JOINT SIMILARITY AND DILATIONS FOR NONCONTRACTIVE SEQUENCES OF OPERATORS

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ABSTRACT. A characteristic function $\Theta_{\mathcal{T}}$ is defined, in terms of multi-analytic operators on Fock spaces, for any noncontractive sequence $\mathcal{T} := (T_1, \dots, T_d)$ ($d \in \mathbb{N}$ or $d = \infty$) of operators on a Hilbert (resp. Kreĭn) space $\mathcal{H}$. It is shown that if $\Theta_{\mathcal{T}}$ is bounded, then it is unitarily equivalent to a compression of an orthogonal projection (on Kreĭn spaces). This leads to a generalization of a theorem of Ch. Davis and C. Foiaş, to multivariable setting. More precisely, it is proved that if $\mathcal{T}$ has bounded characteristic function, then it is jointly similar to a contractive sequence of operators, i.e., there exists a similarity $S \in \mathcal{B}(\mathcal{H})$ such that the operator defined by the row matrix $[ST_1S^{-1}\ ST_2S^{-1}\ \dots\ ST_dS^{-1}]$ is a contraction.

Motivated by the similarity problem, a multivariable dilation theory on Fock spaces with indefinite metric is developed for noncontractive d-tuples of operators. Wold type decompositions for sequences of bounded isometries on Kreĭn spaces and Fourier representations for d-orthogonal shifts are obtained and used to study the geometry of the canonical minimal isometric dilation associated with a sequence $\mathcal{T}$ of operators on a Hilbert space.

1. INTRODUCTION

The starting point of this paper is an old result of Ch. Davis and C. Foiaş [6] establishing that any operator with bounded characteristic function is similar to a contraction. In this paper, we obtain a multivariable version of their result. To reach this goal, we develop a multivariable dilation theory on Fock spaces with indefinite metric for noncontractive sequences $\mathcal{T} := (T_1, \dots, T_d)$ ($d \in \mathbb{N}$ or $d = \infty$) of operators on a Hilbert (resp. Kreĭn) space $\mathcal{H}$.

To put our work in perspective, we recall [19] that the noncommutative disk algebra $\mathcal{A}_d$ is the norm closed algebra generated by the left creation operators $S_1, \dots, S_d$ on the full Fock space $\mathcal{F}_d^2$, and the identity (see Section 3). Generalizing [15], the second author [22] proved that a d-tuple $\mathcal{T} := (T_1, \dots, T_d)$ of operators on a Hilbert space $\mathcal{H}$ generates a completely bounded representation

$$\pi_{\mathcal{T}} :\ \mathcal{A}_d \to B(\mathcal{H}),$$

by setting $\pi_{\mathcal{T}}(p(S_1, \dots, S_d)) := p(T_1, \dots, T_d)$, if and only if $\mathcal{T}$ is jointly similar to a contractive sequence of operators $(A_1, \dots, A_d)$, i.e., $\sum_{i=1}^{d} A_i A_i^* \le I_{\mathcal{H}}$ and there exists a similarity $S \in \mathcal{B}(\mathcal{H})$ such that $T_i = S^{-1} A_i S$ for any $i = 1, \dots, d$.

Date: April 19, 2001; *Revised*: September 12, 2002.

2000 *Mathematics Subject Classification.* Primary: 47A48, 47A20, 47B50 Secondary: 47L25, 47D03, 46L07.

Key words and phrases. Characteristic function, joint similarity, isometric dilation, Fock space, free semigroup, noncontractive sequence.

Research supported in part by COBASE grant from the NRC.

Following the classical case [23], [6], as well as the multivariable setting of contractive sequences of operators [18], we associate a *characteristic function* $\Theta_{\mathcal{T}}$ with any (noncontractive) d-tuple $\mathcal{T}$ of operators. We prove that if $\Theta_{\mathcal{T}}$ is a bounded analytic operator on Fock spaces, then $\mathcal{T}$ is jointly similar to a contractive sequence of operators, and therefore $\pi_{\mathcal{T}}$ is a completely bounded representation of $\mathcal{A}_d$. The result extends to completely bounded representations of the noncommutative analytic Toeplitz algebra F_d^∞, the weakly closed algebra generated by $\mathcal{A}_d$, for a large class of operators.

This problem is also interesting in view of a tentative noncommutative multivariable systems theory, since the boundedness of the characteristic function is a natural condition that appears in stability. In addition, the characteristic function is observable, in terms of the outer description of the system, while the "main operator" $\mathcal{T} := (T_1, \ldots, T_d)$ is non-observable, in terms of the inner (state-space) description of the system. In this perspective, the joint similarity of the main operator with a row contraction can be viewed as a state-space transformation.

Motivated by the similarity problem, we are also interested in answering the question concerning the extent to which the noncommutative dilation theory for contractive sequences of operators (see [2], [8], [16], [17], [18]), and the dilation theory for noncontractions (see [5], [6], [11], [12], [13], [14], [3], [4]), can be generalized to noncontractive sequences of operators on Hilbert (resp. Kreĭn) spaces.

In Section 2, we present some preliminaries on Kreĭn spaces (see also [1]). The next two sections are devoted to isometries with orthogonal ranges on Kreĭn spaces. We obtain Wold type decompositions for sequences of bounded isometries (see Theorem 3.5) on Kreĭn spaces and Fourier representations for d-orthogonal shifts (see Theorem 4.2 and Theorem 4.6), extending the corresponding results from [11], [12], [13], [14], and [17], [18].

These results are used in Section 6, to study the geometry of the canonical minimal isometric dilation (see Section 5) associated with a sequence $\mathcal{T}$ of operators on a Hilbert space (see [5], [6], [12] for the case $d = 1$, and [17] for the multivariable contractive case).

In Section 7, we associate with any d-tuple $\mathcal{T}$ a characteristic function $\Theta_{\mathcal{T}}$ in terms of multi-analytic operators on Fock spaces. The main theorem of this paper is the similarity result in Theorem 7.4, the generalization of Davis-Foiaş theorem [6]. A few comments on the two proofs that we provide for Theorem 7.4 are necessary. As expected, both of them use heavily the Fourier representation associated with the two wandering subspaces $\mathcal{L}$ and $\mathcal{L}_*$. Thus, in Theorem 7.2 we have to prove that, if $\Theta_{\mathcal{T}}$ is bounded, then it is unitarily equivalent to a compression of an orthogonal projection (on Kreĭn spaces), that is, the generalization of the celebrated theorem of Sz.-Nagy and Foiaş to the multivariable contractive setting as in [17]. To do this, we follow a combination of the approach of the original proof of Davis-Foiaş theorem [6] with the more geometric approach of McEnnis [12]: we generalize to this multivariable setting the characterization of the residual space $\mathcal{R}$ in terms of the Fourier coefficients of $\mathcal{M}_+(\mathcal{L})$, cf. Theorem 6.2. Based on this, we can prove Theorem 7.2 using a shortcut (cf. the proof in [10]) with respect to the orignal proof of McEnnis, more precisely, a simple duality argument that allows us to obtain the boundedness of the Fourier transform M_*. From this point the two proofs of Theorem 7.4 split: the first one follows the approach of McEnnis to show that, by renorming suitably the space $\mathcal{M}_+(\mathcal{L}_*)$, we get the desired similarity, while the latter shows that the criterion of similarity obtained in [16] is met.

We remark that all the results of this paper remain true if $d = \infty$, in a slightly adapted version.

The results of this paper lead to a multivariable commutant lifting theorem for noncontractive sequences of operators and non-analytic interpolation in several variables, as well as to a model theory for d-tuples of operators with bounded characteristic function. These two topics will be considered in future papers.

2. Preliminaries on Kreĭn Spaces

2.1. **Operators on Kreĭn Spaces.** A *Kreĭn space* is a complex linear space $\mathcal{K}$ equipped with a Hermitian sesquilinear form $[\cdot,\cdot]_{\mathcal{K}}$, which admits a decomposition of the form

$$\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-,$$

where $\mathcal{K}^+$ and $\mathcal{K}^-$ are linear manifolds in $\mathcal{K}$ such that $(\mathcal{K}^\pm, \pm[\cdot,\cdot]_{\mathcal{K}})$ are Hilbert spaces and $[\mathcal{K}^+,\mathcal{K}^-]_{\mathcal{K}} = \{0\}$. This kind of decomposition is called *fundamental decomposition* of the Kreĭn space $\mathcal{K}$ and, if $J^\pm : \mathcal{K} \to \mathcal{K}$ are the orthogonal projections from $\mathcal{K}$ onto $\mathcal{K}^\pm$, then $J := J^+ - J^-$ is the corresponding *fundamental symmetry.* The form $\langle x, y\rangle := [Jx, y]_{\mathcal{K}}$, $x, y \in \mathcal{K}$, defines a positive definite inner product on $\mathcal{K}$, which turns $\mathcal{K}$ into a Hilbert space. The norm associated with $\langle \cdot,\cdot\rangle$ is called *unitary norm* and it depends on the fundamental symmetry. However, all the unitary norms are equivalent and therefore, define the same topology on $\mathcal{K}$, called the *strong topology.*

The dimensions of the subspaces $\mathcal{K}^\pm$ are the same for each fundamental decomposition of the Kreĭn space $\mathcal{K}$, and the cardinal numbers $\kappa^\pm(\mathcal{K}) := \dim \mathcal{K}^\pm$ are called the *positive* (resp. *negative) signature* of the Kreĭn space $\mathcal{K}$. The cardinal number $\kappa(\mathcal{K}) := \min\{\kappa^+(\mathcal{K}), \kappa^-(\mathcal{K})\}$ is called the *rank of indefiniteness* of the space $\mathcal{K}$.

