots, e_i - e_{i-1}, e_{i+1} - e_i, \dots, e_{k+1} - e_i$). This orientation of σ^k is referred to as the canonical one. The boundary of σ^k consists of $k+1$ many simplices σ_j^{k-1} with $1 \le j \le k+1$. Here

$$\sigma_j^{k-1} = \{x \in \sigma^k : x_j = 0\}.$$

The simplex σ_j^{k-1} is the standard simplex in the space $\mathbb{R}^k$ if one identifies $\mathbb{R}^k$ with the image in $\mathbb{R}^{k+1}$ of the embedding

$$\iota_{k,j} : \mathbb{R}^k \to \mathbb{R}^{k+1}$$

defined by

$$\iota_{k,j}(y_1, \dots, y_k) = (y_1, \dots, y_{j-1}, 0, y_j, \dots, y_k).$$

The standard simplex σ^{k-1} is endowed with the canonical orientation, and the mapping

$$\iota_{k,j} : \sigma^{k-1} \to \sigma_j^{k-1}$$

then induces an orientation on σ_j^{k-1}. For $2 \le j \le k+1$, this orientation is represented by the basis $e_2 - e_1, \dots, e_{j-1} - e_1, e_{j+1} - e_1, \dots, e_{k+1} - e_1$ whereas for $j = 1$, it is represented by $e_3 - e_2, \dots, e_{k+1} - e_2$. On the other hand, σ_j^{k-1} is a part of the boundary of σ^k, and hence, according to section 1.3, has an orientation induced from σ^k. One can verify in straightforward way that for $2 \le j \le k+1$ the orientation of σ_j^{k-1} is represented by the basis $\mathcal{B}_{kj} = \{e_2 - e_1, \dots, e_{j-1} - e_1, e_{j+1} - e_1, \dots, \varepsilon_{kj}(e_{k+1} - e_2)\}$, where $\varepsilon_{kj} = \pm 1$ is chosen in such a way that $-(e_j - e_1)$, $\mathcal{B}_{kj}$ represents the standard basis of σ^k. Hence $\varepsilon_{kj} = (-1)^{j+1}$. In the case $j = 1$, the orientation of σ_1^{k-1} is represented by $e_3 - e_2, \ldots, e_{k+1} - e_2$. This means that the two orientations coincide if j is an odd number, and they are opposite to each other if j is even. As a result, for any $1 \leq k \leq n$, the oriented boundary $\partial\sigma^k$ of σ^k can be written as the union of $(k+1)$ many oriented $(k-1)$-dimensional standard simplices. We define

$$\partial\sigma^k := \sum_{j=1}^{k+1} (-1)^{j+1} \sigma_j^{k-1}. \tag{1.20}$$

For $2 \leq k \leq n$ and $i \neq j$, we set $\sigma_{ij}^{k-2} = \{x \in \sigma^k : x_i = x_j = 0\}$. It follows from (1.20) that

$$\begin{aligned} \partial(\partial\sigma^k) &= \sum_{j=1}^{k+1}\sum_{i=1}^{j-1}(-1)^{i+j}\sigma_{ij}^{k-2} + \sum_{j=1}^{k+1}\sum_{i=j+1}^{k+1}(-1)^{i+j-1}\sigma_{ij}^{k-2} \\ &= \sum_{i<j}(-1)^{i+j}\sigma_{ij}^{k-2} - \sum_{i>j}(-1)^{i+j}\sigma_{ij}^{k-2} \\ &= 0 \end{aligned} \tag{1.21}$$

because $\sigma_{ij}^{k-2} = \sigma_{ji}^{k-2}$.

Let φ be a homeomorphism of σ^k into a k-dimensional submanifold of a manifold M which is smooth. Then the image $\varphi(\sigma^k)$ is called a smooth k-simplex in M; the image $\varphi(\mathring{\sigma}^k)$ is a smooth open k-simplex in M. The homeomorphism φ defines not only a smooth k-simplex as a set; it also induces an orientation on $\varphi(\sigma^k)$. An oriented simplex is a simplex, together with its orientation. A (finite) smooth triangulation of a manifold M is a representation of M as a disjoint union

$$M = \bigsqcup_{k=0}^{n}\bigsqcup_{a=1}^{p_k} \mathring{\sigma}_a^k, \tag{1.22}$$

where $\mathring{\sigma}_a^k$ is a smooth open k-simplex in M, such that the (geometric) boundary of each $\mathring{\sigma}_a^k$ is the union of smooth open simplexes of smaller dimensions. It is known that every closed manifold admits a smooth triangulation – see [Ca] , [Wh], [Whit]. A triangulation will be called oriented if every simplex σ_a^k is endowed with an orientation (these orientations need not be compatible with each other).