We denote by $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ the set of all bounded linear operators from the Kreĭn space $\mathcal{K}_1$ to the Kreĭn space $\mathcal{K}_2$, with respect to the strong topology. If $T \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$, then $T^\sharp \in \mathcal{B}(\mathcal{K}_2, \mathcal{K}_1)$ stands for the *adjoint* of T with respect to the indefinite innner products $[\cdot,\cdot]_{\mathcal{K}_i}$ on $\mathcal{K}_i$, i.e.,

$$[T^\sharp x, y]_{\mathcal{K}_1} = [x, Ty]_{\mathcal{K}_2}, \quad x \in \mathcal{K}_2, \quad y \in \mathcal{K}_1.$$

If J_i is any fundamental symmetry on $\mathcal{K}_i$, then $T^\sharp = J_1 T^* J_2$, where T^* stands for the adjoint of T with respect to the Hilbert spaces $(\mathcal{K}_i, \langle\cdot,\cdot\rangle_{J_i})$. We note that a fundamental symmetry J on a Kreĭn space $\mathcal{K}$ belongs to $\mathcal{B}(\mathcal{K})$ and $J = J^{-1} = J^\sharp = J^*$.

Throughout this paper, *selfadjoint* operators are considered with respect to the involution $\sharp$. An operator $U \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ is called *unitary* if it is boundedly invertible and $U^\sharp = U^{-1}$. An operator $T \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ is called *contraction* if $[Tx, Tx]_{\mathcal{K}_2} \le [x, x]_{\mathcal{K}_2}$ for all $x \in \mathcal{K}_1$, equivalently, $J_1 - T^* J_2 T \ge 0$, where J_i is a fundamental symmetry on $\mathcal{K}_i$.

If $T\colon \mathrm{Dom}\,(T)(\subset \mathcal{K}_1) \to \mathcal{K}_2$ is a densely defined operator, then one can uniquely define its adjoint $T^\sharp\colon \mathrm{Dom}\,(T^\sharp) \to \mathcal{K}_1$ by

$$[Tx, y]_{\mathcal{K}_2} = [x, T^\sharp y]_{\mathcal{K}_1}, \quad x \in \mathrm{Dom}\,(T),\ y \in \mathrm{Dom}\,(T^\sharp).$$

Equivalently, $T^\sharp = J_1 T^* J_2$, in the sense that $J_2 \mathrm{Dom}\,(T^\sharp) = \mathrm{Dom}\,(T^*)$ and $T^\sharp y = J_1 T^* J_2 y$ for any $y \in \mathrm{Dom}\,(T^\sharp)$. In this way, most of the theory of unbounded operators on Hilbert spaces is extended to unbounded operators on Kreĭn spaces.

We will use the notion of *isometric* operator in a very general sense, namely: an operator T with domain in $\mathcal{K}_1$ and values in $\mathcal{K}_2$ is isometric if $[Tx, Ty]_{\mathcal{K}_2} = [x, y]_{\mathcal{K}_1}$ for all $x, y \in \mathrm{Dom}\,(T)$. We recall that, in general, an isometric operator is not injective unless its domain is nondegenerate. Moreover, a densely defined and isometric operator is not necessarily continuous.

2.2. **Subspaces of a Kreĭn Space.** Let $\mathcal{L}$ be a *subspace* of a Kreĭn space $\mathcal{K}$, that is, $\mathcal{L}$ is a closed linear manifold of $\mathcal{K}$. Then the *orthogonal* subspace associated with $\mathcal{L}$ is $\mathcal{L}^{\perp} := \{y \in \mathcal{K} \mid [x, y]_{\mathcal{K}} = 0, \text{ for any } x \in \mathcal{L}\}$. The subspace $\mathcal{L}^0 := \mathcal{L} \cap \mathcal{L}^{\perp}$ is called the *isotropic subspace* of $\mathcal{L}$ and has the property that the inner product $[\cdot, \cdot]_{\mathcal{K}}$ vanishes on it. The subspace $\mathcal{L}$ is called *nondegenerate* if $\mathcal{L}^0 = \{0\}$. For two subspaces $\mathcal{A}$ and $\mathcal{B}$ of a Kreĭn space, the notation $\mathcal{A} \oplus \mathcal{B}$ is used whenever the algebraic sum $\mathcal{A} + \mathcal{B}$ is closed, $\mathcal{A} \perp \mathcal{B}$, and $\mathcal{A} \cap \mathcal{B} = \{0\}$.

A subspace $\mathcal{L}$ is called *nonpositive* (resp. *negative*) if $[x, x]_{\mathcal{K}} \leq 0$ for all $x \in \mathcal{L}$ (resp. $[x, x]_{\mathcal{K}} < 0$ for all $x \in \mathcal{L} \setminus \{0\}$). The subspace $\mathcal{L}$ is called *uniformly negative* if there exists $\delta > 0$ such that $[x, x]_{\mathcal{K}} \leq \delta \langle x, x \rangle_J$ for all $x \in \mathcal{L}$. Similarly one defines *nonnegative, positive*, and *uniformly positive* subpaces.

A special rôle in the geometry of Kreĭn spaces is played by those subspaces $\mathcal{L}$ of a given Kreĭn space $\mathcal{K}$ which become Kreĭn spaces under the induced inner product and strong topology. It is clear that such subspaces must be nondegenerate but, it turns out that, for the case of genuine Kreĭn spaces, nondegeneracy is not sufficient.

Let $\mathcal{L}$ be a subspace of the Kreĭn space $\mathcal{K}$. It is well-known that the following statements are equivalent:

(i) $\mathcal{K} = \mathcal{L} \oplus \mathcal{L}^{\perp}$.
(ii) There exists a (uniquely determined) selfadjoint projection $P \in \mathcal{B}(\mathcal{K})$ such that $P\mathcal{K} = \mathcal{L}$.
(iii) $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$, where $\mathcal{L}_+$ is a uniformly positive subspace and $\mathcal{L}_-$ is a uniformly negative subspace.
(iv) There exists a fundamental symmetry J on $\mathcal{K}$ such that $J\mathcal{L} \subseteq \mathcal{L}$ (equivalently, $J\mathcal{L} = \mathcal{L}$).
(v) There exists a unitary norm $\|\cdot\|$ on $\mathcal{K}$ such that

$$\|x\| = \sup_{\|y\| \leq 1,\ y \in \mathcal{L}} |[x, y]_{\mathcal{K}}|, \quad x \in \mathcal{L}.$$

A subspace $\mathcal{L}$ of a Kreĭn space $\mathcal{K}$ which satisfies one (hence all) of the conditions (i)–(iv) is called *regular*. Regular subspaces are also called *Kreĭn subspaces*. If $\{\mathcal{L}_i\}_{i=1}^n$ is a finite family of mutually orthogonal regular subspaces of $\mathcal{K}$, then the subspace $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_n$ is regular. Attention should be paid to sums of infinite families of mutually orthogonal regular subspaces when the regularity is not automatic.

Two subspaces $\mathcal{L}$ and $\mathcal{M}$ of (possibly different) Kreĭn spaces $\mathcal{K}_1$ and $\mathcal{K}_2$, are called *isometrically isomorphic* if there exists a bounded linear operator $V \colon \mathcal{L} \to \mathcal{M}$ which is boundedly invertible and $[Vx, Vy]_{\mathcal{K}_2} = [x, y]_{\mathcal{K}_1},\ x, y \in \mathcal{L}$. If a subspace $\mathcal{L}$ is isometrically isomorphic to a regular subspace, then $\mathcal{L}$ is also regular. Two regular subspaces $\mathcal{L}$ and $\mathcal{M}$ are isometrically isomorphic if and only if $\kappa^{\pm}(\mathcal{L}) = \kappa^{\pm}(\mathcal{M})$.

3. Isometries with Orthogonal Ranges on Kreĭn Spaces

Let $d \in \mathbb{N}$ and consider the free semigroup $\mathbb{F}_d^+$ on d generators $g_1, g_2, \ldots, g_d$, and unit g_0. For any $\sigma \in \mathbb{F}_d^+$, we denote by $|\sigma|$ its length, i.e., $|\sigma| = n$ if $\sigma = g_{i_1} \cdots g_{i_n}$, $i_j \in \{1, 2, \ldots, d\}$, and $|g_0| = 0$.

Let H_d be a Hilbert space of dimension d, with orthogonal basis $\{e_1, \dots, e_d\}$. We consider the full Fock space

$$\mathcal{F}_d^2 := \mathbb{C}e_0 \oplus \bigoplus_{n\geq 1} H_d^{\otimes n}, \tag{3.1}$$

where e_0 is the so-called "vacuum vector". The set $\{e_\sigma\}_{\sigma\in\mathbb{F}_d^+}$ is an orthonormal basis of $\mathcal{F}_d^2$, where $e_\sigma := e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$, if $\sigma := g_{i_1}g_{i_2}\dots g_{i_n} \in \mathbb{F}_d^+$, and $e_{g_0} := e_0$. Whenever $\mathcal{H}$ is a Hilbert space and $\mathcal{T} := (T_j)_{j=1}^d$ is a d-tuple of bounded operators on $\mathcal{H}$ we denote $T_0 := I_{\mathcal{H}}$ and $T_\sigma := T_{i_1}T_{i_2}\cdots T_{i_n}$, if $\sigma := g_{i_1}\cdots g_{i_n}$.