Let (1.22) be an oriented smooth triangulation of M. For any $0 \le k \le n$, a k-chain in M (with coefficients in $\mathbb{C}$) is a formal linear combination

$$c = \sum_{a=1}^{p_k} c_a \sigma_a^k, \ \ c_a \in \mathbb{C}.$$

For any $0 \le k \le n$, the set of all k-chains form a linear space that is denoted by $C_k(M)$. The definition of the boundary operator for the standard k-simplex is used to define the boundary operator for σ_a^k. For any $1 \le k \le n$ we extend it by linearity to all of $C_k(M)$,

$$\partial_k : C_k(M) \to C_{k-1}(M).$$

Property (1.21) implies that for any $2 \le k \le n$

$$\partial_{k-1} \circ \partial_k = 0. \tag{1.23}$$

A cochain on M is a linear functional on $C_k(M)$. For any $0 \le k \le n$, the space of cochains is denoted by $C^k(M)$. The transpose of ∂_k is the operator denoted by $(1 \le k \le n)$

$$\delta_{k-1} : C^{k-1}(M) \to C^k(M)$$

and is called the coboundary operator. By duality, for any $1 \le k \le n$

$$\delta_k \circ \delta_{k-1} = 0. \tag{1.24}$$

In this way one obtains the cochain complex

$$0 \to C^0(M) \stackrel{\delta_0}{\to} C^1(M) \to \ldots \stackrel{\delta_n}{\to} C^n(M) \to 0.$$

A cocycle is a cochain with zero coboundary; the space of k-cocycles is denoted by $Z^k(M)$. A k-coboundary is a cochain that lies in the range of δ_{k-1}; the space of k-coboundaries is denoted by $B^k(M)$. Identity (1.24) is saying that

$$B^k(M) \subset Z^k(M).$$

The space of k-th simplicial cohomologies of M is defined as

$$H^k(M) = Z^k(M)/B^k(M).$$

A differential form $\omega \in \Omega^k(M)$ defines a k-cochain $\int \omega$ by the formula

$$\langle \int \omega, \sum_{a=1}^{p_k} c_a \sigma_a^k \rangle = \sum_{a=1}^{p_k} c_a \int_{\sigma_a^k} \omega \tag{1.25}$$

where $\int_{\sigma_a^k} \omega$ is defined as the integral over σ^k of the pullback of ω to the standard k-simplex σ^k – see section 1.3. Suppose that $d\omega = 0$. Then, by the definition of δ_{k+1} and (1.25), one has for any element $\sum_{a=1}^{p} c_a \sigma_a^{k+1}$ in $C^{k+1}(M)$

$$\langle \delta_k \int \omega, \sum_{a=1}^{p_{k+1}} c_a \sigma_a^{k+1} \rangle = \langle \int \omega, \sum_{a=1}^{p_{k+1}} c_a \partial \sigma_a^{k+1} \rangle = \sum_{a=1}^{p_k} c_a \int_{\partial \sigma_a^{k+1}} \omega.$$

Hence by Stokes's theorem

$$\langle \delta_k \int \omega, \sum_{a=1}^{p} c_a \sigma_a^{k+1} \rangle = \sum_{a=1}^{p_k} c_a \int_{\sigma_a^{k+1}} d\omega = 0.$$

Therefore, we have proved that for any $\omega \in Z_{dR}^k(M)$,

$$\int \omega \in Z^k(M). \tag{1.26}$$

Similarly, if $\omega = d\eta, \eta \in \Omega^{k-1}(M)$ then, arguing as above, we get

$$\begin{aligned} \langle \int \omega, \sum_{a=1}^{p_k} c_a \sigma_a^k \rangle &= \sum_{a=1}^{p_k} c_a \int_{\sigma_a^k} \omega = \sum_{a=1}^{p_k} c_a \int_{\sigma_a^k} d\eta \\ &= \sum_a c_a \int_{\partial \sigma_a^k} \eta = \langle \int \eta, \partial \left(\sum_{a=1}^{p_k} c_a \sigma_a^k \right) \rangle \\ &= \langle \delta \int \eta, \sum_{a=1}^{p_k} c_a \sigma_a^k \rangle. \end{aligned}$$

Hence we have shown that for $\omega \in B_{dR}^k(M)$, one has $\int \omega \in B^k(M)$. Combined with (1.26) it then follows that the integration of smooth forms over smooth simplices induces for any $0 \leq k \leq n$ a map

$$\int^k : H_{dR}^k(M) \to H^k(M).$$

De Rham's Theorem (see e.g. [Wa]) says that the maps $\int^k$ are isomorphisms. As a consequence, the spaces of de Rham cohomologies are finite dimensional and the spaces $H^k(M)$ are independent of the choice of triangulation.

We saw that the exterior product of differential forms induces the structure of a graded commutative algebra on $H^\bullet_{dR}(M)$. The isomorphism induced by the integration map $\int^k$ between $H^k_{dR}(M)$ and $H^k(M)$ pushes this structure over to $H^\bullet(M)$. One can define a product on $H^\bullet(M)$ in a purely combinatorial way; then $\int^\bullet$ becomes an isomorphism of algebras.

1.5 Hodge decomposition

In this section we review the classical Hodge theory. A detailed exposition can be found e.g. in [Wa].

Let M be a closed manifold of dimension n. We endow M with a Riemannian metric g. It means that at any point $x \in M, g(x)$ is a scalar product $(\cdot,\cdot)$ on T_xM depending smoothly on x. Given local coordinates $x_1,\ldots,x_n$, let

$$g_{ij}(x) := \left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)\ (1 \leq i,j \leq n).$$

Then the functions $g_{ij}(x)$ are smooth, and the matrix $(g_{ij}(x))$ is positive definite. Using $(\cdot,\cdot)$ to identify T_xM with its dual T^*_xM, the scalar product on T_xM induces a scalar product on the cotangent space T^*_xM; in local coordinates,

$$(dx_i, dx_j) = g^{ij}(x)$$

where the matrix $(g^{ij}(x))$ is the inverse of the matrix $(g_{ij}(x))$. This scalar product extends to the scalar product on the tensor powers $(T^*_xM)^{\otimes k}$ by the formula

$$(\xi_1 \otimes \ldots \otimes \xi_k, \eta_1 \otimes \ldots \otimes \eta_k) = \frac{1}{k!}(\xi_1,\eta_1)\cdots(\xi_k,\eta_k)$$

for arbitrary $\xi_i, \eta_i \in T^*_xM$ $(1 \leq i \leq k)$. The normalization factor $1/k!$ in the above formula is introduced to make formulas below simpler. The space $\Lambda^k_x(M)$ is a subspace of $(T^*_xM)^{\otimes k}$ consisting of antisymmetric tensors. In particular, the element $dx_{i_1} \wedge \ldots \wedge dx_{i_k} \in \Lambda^k_x(U')$, defined on anopen subset $U' \subseteq \mathbb{R}^n$ is given by $\sum_{\sigma \in S_k}(-1)^{\mathrm{sgn}\sigma} dx_{i_{\sigma(1)}} \otimes \ldots \otimes dx_{i_{\sigma(k)}}$. The metric g

induces a scalar product in each $\Lambda_x^k(M)$ which we also denote by $(\cdot,\cdot)$. In local coordinates, it is given by the formula

$$
\begin{aligned}
&(dx_{i_1}\wedge\ldots\wedge dx_{i_k}, dx_{j_1}\wedge\ldots\wedge dx_{j_k})\\
&=\sum_{\sigma\in S_k}(-1)^{\operatorname{sgn}\sigma} g^{i_1 j_{\sigma(1)}}(x)\cdots g^{i_k j_{\sigma(k)}}(x)
\end{aligned}
\tag{1.27}
$$

where S_k is the symmetric group of k elements. In addition, a Riemannian metric $(g_{ij}(x))$ induces a measure vol_g on M. In local coordinates it is defined by

$$
d\mathrm{vol}_g=\sqrt{\det(g_{ij}(x))}\;dx_1\ldots dx_n.
$$

It is easy to check that neither the scalar product (1.27) nor the measure vol_g depend on the choice of local coordinates.

The scalar product on the fibers $\Lambda_x^k M$ of the bundle $\Lambda^k(M)$, together with the measure vol_g, give rise to a scalar product on the space of differential forms $\Omega^k(M)$, referred to as the L^2-inner product

$$
(\omega_1,\omega_2)=\int_M(\omega_1(x),\omega_2(x))d\mathrm{vol}_g. \tag{1.28}
$$

The norm corresponding to the L^2-inner product is called the L^2-norm of a k-form ω and denoted by $\|\omega\|, \|\omega\|=(\omega,\omega)^{1/2}$. The completion of $\Omega^k(M)$ with respect to the scalar product $(\cdot,\cdot)$ is the Hilbert space of L^2-forms; we denote it by $L_2^k(M)$. The operator d_k, defined initially on the space $\Omega^k(M)$ of smooth forms is a closable operator, if viewed as an (unbounded) operator from $L_2^k(M)$ to $L_2^{k+1}(M)$; its closure will be also denoted by d_k. Let $d_k^*: L_2^{k+1}(M)\to L_2^k(M)$ be the adjoint to d_k. This is an unbounded operator with the domain

$$
\{\omega\in L_2^{k+1}(M):\exists c>0\ \text{ such that }\ |(d_k\eta,\omega)|\le c\|\eta\|\ \forall\eta\in\Omega^k(M)\}.
$$

Here $\|\eta\|$ is the L^2-norm introduced above of the k-form η .