For each $j \in \{1, \dots, d\}$, we consider the *left creation operator* S_j defined by

$$S_j\xi := e_j \otimes \xi, \quad \xi \in \mathcal{F}_d^2. \tag{3.2}$$

Notice that the isometric operators $S_1, \dots, S_d$ have mutually orthogonal ranges, and $\mathcal{L} = \mathbb{C}e_0$ is a *wandering subspace*, i.e., $S_\sigma\mathcal{L} \perp S_\tau\mathcal{L}$ for all $\sigma, \tau \in \mathbb{F}_d^+$, $\sigma \neq \tau$. It is easy to see that $\mathcal{L}$ is *generating* the Fock space $\mathcal{F}_d^2$, that is,

$$\bigoplus_{\sigma\in\mathbb{F}_d^+} S_\sigma\mathcal{L} = \mathcal{F}_d^2.$$

If $\mathcal{H}$ is a Hilbert space, then $\{S_1 \otimes I_{\mathcal{H}}, \dots, S_d \otimes I_{\mathcal{H}}\}$ is a d-orthogonal shift on $\mathcal{F}_d^2 \otimes \mathcal{H}$, and $\mathcal{L} = \mathbb{C} \otimes \mathcal{H}$ is the generating subspace. This model orthogonal shift has played an important rôle in the noncommutative dilation theory for row contractions (see [16], [17], [18]). In what follows, we find a "model" d-orthogonal shift in the Kreĭn space setting. However, as we shall see subsequently, the notion of d-orthogonal shift on a Kreĭn space is different in many aspects from its counter-part in the Hilbert space setting, and many pathologies will show up.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Kreĭn spaces and fix two fundamental symmetries J_1 and J_2 on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. There is a natural Kreĭn space structure on the Hilbert space direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ (with the positive inner product $\langle\cdot,\cdot\rangle$), defined by the indefinite inner product

$$[\xi, \eta]_{\mathcal{H}_1\oplus\mathcal{H}_2} := \langle (J_1 \oplus J_2)\xi, \eta\rangle, \quad \xi, \eta \in \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Similarly, there is a natural Kreĭn space structure on the Hilbert space tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ (with the positive inner product $\langle\cdot,\cdot\rangle$), defined by the indefinite inner product

$$[\xi, \eta]_{\mathcal{H}_1\otimes\mathcal{H}_2} := \langle (J_1 \otimes J_2)\xi, \eta\rangle, \quad \xi, \eta \in \mathcal{H}_1 \otimes \mathcal{H}_2.$$

Both constructions do not depend on the fundamental symmetries, up to a canonical unitary isomorphisms.

We need to make a few remarks that will apply throughout this paper when $d = \infty$. If $\mathcal{T} := (T_1, T_2, \dots)$ is an infinite sequence of operators on a Hilbert space $\mathcal{H}$, then we always assume that the row matrix $T := [T_1\ T_2\ \dots]$ is a bounded operator from $\bigoplus_{i=1}^{\infty} \mathcal{H}$ to $\mathcal{H}$, i.e., the series $\sum_{i=1}^{\infty} T_iT_i^*$ is strongly convergent. In this case, we have

$$\sum_{|\alpha|=n} T_\alpha T_\alpha^* = \sum_{i=1}^{\infty} T_i\Big(\sum_{|\beta|=n-1} T_\beta T_\beta^*\Big)T_i^*$$

and

$$\sum_{|\alpha|=n} T_\alpha T_\alpha^* \leq \| \sum_{i=1}^{\infty} T_i T_i^* \|^n \; I_{\mathcal{H}}.$$

This shows that, if we consider the lexicographic order on $\{\alpha \in \mathbb{F}_d^+;\ |\alpha| = n\}$, then the row matrix $[T_\alpha]_{|\alpha|=n}$ is a bounded operator acting from $\bigoplus_{|\alpha|=n} \mathcal{H}$ to $\mathcal{H}$. Therefore, if $\bigoplus_{|\alpha|=n} h_\alpha \subset \bigoplus_{|\alpha|=n} \mathcal{H}$, then $\sum_{|\alpha|=n} T_\alpha h_\alpha$ is convergent.

Let $\mathcal{V} := (V_j)_{j=1}^d$ be a d-tuple of bounded isometries on a Kreĭn space $\mathcal{H}$. A subspace $\mathcal{L} \subset \mathcal{H}$ is called *wandering* for $\mathcal{V}$ if it is regular and $V_\sigma \mathcal{L} \perp V_\tau \mathcal{L}$ for any distinct $\sigma, \tau \in \mathbb{F}_d^+$. In this case, we define the subspace $\mathcal{M}_+(\mathcal{L})$ by setting

$$\mathcal{M}_+(\mathcal{L}) := \bigvee_{\sigma \in \mathbb{F}_d^+} V_\sigma \mathcal{L}. \tag{3.3}$$

Let us first note that if $\mathcal{L}$ is a wandering subspace for $\mathcal{V}$, then $V_j\mathcal{L}$ has the same property. Moreover, if $i, j \in \{1, 2, \ldots, d\}$ with $i \neq j$, then $V_i \mathcal{M}_+(\mathcal{L}) \perp V_j \mathcal{M}_+(\mathcal{L})$ and $\mathcal{L} \perp V_i M_+(\mathcal{L})$. We need to pay attention to the case $d = \infty$, when we also assume that $[V_1\ V_2\ \ldots]$ is a bounded operator. Since the row matrix $[V_\alpha]_{|\alpha|=n}$ is a bounded isometry when restricted to the Kreĭn space $\bigoplus_{|\alpha|=n} \mathcal{L}$ and with values in $\mathcal{H}$, we infer that $\bigoplus_{|\alpha|=n} V_\alpha \mathcal{L}$ is a regular subspace of $\mathcal{H}$. Now, as in [17], one can easily verify the following properties.

Lemma 3.1. *Let $\mathcal{V} := (V_j)_{j=1}^d$ be a d-tuple of bounded isometries on a Kreĭn space $\mathcal{H}$ and let $\mathcal{L}$ be a wandering subspace for $\mathcal{V}$. Then*

(a) $\mathcal{M}_+(\mathcal{L}) = \mathcal{L} \oplus (V_1 \mathcal{M}_+(\mathcal{L}) \vee \cdots \vee V_d \mathcal{M}_+(\mathcal{L}))$;

(b) *For any $n \geq 1$, the subspace $\mathcal{M}_+(\mathcal{L})$ has the decomposition*

$$\mathcal{M}_+(\mathcal{L}) = \bigoplus_{|\sigma| \leq n-1} V_\sigma \mathcal{L} \oplus \bigvee_{|\sigma|=n} V_\sigma \mathcal{M}_+(\mathcal{L}); \tag{3.4}$$

(c) *$\mathcal{M}_+(\mathcal{L})$ is nondegenerate if and only if* $\mathcal{L} = \mathcal{M}_+(\mathcal{L}) \cap \left(\bigvee_{j=1}^d V_j \mathcal{M}_+(\mathcal{L}) \right)^\perp$;

(d) $\mathcal{M}_+(\mathcal{L})^0 = \bigcap_{n \geq 0} \bigvee_{|\sigma|=n} V_\sigma \mathcal{M}_+(\mathcal{L})$.

Notice that, according to the equation (3.4), one can associate with each $h \in \mathcal{M}_+(\mathcal{L})$ a unique family $\{l_\sigma\}_{\sigma \in \mathbb{F}_d^+}$ of vectors in $\mathcal{L}$ such that

$$h - \sum_{|\tau| \leq n-1} V_\tau l_\tau \in \bigvee_{|\tau|=n} V_\tau \mathcal{M}_+(\mathcal{L}), \quad n \in \mathbb{N}. \tag{3.5}$$

The sequence $\{l_\sigma\}_{\sigma \in \mathbb{F}_d^+}$ is called the sequence of *Fourier coefficients* associated with $h \in \mathcal{M}_+(\mathcal{L})$.

Lemma 3.2. *Let $\mathcal{V} := (V_j)_{j=1}^d$ be a d-tuple of bounded isometries on a Kreĭn space $\mathcal{H}$ and let $\mathcal{L}$ be a wandering subspace of $\mathcal{V}$. Then, for any $h \in \mathcal{M}_+(\mathcal{L})$, the family of its Fourier coefficients $\{l_\sigma\}_{\sigma \in \mathbb{F}_d^+}$ is given by*

$$l_\sigma = P_{\mathcal{L}} V_\sigma^\sharp h, \quad \sigma \in \mathbb{F}_d^+,$$

where $P_{\mathcal{L}}$ is the selfadjoint projection onto the regular subspace $\mathcal{L}$.

Proof. If $h \in \mathcal{M}_+(\mathcal{L})$, we clearly have $l_0 = P_{\mathcal{L}} h$. Let $n \geq 1$ and $\alpha \in \mathbb{F}_d^+$, $|\alpha| = n$. According to (3.5), there exists a family of vectors $\{h'_\beta\}_{|\beta|=n}$ in $\mathcal{M}_+(\mathcal{L})$ such that

$$h - \sum_{|\beta| \leq n-1} V_\beta l_\beta = \sum_{|\beta|=n} V_\beta h'_\beta.$$

Applying the operator $P_{\mathcal{L}} V_\alpha^\sharp$ to both sides and taking into account that $P_{\mathcal{L}} V_\alpha^\sharp V_\beta l_\beta = 0$ for all $\beta \in \mathbb{F}_d^+$ with $|\beta| \leq n-1$, we get

$$P_{\mathcal{L}} V_\alpha^\sharp h = P_{\mathcal{L}} V_\alpha^\sharp \sum_{|\beta|=n} V_\beta h'_\beta = P_{\mathcal{L}} h'_\alpha.$$

Therefore,

$$h - \sum_{|\beta| \leq n-1} V_\beta l_\beta - \sum_{|\beta|=n} V_\beta P_{\mathcal{L}} V_\beta^\sharp h = \sum_{|\beta|=n} V_\beta (h'_\beta - P_{\mathcal{L}} V_\beta^\sharp h) = \sum_{|\beta|=n} V_\beta (I - P_{\mathcal{L}}) h'_\beta.$$

Since $h'_\beta \in \mathcal{M}_+(\mathcal{L})$, by Lemma 3.1.(a), it follows that

$$(I - P_{\mathcal{L}}) h'_\beta \in \bigvee_{j=1}^{d} V_j \mathcal{M}_+(\mathcal{L}),$$

and hence

$$\sum_{|\beta|=n} V_\beta (I - P_{\mathcal{L}}) h'_\beta \in \bigvee_{|\beta|=n+1} V_\beta \mathcal{M}_+(\mathcal{L}).$$