Let us find an expression for d_k^* in local coordinates. We will use notations from section 1.2. Recall that in local coordinates,

$$
d=\sum_{j=1}^n\Lambda_j\frac{\partial}{\partial x_j}=\sum_{j=1}^n\frac{\partial}{\partial x_j}\Lambda_j \tag{1.29}
$$

hence

$$d^* = \sum_{j=1}^n \left(\frac{\partial}{\partial x_j}\right)^* \Lambda_j^* = \sum_{j=1}^n \Lambda_j^* \left(\frac{\partial}{\partial x_j}\right)^*. \tag{1.30}$$

We view Λ_j and $\frac{\partial}{\partial x_j}$ as a linear operator $\Omega^k(U) \to \Omega^{k+1}(U)$ and $\Omega^k(U) \to \Omega^k(U)$ respectively where U is a coordinate chart of M. We start with analyzing Λ_j^*. The operator Λ_j is a linear operator $\Lambda^k(U) \to \Lambda^{k+1}(U)$, so Λ_j^* is a linear operator $\Lambda^{k+1}(U) \to \Lambda^k(U)$. Take local coordinates near $x_0 \in M$ such that $g_{ab}(x_0) = \delta_{ab}$. Then for any $1 \leq \ell \leq n$ the tensors $dx_{i_1} \wedge \ldots \wedge dx_{i_\ell}$ form an orthonormal basis in $\Lambda_{x_0}^\ell(U)$ with respect to the scalar product given by (1.27). An element $\omega \in \Lambda_{x_0}^{k+1}(U)$ can be represented as

$$\omega = \omega_0 + dx_j \wedge \omega_1$$

where $\omega_0 \in \Lambda_{x_0}^{k+1}(U)$ and $\omega_1 \in \Lambda_{x_0}^k(U)$ do not contain dx_j. Clearly, for $\eta \in \Lambda_{x_0}^k(U)$,

$$(\eta, \Lambda_j^* \omega) = (dx_j \wedge \eta, \omega) = \begin{cases} (\eta, \omega_1) & \text{if } \eta \text{ does not contain } dx_j. \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, at x_0, $\Lambda_j^* \omega = \omega_1 = \iota_j \omega$ where ι_j is the operator of interior multiplication by $\partial/\partial x_j$, i.e.

$$(\iota_j \omega)(X_1, \ldots, X_k) = \omega\left(\frac{\partial}{\partial x_j}, X_1, \ldots, X_k\right)$$

for any elements $X_1, \ldots, X_k \in T_{x_0}M$. (For convenience we set $i_j \big|_{\Lambda^0(U)} = 0$.) In an arbitrary coordinate system

$$\Lambda_j^* = \sum_{i=1}^n g^{ji} \iota_i. \tag{1.31}$$

Let us turn our attention to computing $(\frac{\partial}{\partial x_j})^*$. Let ω and η be in $\Omega^k(M)$ with support in the coordinate neighborhood U of M. Written in coordinates in multi-index notation, ω and η are of the form $\omega = \sum_{|I|=k} \omega_I dx_I$ and $\eta = \sum_{|J|=k} \eta_J dx_J$. Then

$$(\partial_{x_j}\omega, \eta) = \sum (-1)^{\mathrm{sgn}\sigma} \int_{\mathbb{R}^n} \partial_{x_j}(\omega_I) \overline{\eta_J} g^{i_1 j_{\sigma(1)}} \cdots g^{i_k j_{\sigma(k)}} \sqrt{\det(g_{ij})} dx_1 \ldots dx_n \tag{1.32}$$

where the sum is taken over all permutations $\sigma \in S_k$ and all k-tuples $I = \{i_1 < \ldots < i_k\}, J = \{j_1 < \ldots < j_k\}$. When one integrates by parts in (1.32), one gets three terms. The first one in which the components η_J are differentiated, equals

$$-(\omega, \partial_{x_j}\eta).$$

The second one, in which $|g| := \det(g_{ij})$ is differentiated, equals

$$-\frac{1}{2}(\omega, \partial_{x_j}(\log|g|)\eta).$$

The third term equals

$$-\sum(-1)^{\mathrm{sgn}\sigma}\int_{\mathbb{R}^n}\omega_I\eta_J g^{i_1 j_{\sigma(1)}}\cdots\partial_{x_j}(g^{i_p j_{\sigma(p)}})\cdots g^{i_k j_{\sigma(k)}}\sqrt{|g|}\,dx_1\ldots dx_n$$

where the sum extends over $1 \le p \le k$, all permutations $\sigma \in S_k$, and all k-tuples $I = \{i_1 < \ldots < i_k\}, J = \{j_1 < \ldots < j_k\}$. It is not difficult to show that the latter expression is equal to

$$-\sum_{a,b}\int_{\mathbb{R}^n}(\iota_a\omega, \iota_b\eta)\partial_{x_j}(g^{ab})\sqrt{|g|}\,dx_1\ldots dx_n.$$

Combining the computations above we get in arbitrary local coordinates

$$\left(\frac{\partial}{\partial x_j}\right)^* = -\frac{\partial}{\partial x_j} - \frac{1}{2}\partial_{x_j}(\log|g|) - \sum_{a,b}\partial_{x_j}(g^{ab})\iota_a^*\iota_b. \tag{1.33}$$