Now, using the definition of the Fourier coefficients, one can easily see that $l_\alpha = P_{\mathcal{L}} V_\alpha^\sharp h$ for any $\alpha \in \mathbb{F}_d^+$. This completes the proof. □

Let us now assume that $\mathcal{V} := (V_1, \ldots, V_d)$ is a d-tuple of isometric operators on a Kreĭn space $\mathcal{H}$, with *mutually orthogonal ranges.* Since $V_\alpha \mathcal{H} \perp V_\beta \mathcal{H}$ for any $\alpha, \beta \in \mathbb{F}_d^+$, $\alpha \neq \beta$, with $|\alpha| = |\beta| \geq 1$, and the ranges of bounded isometric operators are regular subspaces, we infer that $V_\alpha \mathcal{H}$, $\alpha \in \mathbb{F}_d^+$, and $\bigoplus_{|\sigma|=n} V_\sigma \mathcal{H}$ are regular subspaces. The equation (3.4) can be written as

$$\mathcal{M}_+(\mathcal{L}) = \bigoplus_{|\sigma| \leq n-1} V_\sigma \mathcal{L} \oplus \bigoplus_{|\sigma|=n} V_\sigma \mathcal{M}_+(\mathcal{L}). \tag{3.6}$$

On the other hand, since $V_\alpha V_\alpha^\sharp$ is a selfadjoint projection onto the range of the isometry V_α, the operators

$$P_n := I - \sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp, \quad n \geq 1, \tag{3.7}$$

are also selfadjoint projections. Notice that the sequence of selfadjoint projections $\{P_n\}_{n \geq 0}$, where $P_0 = 0$, is nondecreasing, i.e., the range of P_n is included in the range of P_{n+1}. Furthermore, it is easy to see that

$$\mathcal{L} := (V_1 \mathcal{H} \oplus \cdots \oplus V_d \mathcal{H})^\perp, \tag{3.8}$$

is a wandering subspace of $\mathcal{V}$ and

$$\mathcal{M}_+(\mathcal{L}) = \bigvee_{n\geq 0} P_n\mathcal{H}. \tag{3.9}$$

Remark 3.3. As shown in [9], the regularity of a subspace is related to the graph convergence. We recall the following definitions only for bounded operators: the sequence $(C_n)_{n\in\mathbb{N}}$, of bounded operators on a Hilbert space $\mathcal{H}$, is said to converge in the *strong* (*weak*) *graph sense* to $C \in \mathcal{B}(\mathcal{H})$ if for any $x \in \mathcal{H}$ there exists a sequence (x_n) of vectors in $\mathcal{H}$ such that $x_n \to x$ strongly (weakly) and $C_n x_n \to Cx$ strongly (weakly).

Let $\mathcal{V} := \{V_1, \dots , V_d\}$ be a family of isometric operators with mutually orthogonal ranges and consider the wandering subspace $\mathcal{L}$, as defined in (3.8). Then, the following statements are equivalent:

(a) $\{\sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp\}_{n\geq 0}$ converges in the strong graph sense;
(b) $\{\sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp\}_{n\geq 0}$ converges in the weak graph sense;
(c) $\mathcal{M}_+(\mathcal{L})$ is a regular subspace.

These equivalences can be proved using (3.7), (3.9), the observation that the sequence of selfadjoint projections $\{P_n\}_{n\geq 0}$ is nondecreasing, and Corollary 3.3 from [9].

A d-tuple $\mathcal{V} := (V_j)_{j=1}^d$ of bounded isometries on the Kreĭn space $\mathcal{H}$ is called a *d-orthogonal shift* if there exists a subspace $\mathcal{L}$ of $\mathcal{H}$, which is wandering for $\mathcal{V}$ and such that $\mathcal{H} = \mathcal{M}_+(\mathcal{L})$. In this case, the wandering space $\mathcal{L}$ is called *generating* for $\mathcal{V}$. As a consequence of Proposition 3.1, it follows that the generating subspace of a d-orthogonal shift $\mathcal{V}$ is uniquely determined and given by (3.8). If $x \in \mathcal{H} = \mathcal{M}_+(\mathcal{L})$ and $\{l_\alpha\}_{\alpha\in\mathbb{F}_d^+}$ are its Fourier coefficients, then we write $x \sim \sum_{\alpha\in\mathbb{F}_d^+} V_\alpha l_\alpha$.

Example 3.4. A class of d-orthogonal shifts on Kreĭn spaces is obtained as follows. Let $\mathcal{H}$ be a Kreĭn space and let $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator of Kreĭn space. The d-tuple $\mathcal{V} := (S_1 \otimes U, \dots , S_d \otimes U)$ is a d-orthogonal shift on the Kreĭn space $\mathcal{F}_d^2 \otimes \mathcal{H}$, with generating space $\mathbb{C}e_0 \otimes \mathcal{H}$.

We say that U is fundamentally reducible if $JU = UJ$ for some fundamental symmetry J of $\mathcal{H}$, equivalently, $U^* = U^\sharp$, where U^* is considered with respect to the positive definite inner product $\langle\cdot,\cdot\rangle_J$. Then $\mathcal{V}$ is a d-orthogonal shift with respect to both the Kreĭn space structure and the canonical Hilbert space structure of $\mathcal{F}_d^2 \otimes \mathcal{H}$ induced by the fundamenal symmetry $I_{\mathcal{F}_d^2} \otimes J$. In particular, this is true for $U = I_\mathcal{H}$. In this case, the d-orthogonal shift $\mathcal{S} := \{S_1 \otimes I_\mathcal{H}, \dots , S_d \otimes I_\mathcal{H}\}$ is called the *canonical d-orthogonal shift* on $\mathcal{F}_d^2 \otimes \mathcal{H}$.

In a certain sense, the canonical d-orthogonal shift, as described above, might be considered as a "model" even in our indefinite setting. If $\mathcal{V} := (V_j)_{j=1}^d$ is a sequence of bounded isometric operators on a Kreĭn space $\mathcal{H}$, then a decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is called *Wold-von Neumann decomposition* if, for each $i = 0, 1$, the following conditions are satisfied:

(i) $\mathcal{H}_i$ is a regular subspace of $\mathcal{H}$;
(ii) $\mathcal{H}_i$ is invariant under each V_j, $j = 1, 2, \dots , d$;

(iii) $(I - \sum_{j=1}^{d} V_j V_j^\sharp)|\mathcal{H}_1 = 0$;

(iv) $\{V_j|\mathcal{H}_0\}_{j=1}^d$ is a d-orthogonal shift on $\mathcal{H}_0$.

There is an analogue of Wold-von Neumann theorem (see [23]) in our setting. The proof is essentially the same as that from [17], so we shall omit it.

Theorem 3.5. *Let $\mathcal{V} := \{V_i\}_{i=1}^d$ be a sequence of bounded isometries on a Kreĭn space $\mathcal{H}$, having orthogonal ranges, and let $\mathcal{L}$ be the wandering subspace defined by $\mathcal{L} := \left(\bigoplus_{i=1}^{d} V_i \mathcal{H}\right)^{\perp}$. Then $\mathcal{V}$ admits a Wold-von Neumann decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ if and only if the subspace $\mathcal{M}_+(\mathcal{L}) := \bigvee_{\sigma \in \mathbb{F}_d^+} V_\sigma \mathcal{L}$ is regular. In this case, $\mathcal{H}_0 = \mathcal{M}_+(\mathcal{L})$ and*

$$\mathcal{H}_1 = \bigcap_{n \geq 0} \Big(\bigoplus_{|\alpha|=n} V_\alpha \mathcal{H} \Big).$$

Moreover, if a Wold-von Neumann decomposition exists, then it is unique.

A delicate question in this setting is related to the possibility of recovering the vectors in $\mathcal{M}_+(\mathcal{L})$ from their Fourier coefficients.

Proposition 3.6. *Let $\mathcal{V} := (V_1, \dots, V_d)$ be a d-tuple of bounded isometric operators on a Kreĭn space $\mathcal{H}$ with mutually orthogonal ranges, and let $\mathcal{L}$ be the wandering subspace associated with $\mathcal{V}$. Then, the following statements are equivalent:*

(a) *The sequence $\{\sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp\}_{n \geq 0}$ is uniformly bounded.*

(b) *The sequence $\{\sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp\}_{n \geq 0}$ converges in the strong operator topology.*

(c) *The sequence $\{\sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp\}_{n \geq 0}$ converges in the weak operator topology.*

(d) *$\mathcal{M}_+(\mathcal{L})$ is regular and, for every vector $h \in \mathcal{M}_+(\mathcal{L})$, we have*

$$h = \sum_{n \geq 0} \sum_{|\alpha|=n} V_\alpha l_\alpha, \tag{3.10}$$

where $\{l_\alpha\}_{\alpha \in \mathbb{F}_d^+}$ is the sequence of Fourier coefficients of h and the series converges strongly.

In addition, if either one of the above statements holds, then for any $h, h' \in \mathcal{M}_+(\mathcal{L})$, we have

$$[h, h']_{\mathcal{K}} = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} [l_\alpha, l'_\alpha]_{\mathcal{K}}, \tag{3.11}$$

where $h \sim \sum_{\alpha \in \mathbb{F}_d^+} V_\alpha l_\alpha$ and $h' \sim \sum_{\alpha \in \mathbb{F}_d^+} V_\alpha l'_\alpha$.

Proof. To prove the equivalence of (a), (b), and (c), one can use (3.7), (3.9), the observation that the sequence of selfadjoint projections $\{P_n\}_{n \geq 0}$ is nondecreasing, and Lemma 3.4 in [9].