It remains to compute ι_a^* in formula (1.33) Taking the adjoint of both sides of (1.31), one gets

$$\Lambda_j = \sum_\ell g^{j\ell}\iota_\ell^*.$$

Therefore,

$$\iota_a^* = \sum g_{a\ell}\Lambda_\ell. \tag{1.34}$$

Substituting (1.34) into (1.33), we finally obtain

$$\left(\frac{\partial}{\partial x_j}\right)^* = -\frac{\partial}{\partial x_j} - \frac{1}{2}\partial_{x_j}(\log|g|) - \sum_{a,b,\ell}\partial_{x_j}(g^{ab})g_{a\ell}\Lambda_\ell\iota_b, \tag{1.35}$$

and hence, in local coordinates,

$$\begin{aligned} d^* = -\sum_{j,m} g^{jm}\iota_m \frac{\partial}{\partial x_j} - \frac{1}{2}\sum_{j,m} g^{jm}\partial_{x_j}(\log|g|)\iota_m \\ - \sum_{j,a,b,\ell,m} g^{jm}\partial_{x_j}(g^{ab})g_{a\ell}\iota_m\Lambda_\ell\iota_b. \end{aligned} \tag{1.36}$$

1.10 Remark. In case the manifold M is *orientable*, the operator d_k^* can be defined in terms of the Hodge $*$ operator

$$*_k : \Omega^k(M) \to \Omega^{n-k}(M)$$

associated to a volume form. Recall that a volume form on M is a smooth n-form $\nu(x)$ such that $(\nu(x), \nu(x)) = 1$ for any $x \in M$. Using a partition of unity subordinate to a covering of M by open coordinate neighborhoods and using the assumption that M is orientable one can easily show that such a volume form always exists. For $\beta \in \Omega^k(M)$, the form $*_k\beta \in \Omega^{n-k}(M)$ is defined in such a way that

$$(\alpha(x), \beta(x))\nu(x) = \alpha(x) \wedge (*_k\beta)(x) \tag{1.37}$$

for every point $x \in M$ and for every form $\alpha \in \Omega^k(M)$. It follows that

$$(\alpha, \beta) = \int_M \alpha \wedge *_k\beta$$

where $\int_M \alpha \wedge *_k\beta$ is the integral of the n-form $\alpha \wedge *_k\beta$ with respect to the orientation determined by the volume form ν. One can easily check that

$$*_{n-k}*_k = (-1)^{k(n-k)}. \tag{1.38}$$

(Note that if n is odd then $(-1)^{k(n-k)} = 1 \ \forall 0 \le k \le n$.) To do that choose local coordinates near an arbitrary point $x_0 \in M$ such that $g_{ij}(x_0) = \delta_{ij}$. The above identity then follows from straightforward combinatorics.

Given a form $\gamma \in \Omega^k(M)$ we get by (1.38) for any $\alpha \in \Omega^{k-1}(M)$

$$(\alpha, d^*\gamma) = (d\alpha, \gamma) = \int_M d\alpha \wedge (*_k\gamma) = (-1)^k \int_M \alpha \wedge d(*_k\gamma)$$

where for the latter identity we used Stokes's theorem and Leibniz rule (Proposition 1.7 (ii)). By (1.38) we then get

$$\int_M \alpha \wedge d *_k \gamma = (-1)^{(k-1)(n-k+1)} \int_M \alpha \wedge *_{k-1}(*_{n-k+1} d *_k \gamma)$$
$$= (-1)^{(k-1)(n-k+1)}(\alpha, (*_{n-k+1} d *_k)\gamma).$$

Combining the two identities displayed above it follows that for any $\alpha \in \Omega^{k-1}(M)$,

$$(\alpha, d^*\gamma) = (-1)^{k+(k-1)(n-k+1)}(\alpha, *_{n-k+1} d *_k \gamma).$$

As

$$k + (k-1)(n-k+1) \equiv n(k-1) + 1 \pmod 2$$

we obtain the formula

$$d_k^* = (-1)^{n(k-1)+1} *_{n-k+1} d_{n-k} *_k .$$

The Laplacian acting on k-forms is defined by the formula

$$\Delta_k = d_k^* \circ d_k + d_{k-1} \circ d_{k-1}^*. \tag{1.39}$$

It is a second order differential operator; according to the formulas (1.29) and (1.36) for d_k and d_k^*, respectively, its leading part, i.e. the sum of all terms containing second derivatives, equals in arbitrary local coordinates

$$-\sum_{m,p} g^{jm}(\iota_m \Lambda_p + \Lambda_p \iota_m)\frac{\partial^2}{\partial x_j \partial x_p}. \tag{1.40}$$