(c)⇒(d). Let h be a vector in $\mathcal{M}_+(\mathcal{L})$ and let $\{l_\alpha\}_{\alpha\in\mathbb{F}_d^+}$ be its sequence of Fourier coefficients. By Lemma 3.2, we have

$$l_\alpha = P_{\mathcal{L}} V_\alpha^\sharp h = (I - \sum_{j=1}^d V_j V_j^\sharp) V_\alpha^\sharp h = V_\alpha^\sharp h - \sum_{j=1}^d V_j V_j^\sharp V_\alpha^\sharp h, \quad \alpha \in \mathbb{F}_d^+.$$

Therefore, for arbitrary $n \geq 1$, we have

$$\sum_{|\alpha|=n} V_\alpha l_\alpha = \sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp h - \sum_{j=1}^d \sum_{|\alpha|=n} (V_\alpha V_j)(V_\alpha V_j)^\sharp h,$$

and hence, for fixed $N \geq 1$, we have

$$\sum_{n=0}^N \sum_{|\alpha|=n} V_\alpha l_\alpha = \sum_{n=0}^N \Big(\sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp h - \sum_{j=1}^d \sum_{|\alpha|=n} (V_\alpha V_j)(V_\alpha V_j)^\sharp h \Big). \tag{3.12}$$

Noticing that the sum in the right hand side of (3.12) is telescopic, we obtain

$$\sum_{|\alpha|\leq N} V_\alpha l_\alpha = \big(I - \sum_{|\alpha|=N+1} V_\alpha V_\alpha^\sharp\big) h = P_N h. \tag{3.13}$$

Since the sequence $\{\sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp\}_{n\geq 0}$ converges in the strong operator topology, it follows that the sequence $\{P_n\}_{n\geq 0}$ converges to the bounded selfadjoint projection onto $\mathcal{M}_+(\mathcal{L})$. Therefore, the subspace $\mathcal{M}_+(\mathcal{L})$ is regular and, in addition,

$$h = \lim_{n\to\infty} P_n h = \sum_{n=0}^\infty \sum_{|\alpha|=n} V_\alpha l_\alpha. \tag{3.14}$$

(d)⇒(c) Assume that $\mathcal{M}_+(\mathcal{L})$ is regular and the expansion (3.10) holds for any $h \in \mathcal{M}_+(\mathcal{L})$. By (3.13), it follows that the sequence $\{\sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp h\}_{n\geq 0}$ converges strongly. Taking into account that $\mathcal{M}_+(\mathcal{L})$ is regular and using Theorem 3.5, we infer that

$$k = \sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp k, \quad k \in \mathcal{M}_+(\mathcal{L})^\perp, \ n \geq 1.$$

Since $\mathcal{H} = \mathcal{M}_+(\mathcal{L}) \oplus \mathcal{M}_+(\mathcal{L})^\perp$, this implies that the sequence $\{\sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp\}_{n\geq 0}$ converges in the strong operator topology. □

Remark 3.7. If the sequence $\{\sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp\}_{n\geq 0}$ is uniformly bounded, then the subspaces in the Wold decomposition have the form

$$\mathcal{H}_0 = \{k \in \mathcal{H} : \lim_{n\to\infty} \sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp k = 0\}$$

and

$$\mathcal{H}_1 = \{k \in \mathcal{H} : \sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp k = k \text{ for any } n = 0, 1, \ldots\}.$$

In particular, if $\sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp \to 0$ strongly as $n \to \infty$, then $(V_1, \ldots, V_d)$ is a d-orthogonal shift.

Recall that a regular subspace $\mathcal{L}$ is uniquely determined, modulo an isometric isomorphism, by its signature $(\kappa^+(\mathcal{L}), \kappa^-(\mathcal{L}))$ and hence, the signature of the generating subspace $\mathcal{L}$ might be called the *multiplicity* of the d-orthogonal shift $\mathcal{V}$. But, contrary to the positive definite case, when the multiplicity of the generating space determines uniquely the d-orthogonal shift, up to unitary equivalence, the next proposition shows that this is not true in the setting of genuine Kreĭn spaces.

Two d-orthogonal shifts $\mathcal{V}$ and $\mathcal{V}'$, on Kreĭn spaces $\mathcal{K}_1$ (resp. $\mathcal{K}_2$) are called *unitary equivalent* if there exists a bounded unitary operator $\Psi \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ such that $\Psi V_j = V_j' \Psi$ for all $j = 1, 2, \ldots, d$.

Proposition 3.8. *If the Kreĭn space $\mathcal{H}$ is indefinite, that is, it contains positive as well as negative vectors, then there exist two d-orthogonal shifts on the Kreĭn space $\mathcal{F}_d^2 \otimes \mathcal{H}$, having the same generating space $e_0 \otimes \mathcal{H}$, and such that they are not unitarily equivalent.*

Proof. Let $\mathcal{H}$ be a Kreĭn space and let $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator of Kreĭn space. Define $\mathcal{V} := (V_1, \ldots, V_d)$, where $V_i := S_i \otimes U$, $i = 1, 2, \ldots, d$, and notice that $\mathcal{V}$ is a d-orthogonal shift on $\mathcal{K} = \mathcal{F}_d^2 \otimes \mathcal{H}$, with the generating space $e_0 \otimes \mathcal{H}$. On the other hand, we have $V_\alpha = S_\alpha \otimes U^{|\alpha|}$ and $\sup_{\alpha \in \mathbb{F}_d^+} \|V_\alpha\| = \sup_{n \in \mathbb{N}} \|U^n\|$.

Now we show that, if $\mathcal{H}$ is indefinite, then there always exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $\sup_{n \in \mathbb{N}} \|U^n\| = \infty$. Indeed, it is easy to see that it is sufficient to prove this statement for the particular case when $\mathcal{H} = \mathbb{C} \oplus \mathbb{C}$, with the indefinite inner product $[h_1 \oplus k_1, h_2 \oplus k_2] := h_1 \overline{h}_2 - k_1 \overline{k}_2$. In this case, fix $r \in (0, 1)$ and define

$$U := \frac{1}{\sqrt{1 - r^2}} \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}. \tag{3.15}$$

Then U is a unitary operator on the Kreĭn space $\mathcal{H}$ and

$$\|U^n\| = \left(\frac{1 + r}{1 - r} \right)^{\frac{n}{2}} \to \infty, \quad \text{as } n \to \infty.$$

Since $\lim_{|\alpha| \to \infty} \|S_\alpha\| < \infty$ and $\lim_{|\alpha| \to \infty} \|V_\alpha\| = \infty$, the d-orthogonal shift $\mathcal{V}$ is not unitarily equivalent to the canonical d-orthogonal shift $\mathcal{S} := \{S_1 \otimes \mathcal{H}, \ldots, S_d \otimes I_{\mathcal{H}}\}$, although they have the same generating space $e_0 \otimes \mathcal{H}$. □

4. The Fourier Representation of d-Orthogonal Shifts

Let $\mathcal{V} := (V_1, \ldots, V_d)$ be a d-orthogonal shift on the Kreĭn space $\mathcal{K} = \mathcal{M}_+(\mathcal{L})$, with generating space $\mathcal{L}$. With the notation as in the previous sections, we consider the Kreĭn space

$$\mathcal{F}_d^2 \otimes \mathcal{L} := \{ \bigoplus_{\alpha \in \mathbb{F}_d^+} e_\alpha \otimes l_\alpha \mid (l_\alpha)_{\alpha \in \mathbb{F}_d^+} \subset \mathcal{L}, \ \sum_{\alpha \in \mathbb{F}_d^+} \|l_\alpha\|^2 < \infty \}, \tag{4.1}$$

where $\|\cdot\|$ is a fixed unitary norm on $\mathcal{K}$. For any vectors $x, y \in \mathcal{F}_d^2 \otimes \mathcal{L}$ given by

$$x = \bigoplus_{\alpha \in \mathbb{F}_d^+} e_\alpha \otimes x_\alpha, \quad y = \bigoplus_{\alpha \in \mathbb{F}_d^+} e_\alpha \otimes y_\alpha,$$

we have

$$[x, y]_{\mathcal{F}_d^2 \otimes \mathcal{L}} = \sum_{j=0}^{\infty} \sum_{|\alpha|=j} [x_\alpha, y_\alpha]_{\mathcal{K}}. \tag{4.2}$$

We define the linear operator $\Phi_0 \colon \operatorname{Dom}(\Phi_0) \to \mathcal{F}_d^2 \otimes \mathcal{L}$ as follows:

$$\operatorname{Dom}(\Phi_0) := \{ \sum_{|\alpha| \leq n} V_\alpha l_\alpha \mid n \in \mathbb{N},\ (l_\alpha)_{|\alpha| \leq n} \subset \mathcal{L} \} \subseteq \mathcal{K}, \tag{4.3}$$

$$\Phi_0 (\sum_{|\alpha| \leq n} V_\alpha l_\alpha) := \bigoplus_{|\alpha| \leq n} e_\alpha \otimes l_\alpha, \quad (l_\alpha)_{|\alpha| \leq n} \subset \mathcal{L},\ n \in \mathbb{N}. \tag{4.4}$$

Lemma 4.1. *The linear operator Φ_0 is densely defined, closable, isometric, with dense range, and*

$$\Phi_0 V_j = (S_j \otimes I_{\mathcal{L}}) \Phi_0, \tag{4.5}$$

for all $j = 1, \ldots, d$.