Let us compute the operator $\iota_m \Lambda_p + \Lambda_p \iota_m$ in the expression above. If $m \neq p$ then a form $\omega \in \Omega^k(U)$ can be represented as

$$\omega = \omega_0 + dx_m \wedge \omega_1 + dx_p \wedge \omega_2 + dx_m \wedge dx_p \wedge \omega_3$$

where the forms $\omega_0, \omega_1, \omega_2, \omega_3$ do not contain dx^m and dx^p. One has

$$\Lambda_p \omega = dx_p \wedge \omega_0 + dx_p \wedge dx_m \wedge \omega_1,$$
$$\iota_m \Lambda_p \omega = -dx_p \wedge \omega_1,$$

and

$$\iota_m \omega = \omega_1 + dx_p \wedge \omega_3,$$
$$\Lambda_p \iota_m \omega = dx_p \wedge \omega_1.$$

Therefore, in the case $m \neq p$, $(\iota_m \Lambda_p + \Lambda_p \iota_m)\omega = 0$ for every form ω.
If $p = m$ then, for a form

$$\omega = \omega_0 + dx_p \wedge \omega_1,$$

one gets

$$(\iota_p \Lambda_p + \Lambda_p \iota_p)\omega = \iota_p(dx_p \wedge \omega_0) + \Lambda_p \omega_1 = \omega_0 + dx_p \wedge \omega_1 = \omega.$$

We conclude that

$$\iota_m \Lambda_p + \Lambda_p \iota_m = \begin{cases} 0 & \text{if } p \neq m \\ Id_k & \text{if } p = m \end{cases} \tag{1.41}$$

where Id_k is the identity operator on $\Omega^k(U)$. Substituting (1.41) into (1.40) it follows that for any $0 \leq k \leq n$ in arbitrary local coordinates the *leading part* of the Laplacian Δ_k equals

$$-\sum_{j,m} g^{jm}(x) \frac{\partial^2}{\partial x_j \partial x_m} \tag{1.42}$$

1.11 Remark. The Laplacian acting on functions: If $f(x) \in C^\infty(M) = \Omega^0(M)$, then in local coordinates, $\Delta_0 f(x) = d_0^* d_0 f$ can be computed as follows

$$\begin{aligned} &-\sum_{j,m} g^{jm} \partial_{x_j} \partial_{x_m} f - \frac{1}{2} \sum_{j,m} g^{jm} \partial_{x_j} (\log|g|) \partial_{x_m} f \\ &- \sum_{j,a,b,\ell,m,p} g^{jm} \partial_{x_j} (g^{ab}) g_{a\ell} \iota_m \Lambda_\ell \iota_b \Lambda_p \partial_{x_p} f. \end{aligned} \tag{1.43}$$

Notice that, for a function $h(x)$,

$$\iota_b \Lambda_p h = \begin{cases} h & \text{if } b = p; \\ 0 & \text{otherwise}, \end{cases}$$

so the last term in (1.43) equals

$$\begin{aligned} \sum_{j,m,a,p} g^{jm} \partial_{x_j} (g^{ap}) g_{am} \partial_{x_p} f &= \sum_{j,a,p} \delta_{ja} \partial_{x_j} (g^{ap}) \partial_{x_p} f \\ &= \sum_{j,p} \partial_{x_j} (g^{jp}) \partial_{x_p} f. \end{aligned}$$

As a consequence,

$$\begin{aligned}\Delta_0 &= -\sum_{j,m}\left[g^{jm}\frac{\partial^2}{\partial x_j \partial x_m} + \frac{1}{2}g^{jm}\partial_{x_j}(\log|g|)\frac{\partial}{\partial x_m} + \partial_{x_j}(g^{jm})\frac{\partial}{\partial x_m}\right] \\ &= -\frac{1}{\sqrt{|g|}}\sum_{j,m}\frac{\partial}{\partial x_j}\left(g^{jm}\sqrt{|g|}\frac{\partial}{\partial x_m}\right).\end{aligned} \tag{1.44}$$

The *principal symbol* of the Laplacian Δ_k is given for any $\xi \in T_x^*M, x \in M$ by

$$\|\xi\|^2 Id_k$$

where in local coordinates, $\|\xi\| = (\sum_{j,m} g^{jm}(x)\xi_j\xi_m)^{1/2}$ is the norm defined by (1.27) with ξ_j denoting the components of ξ with respect to the basis $dx_1, \ldots, dx_n$ of T_x^*M and where $Id_k : \Lambda^k(M) \to \Lambda^k(M)$ is the identity operator. (By definition, to form the principal symbol in local coordinates one replaces $\partial/\partial x_j$ by $\sqrt{-1}\ \xi_j$ in the leading part of the Laplacian (1.42) – see Appendix A for further details.) As the principal symbol is an invertible linear operator on $\Lambda^k(M)$ when $\xi \neq 0$, Δ_k is by definition an elliptic operator. Moreover, Δ_k is non-negative with respect to the L^2-inner product (1.28) because

$$\begin{aligned}(\Delta_k\omega, \omega) &= (d_k^* \circ d_k\omega, \omega) + (d_{k-1} \circ d_{k-1}^*\omega, \omega) \\ &= \|d_k\omega\|^2 + \|d_{k-1}^*\omega\|^2 \geq 0\end{aligned} \tag{1.45}$$

and it is an unbounded symmetric operator on $L_2^k(M)$. It is closable and its closure is selfadjoint and also denoted by Δ_k. Let us summarize a few properties of the operator Δ_k which follow from the theory of elliptic differential operators – see Appendix A for more explanations.