Proof. Clearly, the linear manifold $\operatorname{Dom}(\Phi_0)$ is dense in $\mathcal{K}$ and $\Phi_0 \operatorname{Dom}(\Phi_0)$ is dense in $\mathcal{F}_d^2 \otimes \mathcal{L}$. On the other hand, taking into account that $\mathcal{V}$ is a d-orthogonal shift with generating space $\mathcal{L}$ and the definition of the inner product $[\cdot, \cdot]_{\mathcal{F}_d^2 \otimes \mathcal{L}}$ (see (4.2)), it is easy to see that Φ_0 is isometric, that is,

$$[\Phi_0 x, \Phi_0 y]_{\mathcal{F}_d^2 \otimes \mathcal{L}} = [x, y]_{\mathcal{K}}, \quad x, y \in \operatorname{Dom}(\Phi_0). \tag{4.6}$$

Note that if $x \sim \sum_{\alpha \in \mathbb{F}_d^+} V_\alpha l_\alpha$ is an arbitrary vector in $\mathcal{K}$ and $j \in \{1, \ldots, d\}$, then $V_j x \sim \sum_{\alpha \in \mathbb{F}_d^+} V_{g_j \alpha} l_\alpha$. Thus, $x \in \operatorname{Dom}(\Phi_0)$ if and only if $V_j x \in \operatorname{Dom}(\Phi_0)$ and hence $\operatorname{Dom}(\Phi_0 V_j) = \operatorname{Dom}((S_j \otimes I_{\mathcal{L}}) \Phi_0)$. Moreover, for any $x = \sum_{|\alpha| \leq n} V_\alpha l_\alpha$ in $\operatorname{Dom}(\Phi_0)$, we have

$$(S_j \otimes I_{\mathcal{L}}) \Phi_0 x = \bigoplus_{|\alpha| \leq n} e_{g_j \alpha} \otimes l_\alpha = \Phi_0 V_j x,$$

and hence (4.5) holds.

It remains to prove that Φ_0 is closable. To see this, let $(x_n)_{n \geq 0}$ be a sequence of vectors in $\operatorname{Dom}(\Phi_0)$ such that $x_n \to 0 \in \mathcal{K}$ and $\Phi_0 x_n \to y \in \mathcal{F}_d^2 \otimes \mathcal{L}$, as $n \to \infty$. By definition, $x_n = \sum_{|\alpha| \leq m_n} V_\alpha l_{\alpha,n}$ for some $m_n \in \mathbb{N}$ and $\{l_{\alpha,n}\}_{|\alpha| \leq m_n}$ in $\mathcal{L}$. Then $\Phi_0 x_n = \bigoplus_{|\alpha| \leq m_n} e_\alpha \otimes l_{\alpha,n}$ for all $n \in \mathbb{N}$. For any $\sigma \in \mathbb{F}_d^+$, let $Q_\sigma \colon \mathcal{F}_d^2 \otimes \mathcal{L} \to \mathcal{L}$ be the operator defined by

$$Q_\sigma (\bigoplus_{\alpha \in \mathbb{F}_d^+} e_\alpha \otimes l_\alpha) = l_\sigma. \tag{4.7}$$

Clearly, Q_σ is a bounded linear operator and $Q_\sigma \Phi_0 x_n \to Q_\sigma y$ as $n \to \infty$. Setting $y := \bigoplus_{\alpha \in \mathbb{F}_d^+} e_\alpha \otimes l_\alpha$, where $\{l_\alpha\}_{\alpha \in \mathbb{F}_d^+} \subset \mathcal{L}$ and $\sum_{\alpha \in \mathbb{F}_d^+} \|l_\alpha\|^2 < \infty$, we have $l_{\sigma,n} \to l_\sigma$ as $n \to \infty$.

On the other hand, as a consequence of the regularity of the generating space $\mathcal{L}$ and the decompositin (3.4), there exists a unitary norm $\| \cdot \|$ on $\mathcal{K} = \mathcal{M}_+(\mathcal{L})$ such that, for any $\alpha \in \mathbb{F}_d^+$, $\sigma \neq \alpha$, we have $V_\sigma \mathcal{L} \perp V_\alpha \mathcal{L}$ with respect to the corresponding positive definite inner product on $\mathcal{K}$. In addition, the operator $U_\sigma = V_\sigma|\mathcal{L}\colon \mathcal{L} \to V_\sigma\mathcal{L}$ is a bounded unitary operator and, as a consequence, it has a bounded inverse. Hence, for all $n \in \mathbb{N}$,

$$\|x_n\|^2 = \|\sum_{|\alpha|\le m_n} V_\alpha l_{\alpha,n}\|^2 \ge \|V_\sigma l_{\sigma,n}\|^2 \ge \frac{1}{\|U_\sigma^{-1}\|}\|l_{\sigma,n}\|^2,$$

where we let $l_{\sigma,n} = 0$ if $|\sigma| > m_n$. Since $x_n \to 0$ strongly, we have $l_{\sigma,n} \to 0$ strongly. Thus $y = 0$, and this proves that the operator Φ_0 is closable. □

As a consequence of Lemma 4.1, we can define the *Fourier representation* associated with the d-orthogonal shift $\mathcal{V}$ as the (possibly unbounded) operator Φ, the closure of the operator Φ_0. In addition, another extension $\widetilde{\Phi}\colon \mathrm{Dom}\,(\widetilde{\Phi}) \to \mathcal{F}_d^2 \otimes \mathcal{L}$ of Φ_0 can be defined as follows:

$$\mathrm{Dom}\,(\widetilde{\Phi}) := \{x \in \mathcal{K} \mid \sum_{\alpha\in\mathbb{F}_d^+} \|P_\mathcal{L} V_\alpha^\sharp x\|^2 < \infty\}, \tag{4.8}$$

$$\widetilde{\Phi}(x) := \bigoplus_{\sigma\in\mathbb{F}_d^+} e_\sigma \otimes P_\mathcal{L} V_\sigma^\sharp x, \quad x \in \mathrm{Dom}\,(\widetilde{\Phi}). \tag{4.9}$$

Theorem 4.2. *The Fourier representation Φ associated with the d-orthogonal shift $\mathcal{V}$ is a densely defined closed isometry with dense range, such that*

$$\Phi(x) = \bigoplus_{\alpha\in\mathbb{F}_d^+} e_\alpha \otimes P_\mathcal{L} V_\alpha^\sharp x, \quad x \in \mathrm{Dom}\,(\Phi), \tag{4.10}$$

and, for any $j \in \{1, \dots, d\}$,

$$\Phi V_j = (S_j \otimes I_\mathcal{L})\Phi. \tag{4.11}$$

Proof. By the definition of Φ and the properties of Φ_0, it follows that Φ is a densely defined closed isometry with dense range. Since Φ is the smallest closed extension of Φ_0, and $\widetilde{\Phi}$ is an extension of Φ_0, in order to prove (4.10) it is sufficient to show that $\widetilde{\Phi}$ is closed as well. To see this, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{Dom}\,(\widetilde{\Phi})$ such that $\widetilde{\Phi}x_n \to x$ and $\widetilde{\Phi}x_n \to y$, as $n \to \infty$. A similar argument as in the proof of Lemma 4.1 shows that, for any $\sigma \in \mathbb{F}_d^+$, $P_\mathcal{L} V_\sigma^\sharp x_n \to Q_\sigma y$, as $n \to \infty$, where Q_σ is defined by (4.7). Since $P_\mathcal{L} V_\sigma^\sharp x_n \to P_\mathcal{L} V_\sigma^\sharp x$, we conclude that, for any $\sigma \in \mathbb{F}_d^+$, $P_\mathcal{L} V_\sigma^\sharp x = Q_\sigma y$. Since $y \in \mathcal{F}_d^2 \otimes \mathcal{L}$, we have $x \in \mathrm{Dom}\,(\widetilde{\Phi})$ and $\widetilde{\Phi}(x) = y$. Thus, $\widetilde{\Phi}$ is closed and (4.10) holds.

In order to prove (4.11), fix j in $\{1, \dots, d\}$. It is clear that (4.5) implies $\Phi V_j \supseteq (S_j \otimes I_\mathcal{L})\Phi$. To prove the converse inclusion, let $x \in \mathcal{K}$ be such that $V_j x \in \mathrm{Dom}\,(\Phi)$, that is, there exist $(x_n)_{n\in\mathbb{N}}$ with $x_n \to V_j x$ and $\Phi_0 x_n \to \Phi V_j x$, as $n \to \infty$. Then $V_j^\sharp V_j x_n \to V_j^\sharp V_j x = x$, as $n \to \infty$. Let $l_\alpha = P_\mathcal{L} V_\alpha^\sharp x$ and $l_{\alpha,n} = P_\mathcal{L} V_\alpha^\sharp x_n$ be the Fourier coefficients of x and x_n, respectively. Taking into account that

$$V_j^\sharp V_\alpha = \begin{cases} V_\beta, & \alpha = g_j\beta, \\ 0, & \text{otherwise}, \end{cases}$$

we have

$$V_j^\sharp x_n = \sum_{|\alpha|\le m_n} V_j^\sharp V_\alpha l_{\alpha,n} = \sum_{|\beta|\le m_n-1} V_\beta l_{g_j\beta,n}.$$

Then

$$\begin{aligned}\|\Phi_0 x_n - \Phi V_j x\|^2 &= \| \bigoplus_{|\alpha|\leq m_n} e_\alpha \otimes l_{\alpha,n} - \bigoplus_{\beta\in\mathbb{F}_d^+} e_{g_j\beta}\otimes l_\beta\|^2 \\ &= \sum_{\beta\in\mathbb{F}_d^+} \|l_{g_j\beta,n} - l_\beta\|^2 + \sum_{\alpha\notin g_j\mathbb{F}_d^+} \|l_{\alpha,n}\|^2,\end{aligned}$$

where we set $l_{\alpha,n}=0$ if $|\alpha|>m_n$. Since $\|\Phi_0 x_n - \Phi V_j x\| \to 0$ as $n\to\infty$, we infer that

$$\sum_{\beta\in\mathbb{F}_d^+} \|l_{g_j\beta,n} - l_\beta\|^2 \to 0, \quad n\to\infty.$$

Hence, the vector $h = \bigoplus_{\sigma\in\mathbb{F}_d^+} e_\sigma \otimes l_\sigma$ is in $\mathcal{F}_d^2\otimes\mathcal{L}$. Taking into account that

$$\|\Phi_0 V_j^\sharp x_n - h\|^2 = \sum_{\beta\in\mathbb{F}_d^+} \|l_{g_j\beta,n} - l_\beta\|^2,$$

it follows that $\Phi_0 V_j^\sharp x_n \to h$. Since $V_j^\sharp x_n \to x$, as $n\to\infty$, we infer that $x\in\operatorname{Dom}(\Phi)$ and $h=\Phi x$. This proves that $\operatorname{Dom}(\Phi V_j)\subseteq\operatorname{Dom}(\Phi)$ and hence (4.11) is completely proved. $\square$

We are now interested in characterizations of the boundedness of the Fourier representation associated to a d-orthogonal shift. We first make two observations.

Remark 4.3. We can introduce a partial order on $\mathbb{F}_d^+$. If $\alpha,\beta\in\mathbb{F}_d^+$, then we say that $\alpha\leq\beta$ whenever there exists $\sigma\in\mathbb{F}_d^+$ such that $\beta=\alpha\sigma$. Notice that, if $\alpha,\beta\in\mathbb{F}_d^+$, then

$$V_\alpha V_\alpha^\sharp V_\beta V_\beta^\sharp = \begin{cases} V_\beta V_\beta^\sharp, & \alpha\leq\beta, \\ V_\alpha V_\alpha^\sharp, & \beta\leq\alpha, \\ 0, & \text{all other cases.}\end{cases} \tag{4.12}$$

Now it is easy to see that the selfadjoint projections $V_\alpha V_\alpha^\sharp$ are mutually commutative.

In what follows, we use the notation $A\wedge B = AB$ and $A\vee B = A+B-AB$, whenever A,B are two commuting selfadjoint projections in $\mathcal{B}(\mathcal{K})$. Observe that the operators $A\wedge B$, $A\vee B$ and $I-A$ are selfadjoint projections.

Remark 4.4. Taking into account Remark 4.3, it makes sense to consider the Boolean algebra $\mathfrak{B}$ generated by the mutually commutative selfadjoint projections $\{V_\alpha V_\alpha^\sharp\}_{\alpha\in\mathbb{F}_d^+}$, that is, the smallest family of selfadjoint projections in $\mathcal{B}(\mathcal{K})$ invariant under the operations $\vee$, $\wedge$, and complementation (recall that, since $A\vee B = (I-A)\wedge(I-B)$, only $\wedge$ and complementation are sufficient).

For arbitrary $\alpha\in\mathbb{F}_d^+$, consider the sefadjoint projection Q_α from $\mathcal{K}$ onto the regular subspace $V_\alpha\mathcal{L}$, more precisely,

$$Q_\alpha x = V_\alpha P_{\mathcal{L}} V_\alpha^\sharp x, \quad x\in\mathcal{K}. \tag{4.13}$$

To see that $\{Q_\alpha\}_{\alpha\in\mathbb{F}_d^+}\subset\mathfrak{B}$, we use (4.12) and observe that

$$Q_\alpha = \left(I - \bigvee_{j=1}^{d} V_{\alpha g_j} V_{\alpha g_j}^\sharp\right)\wedge V_\alpha V_\alpha^\sharp. \tag{4.14}$$

Moreover, Q_α are atoms for the Boolean algebra $\mathfrak{B}$, in the sense that, for any $\alpha \in \mathbb{F}_d^+$ and $B \in \mathfrak{B}$ such that $B \leq Q_\alpha$, it follows that $B = 0$. In addition, for any $A \in \mathfrak{B}$ there exists a finite subset $G \subset \mathbb{F}_d^+$, such that $A = \sum_{\alpha \in G} Q_\alpha$, that is, the Boolean algebra $\mathfrak{B}$ is atomic.

We will use the following lemma which is a particular case of Lemma XV.6.2 from [7].

Lemma 4.5. *Let $\mathfrak{B}$ be a Boolean algebra of selfadjoint projections on the Kreĭn space $\mathcal{K}$. Then $\mathcal{B}$ is uniformly bounded if and only if there exists a fundamental symmetry J on $\mathcal{K}$ such that $JB = BJ$ for all $B \in \mathfrak{B}$, equivalently, $B = B^*$ for all $B \in \mathfrak{B}$, where $*$ denotes the involution with respect to the positive inner product $\langle \cdot, \cdot \rangle_J$.*

Theorem 4.6. *Given a d-orthogonal shift $\mathcal{V} := (V_1, \dots, V_d)$ on the Kreĭn space $\mathcal{K}$, with generating subspace $\mathcal{L}$, the following statements are equivalent:*

(1) *Any vector in $\mathcal{K}$ has square summable Fourier coefficients;*
(2) *The Fourier representation Φ is bounded (and hence, $\Phi \colon \mathcal{K} \to \mathcal{F}_d^2 \otimes \mathcal{L}$ is a unitary operator of Kreĭn spaces);*
(3) *The family $\{V_\alpha\}_{\alpha \in \mathbb{F}_d^+}$ and the Boolean algebra $\mathfrak{B}$ generated by the selfadjoint projections $\{V_\alpha V_\alpha^\sharp\}_{\alpha \in \mathbb{F}_d^+}$ are uniformly bounded in $\mathcal{B}(\mathcal{K})$.*

Proof. (1)$\Rightarrow$(2) If all the vectors in $\mathcal{K}$ have square-summable Fourier coefficients then the operator $\widetilde{\Phi}$ given by (4.8) and (4.9) is defined on $\mathcal{K}$. Since $\widetilde{\Phi}$ is closed, by the Closed Graph Theorem, it follows that $\widetilde{\Phi}$ is bounded. Thus, Φ is bounded and equals $\widetilde{\Phi}$. From Theorem 4.2, it follows that Φ is a bounded unitary operator of Kreĭn spaces.

(2)$\Rightarrow$(3) If Φ is bounded, then, by Theorem 4.2, it is a bounded unitary operator of Kreĭn spaces and, for all $\alpha \in \mathbb{F}_d^+$, we have $V_\alpha = \Phi^\sharp (S_\alpha \otimes I_{\mathcal{L}}) \Phi$. Hence, $V_\alpha V_\alpha^\sharp = \Phi^\sharp (S_\alpha \otimes I_{\mathcal{L}})(S_\alpha \otimes I_{\mathcal{L}})\Phi$. The latter implies that the Boolean algebras $\mathfrak{B}$ and $\mathfrak{S}$ are unitarily equivalent. Since $(S_\alpha \otimes I_{\mathcal{L}})^\sharp = (S_\alpha \otimes I_{\mathcal{L}})^*$, both $\{(S_\alpha \otimes I_{\mathcal{L}})\}_{\alpha \in \mathbb{F}_d^+}$ and the Boolean algebra $\mathfrak{S}$ are uniformly bounded by 1. Therefore, the family $\{V_\alpha\}_{\alpha \in \mathbb{F}_d^+}$ and the Boolean algebra $\mathfrak{B}$ are uniformly bounded by $\|\Phi\|^2$.

(3)$\Rightarrow$(1) Assume that both the family $\{V_\alpha\}_{\alpha \in \mathbb{F}_d^+}$ and the Boolean algebra $\mathfrak{B}$ generated by the selfadjoint projections $\{V_\alpha V_\alpha^\sharp\}_{\alpha \in \mathbb{F}_d^+}$ are uniformly bounded in $\mathcal{B}(\mathcal{K})$. By Lemma 4.5, we can choose a fundamental symmetry J of the Kreĭn space $\mathcal{K}$ such that all the projections in $\mathfrak{B}$ are selfadjoint with respect to the positive definite inner product $\langle \cdot, \cdot \rangle_J$, hence of norm 1. In particular, by Proposition 3.6, it follows that

$$x = \sum_{j=0}^{\infty} \sum_{|\beta| = j} V_\beta P_{\mathcal{L}} V_\beta^\sharp x, \quad x \in \mathcal{K},$$

where the series converges strongly. For any $\alpha \in \mathbb{F}_d^+$, let Q_α denote the selfadjoint projection onto the regular subspace $V_\alpha \mathcal{L}$, as defined in (4.13). According to Remark 4.4, $\{Q_\alpha\}_{\alpha \in \mathbb{F}_d^+} \subset \mathfrak{B}$ is a family of disjoint projections, orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_J$. Thus, there is a constant $K > 0$ such that

$$\begin{aligned}\sum_{\alpha\in\mathbb{F}_d^+} \|P_{\mathcal{L}} V_\alpha^\sharp x\|^2 &= \sum_{\alpha\in\mathbb{F}_d^+} \|V_\alpha^\sharp V_\alpha P_{\mathcal{L}} V_\alpha^\sharp x\|^2 \leq K \sum_{\alpha\in\mathbb{F}_d^+} \|V_\alpha P_{\mathcal{L}} V_\alpha^\sharp x\|^2 \\ &= K \sum_{\alpha\in\mathbb{F}_d^+} \|Q_\alpha x\|^2 = K\| \sum_{\alpha\in\mathbb{F}_d^+} Q_\alpha x\|^2 \\ &= K\| \sum_{\alpha\in\mathbb{F}_d^+} V_\alpha P_{\mathcal{L}} V_\alpha^\sharp x\|^2 = K\|x\|^2.\end{aligned}$$

Hence, x has square summable Fourier coefficients and the proof is complete. □

5. Isometric Dilations

Let $\mathcal{T} := (T_j)_{j=1}^d$ be a family of bounded operators on a Kreĭn space $\mathcal{H}$. A pair $(\mathcal{V}, \mathcal{K})$, where $\mathcal{V} := (V_j)_{j=1}^d$ is a d-tuple of bounded isometric operators on a Kreĭn space $\mathcal{K} \supseteq \mathcal{H}$, is called a *minimal isometric dilation* of $\mathcal{T}$ if

(i) $V_i^\sharp V_j = 0$ for all $i \neq j$, $i, j = 1, \ldots, d$;
(ii) $V_j^\sharp | \mathcal{H} = T_j^\sharp$, for all $j = 1, \ldots, d$;
(iii) $\mathcal{K} = \bigvee_{\alpha\in\mathbb{F}_d^+} V_\alpha \mathcal{H}$.