(EDO1) the null space $\mathcal{H}^k(M) = \mathrm{Ker}\Delta_k$ of the operator Δ_k is finite dimensional, and it consists of smooth forms;

(EDO2) for every form $\omega \in L_2^k(M)$ that is orthogonal to $\mathcal{H}^k(M)$, the equation

$$\Delta_k\eta = \omega \tag{1.46}$$

has the unique solution η in $L_2^k(M)$ which is orthogonal to $\mathcal{H}^k(M)$ and there exists a constant C independent of ω such that the solution η satisfies the estimate $\|\eta\| \leq C\|\omega\|$. Combined with (1.46) this leads to

$$\|\Delta_k\eta\| + \|\eta\| \leq (C+1)\|\omega\|. \tag{1.47}$$

With a slight abuse of notation, we denote by Δ_k^{-1} the operator defined on the orthogonal complement of $\mathcal{H}^k(M)$, $L_2^k(M) \ominus \mathcal{H}^k(M)$ that is the inverse of the restriction of Δ_k to this space. Inequality (1.47) means that Δ_k^{-1} is a bounded operator.

Let $Z_2^k(M)$ be the kernel of the operator $d_k : L_2^k(M) \to L_2^{k+1}(M)$, and let $B_2^k(M)$ be the range of the operator $d_{k-1} : L_2^{k-1}(M) \to L_2^k(M)$. The difference between $Z^k(M), B^k(M)$ and $Z_2^k(M), B_2^k(M)$ is that the latter spaces contain non-smooth forms. The subspace $Z_2^k(M) \subseteq L_2^k(M)$ is closed as it is the kernel of a closed operator. By $\tilde{Z}_2^k(M)$ we denote the null-space of the operator $d_{k-1}^* : L_2^k(M) \to L_2^{k-1}(M)$, and by $\tilde{B}_2^k(M)$ we denote the range of the operator $d_k^* : L_2^{k+1}(M) \to L_2^k(M)$. The subspace $\tilde{Z}_2^k(M) \subseteq L_2^k(M)$ is also closed.

1.12 Proposition. *The following statements hold:*

(i) $\mathcal{H}^k(M) = Z_2^k(M) \cap \tilde{Z}_2^k(M)$;

(ii) $B_2^k(M)$ *is orthogonal to* $\tilde{Z}_2^k(M)$ *and* $\tilde{B}_2^k(M)$ *is orthogonal to* $Z_2^k(M)$;

(iii) *the subspaces* $B_2^k(M), \tilde{B}_2^k(M) \subset L_2^k(M)$ *are closed;*

(iv) $L_2^k(M) = Z_2^k(M) \oplus \tilde{B}_2^k(M) = \tilde{Z}_2^k(M) \oplus B_2^k(M)$.

Proof. (i) The claimed identity follows from (1.45).
(ii) Let $\omega = d_{k-1}\eta \in B_2^k(M)$ and $\alpha \in \tilde{Z}_2^k(M)$. Then

$$0 = (d_{k-1}^*\alpha, \eta) = (\alpha, d_{k-1}\eta) = (\alpha, \omega).$$

Similarly, let $\omega = d_k^*\eta \in \tilde{B}_2^k(M)$ and $\alpha \in Z_2^k(M)$. Then

$$0 = (d_k\alpha, \eta) = (\alpha, d_k^*\eta) = (\alpha, \omega).$$

(iii) Let $\omega_j = d_{k-1}\eta_j \in B_2^k(M)$, and suppose that $\omega_j \to \omega$ in $L_2^k(M)$. Clearly, one can choose η_j to be orthogonal to $Z_2^{k-1}(M)$. Then $\eta_j \in \tilde{Z}_2^{k-1}(M)$. Indeed, let $\alpha \in \Omega^{k-1}(M)$ be a smooth form that is orthogonal to $Z_2^{k-1}(M)$. Then, for any smooth form $\beta \in \Omega^{k-2}(M)$,

$$(d_{k-1}^*\alpha, \beta) = (\alpha, d_{k-2}\beta) = 0$$

because $B^{k-1}(M) \subset Z^{k-1}(M)$. The space $\Omega^{k-2}(M)$ is dense in $L_2^{k-2}(M)$, hence $d_{k-1}^*\alpha = 0$. Because $\tilde{Z}_2^k(M)$ is closed, we conclude that

$$Z_2^{k-1}(M)^\perp \subset \tilde{Z}_2^{k-1}(M). \tag{1.48}$$

This proves that $\eta_j \in \tilde{Z}_2^{k-1}(M)$ and therefore $d_{k-2}d_{k-2}^*\eta_j = 0$. According to (EDO2) and item (ii), ω_j is in the domain of Δ_{k-1}. Therefore, we can apply the operator d_{k-1}^* to both sides of the equation $\omega_j = d_{k-1}\eta_j$ to get

$$d_{k-1}^*\omega_j = d_{k-1}^*d_{k-1}\eta_j = (d_{k-1}^*d_{k-1} + d_{k-2}d_{k-2}^*)\eta_j = \Delta_{k-1}\eta_j.$$

Therefore,

$$\eta_j = \Delta_{k-1}^{-1}d_{k-1}^*\omega_j.$$

It follows from (1.47) that the sequence η_j converges to a form $\eta \in L_2^{k-1}(M)$. The operator d_{k-1} is closed, therefore, η belongs to its domain, and $d_{k-1}\eta = \omega$.

The proof of the fact that the space $\tilde{B}_2^k(M)$ is closed is similar.

(iv) We have already shown that $\tilde{B}_2^k(M)$ is orthogonal to $Z_2^k(M)$, and that $\tilde{B}_2^k(M)$ is closed. To prove that $L_2^k(M) = Z_2^k(M) \oplus \tilde{B}_2^k(M)$, it is sufficient to show that $Z_2^k(M) \supset \tilde{B}_2^k(M)^\perp$. Suppose that $\omega \in \tilde{B}_2^k(M)^\perp$. By (i), ω is in the domain of Δ_k, hence of d_k. Then, for any form $\eta \in \Omega^{k+1}(M)$, one has

$$(d_k\omega, \eta) = (\omega, d_k^*\eta) = 0.$$

The space $\Omega^{k+1}(M)$ is dense in $L_2^{k+1}(M)$, so $d_k\omega = 0$. The proof that $L_2^k(M) = \tilde{Z}_2^k(M) \oplus B_2^k(M)$ is similar. ■

The space of exact smooth k-forms, $B^k(M)$ is dense in $B_2^k(M)$, and $B^k(M) \subset Z^k(M)$. Therefore, $B_2^k(M) \subset Z_2^k(M)$. Then, by Proposition 1.12,

$$\begin{aligned} Z_2^k(M) \ominus B_2^k(M) &= Z_2^k(M) \cap B_2^k(M)^\perp \\ &= Z_2^k(M) \cap \tilde{Z}_2^k(M) = \mathcal{H}^k(M), \end{aligned}$$

and, therefore, by Proposition 1.12 (iv)

$$L_2^k(M) = Z_2^k(M) \oplus \tilde{B}_2^k(M) = B_2^k(M) \oplus \mathcal{H}^k(M) \oplus \tilde{B}_2^k(M). \tag{1.49}$$

The decomposition (1.49) of the space $L_2^k(M)$ is called the Hodge decomposition. Any form $\omega \in L_2^k(M)$ can be represented as

$$\omega = d\alpha + \gamma + d^*\beta \tag{1.50}$$

where $\alpha \in L_2^{k-1}(M), \beta \in L_2^{k+1}(M)$, and $\gamma \in \mathcal{H}^k(M)$. The forms α and β are defined uniquely modulo $Z_2^{k-1}(M)$ and $\tilde{Z}_2^{k+1}(M)$, respectively and the form γ is defined uniquely. To define α and β in a unique way, one may require, in addition, $\alpha \in Z_2^{k-1}(M)^\perp$ and $\beta \in \tilde{\mathcal{Z}}_2^{k+1}(M)^\perp$. Then $d^*_{k-2}\alpha = 0$ and $d_{k+1}\beta = 0$. To compute α and β in terms of ω, apply the operator d_k to both sides of (1.50) to get

$$d\omega = dd^*\beta = (dd^* + d^*d\beta) = \Delta\beta,$$

and

$$\beta = \Delta_{k+1}^{-1} d_k \omega. \tag{1.51}$$

Similarly,

$$d^*\omega = d^*d\alpha = \Delta\alpha,$$

and

$$\alpha = \Delta_{k-1}^{-1} d^*\omega. \tag{1.52}$$

Ellipticity of the Laplacian implies that the forms α and β are smooth if the form ω is smooth. Therefore

$$\begin{aligned} \Omega^k(M) &= B^k_{dR}(M) \oplus \mathcal{H}^k(M) \oplus \tilde{B}^k_{dR}(M) \\ &= d\Omega^{k-1}(M) \oplus \mathcal{H}^k(M) \oplus d^*\Omega^{k+1}(M). \end{aligned} \tag{1.53}$$

The decomposition (1.53) is also called the Hodge decomposition. Notice that $Z^k_{dR}(M) = B^k_{dR}(M) \oplus \mathcal{H}^k(M)$, so the space $\mathcal{H}^k(M)$ is a realization of k-th de Rham cohomologies. Each cohomology class contains a unique harmonic form.

The de Rham theorem says that de Rham cohomologies are isomorphic to simplicial cohomologies. In particular, de Rham cohomologies are finite dimensional. Now we see another reason, why they have finite dimension: the space of harmonic forms is the null-space of an elliptic differential operator.

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