We remark that if $d = \infty$, then both row matrices $[T_1 \;\; T_2 \;\; \ldots]$ and $[V_1 \;\; V_2 \;\; \ldots]$ should be bounded operators. Let J be a fundamental symmetry on $\mathcal{H}$ and let $(\mathcal{H}^d, \langle\cdot,\cdot\rangle_{\widetilde{J}})$ be the direct sum of d copies of the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_J)$. The space $\mathcal{H}^d$ is regarded as a direct sum of Kreĭn spaces, where the fundamental symmetry $\widetilde{J}$ is the direct sum of d-copies of J. Let $T := [T_1 \;\; T_2 \;\; \ldots \;\; T_d] \in \mathcal{B}(\mathcal{H}^d, \mathcal{H})$ and consider the operators

$$J_T := \operatorname{sgn}(\widetilde{J} - T^* J T), \quad D_T := |\widetilde{J} - T^* J T|^{1/2}. \tag{5.1}$$

Then J_T is a selfadjoint partial isometry on $\mathcal{H}^d$ which commutes with D_T. The *defect space*

$$\mathcal{D}_T := \overline{D_T \mathcal{H}^d} = J_T \mathcal{H}^d \tag{5.2}$$

is invariant under both J_T and D_T, and J_T is a symmetry on $\mathcal{D}_T$. We define a new inner product on $\mathcal{D}_T$ by setting

$$[\xi, \eta]_{\mathcal{D}_T} := \langle J_T \xi, \eta\rangle, \quad \xi, \eta \in \mathcal{H}^d. \tag{5.3}$$

Consider now the Kreĭn space

$$\mathcal{K} := \mathcal{H} \oplus (\mathcal{F}_d^2 \otimes \mathcal{D}_T), \tag{5.4}$$

with the indefinite inner product defined by the symmetry

$$J_{\mathcal{K}} := J \oplus (I_{\mathcal{F}_d^2} \otimes J_T). \tag{5.5}$$

We consider $\mathcal{H}$ embedded into $\mathcal{K}$ onto the first component. For each $j = 1, \ldots, d$, define the bounded isometry $V_j \colon \mathcal{K} \to \mathcal{K}$ by

$$V_j(h \oplus (\xi \otimes k)) := T_j h \oplus \big(e_0 \otimes D_T(\underbrace{0, \ldots, 0}_{j-1 \text{ times}}, h, 0, \ldots) + (S_j \otimes I_{\mathcal{D}_T})(\xi \otimes k)\big). \tag{5.6}$$

where $h \in \mathcal{H}$, $\xi \in \mathcal{F}_d^2$, $k \in \mathcal{D}_T$, and $(S_j)_{j=1}^d$ is the canonical d-orthogonal shift on $\mathcal{F}_d^2$.

We say that two minimal isometric dilations $(\mathcal{V},\mathcal{K})$ and $(\mathcal{V}',\mathcal{K}')$, of the same d-tuple $\mathcal{T}$ of operators in $\mathcal{H}$, are *unitarily equivalent* if there exists $\Psi\in\mathcal{B}(\mathcal{K},\mathcal{K}')$, a unitary operator of Kreĭn spaces, such that $\Psi|\mathcal{H}=I_{\mathcal{H}}$ and $\Psi V_j=V_j'\Psi$ for all $j=1,\dots,d$.

Theorem 5.1. *Let $T:=(T_j)_{j=1}^d$ be a d-tuple of bounded operators on a Kreĭn space $\mathcal{H}$. The pair $(\mathcal{V},\mathcal{K})$ defined by the relations (5.4) and (5.6) is a minimal isometric dilation of $\mathcal{T}$. Moreover, $\mathcal{T}$ admits a unique minimal isometric dilation, up to unitary equivalence, if and only if the operator $T:=[T_1\ \ T_2\ \dots\ T_d]$ is either contractive or expansive.*

Proof. The proof of the fact that (5.4) and (5.6) define a minimal isometric dilation of $\mathcal{T}$ is essentially the same as the proof of the noncommutative dilation theorem from [16] and therefore, we shall omit the details. We only need to add that if $d=\infty$, then $V:=[V_1\ \ V_2\ \dots]$ is a bounded operator. Indeed, this follows from (5.6) if one takes into account that $T:=[T_1\ \ T_2\ \dots]$ is a bounded operator.

Let us now assume that the operator $T:=[T_1\ \ T_2\ \dots\ T_d]$ is contractive, that is, $\widetilde{J}-T^*JT\geq 0$. Consequently, the operator J_T is the identity on $\mathcal{D}_T$, and the Kreĭn space $\mathcal{D}_T$ is positive definite. Let $(\mathcal{V}',\mathcal{K}')$ be another minimal isometric dilation of $\mathcal{T}$, where $\mathcal{V}'=(V_1',\dots,V_d')$. Since, for arbitrary $N\geq 0$ and $\{h_\alpha\}_{|\alpha|\leq N},\{k_\alpha\}_{|\alpha|\leq N}\subset\mathcal{H}$ we have

$$[\sum_{|\alpha|\leq N}V_\alpha h_\alpha,\sum_{|\alpha|\leq N}V_\alpha k_\alpha]_{\mathcal{K}}=[\sum_{|\alpha|\leq N}V_\alpha' h_\alpha,\sum_{|\alpha|\leq N}V_\alpha' k_\alpha]_{\mathcal{K}'},$$

taking into account of the minimality of the isometric dilation, it follows that the mapping

$$\Psi(\sum_{|\alpha|\leq N}V_\alpha h_\alpha)=\sum_{|\alpha|\leq N}V_\alpha' h_\alpha,\quad N\geq 0,\ \{h_\alpha\}_{|\alpha|\leq N}\subset\mathcal{H}, \tag{5.7}$$

is well-defined, and the linear operator Ψ is isometric, densely defined and with dense range.

Let $\mathcal{H}=\mathcal{H}^+\oplus\mathcal{H}^-$ be a fundamental decomposition of the Kreĭn space $\mathcal{H}$. Note that, since $\mathcal{D}_T$ is positive definite, it follows that the Kreĭn space $\mathcal{F}_d^2\otimes\mathcal{D}_T$ is positive definite and hence, $\mathcal{K}=(\mathcal{H}^+\oplus\mathcal{F}_d^2\otimes\mathcal{D}_T)\oplus\mathcal{H}^-$ is a fundamental decomposition of $\mathcal{K}$. Thus, $\mathcal{H}^-$ is contained in the domain of Ψ and we can apply Theorem VI.3.5 from [1], to conclude that Ψ is bounded, and hence a unitary operator. It is now clear that Ψ is the required unitary equivalence of the minimal isometric dilations $(\mathcal{V},\mathcal{K})$ and $(\mathcal{V}',\mathcal{K}')$. A similar argument holds in the case when T is expansive, that is, $\widetilde{J}-T^*JT\leq 0$.

Let us assume now that T is neither contractive, nor expansive. Then $\mathcal{D}_T$ is an indefinite Kreĭn space, that is, it contains positive as well as negative vectors. As in the proof of Proposition 3.8, there is a unitary operator $U\in\mathcal{B}(\mathcal{D}_T)$ such that the d-orthogonal shift $\{S_j\otimes U\}_{j=1}^d$, defined on the Kreĭn space $\mathcal{F}_d^2\otimes\mathcal{D}_T$, is not unitary equivalent to the d-orthogonal shift $\{S_j\otimes I_{\mathcal{D}_T}\}_{j=1}^d$. Let $\mathcal{K}$ be defined as in (5.4) and let $\mathcal{V}':=(V_1',\dots,V_d')$ be defined by

$$V_j'(h\oplus(\xi\otimes k)):=T_jh\oplus\big(e_0\otimes D_T(\underbrace{0,\dots,0}_{j-1\text{ times}},h,0,\dots)+(S_j\otimes U)(\xi\otimes k)\big). \tag{5.8}$$

where $h\in\mathcal{H}$, $\xi\in\mathcal{F}_d^2$ and $k\in\mathcal{D}_T$. Then $(\mathcal{V}',\mathcal{K})$ is a minimal isometric dilation of $\mathcal{T}$, that is not unitarily equivalent to $(\mathcal{V},\mathcal{K})$. Indeed, suppose that there exists $\Psi\in\mathcal{B}(\mathcal{K})$, a unitary operator of Kreĭn space, such that $\Psi|\mathcal{H}=I_{\mathcal{H}}$ and $\Psi V_j=V_j'\Psi$ for all $j=1,\dots,d$. Since $\Psi\mathcal{H}=\mathcal{H}$, it follows that $\Psi(\mathcal{F}_d^2\otimes\mathcal{D}_T)=\mathcal{F}_d^2\otimes\mathcal{D}_T$. When restricted to $\mathcal{F}_d^2\otimes\mathcal{D}_T$, both $\mathcal{V}$ and $\mathcal{V}'$ become d-orthogonal shifts that are unitarily equivalent by the unitary operator $\Psi|(\mathcal{F}_d^2\otimes\mathcal{D}_T)$. According to Proposition 3.8, this is a contradiction. The proof is complete. $\square$

The isometric dilation $(\mathcal{V}, \mathcal{K})$, defined by (5.4) and (5.6), is called the *canonical minimal isometric dilation* of $\mathcal{T}$.

6. The Geometry of the Canonical Minimal Isometric Dilation

Throughout this section, $\mathcal{H}$ is a *Hilbert space* and $\mathcal{T} := (T_1, \ldots, T_d)$ is a sequence of operators on $\mathcal{H}$. Consider the operator $T \in \mathcal{B}(\mathcal{H}^d, \mathcal{H})$ defined by the row matrix $[T_1 \ \ldots \ T_d]$, where $\mathcal{H}^d$ is the direct sum of d copies of $\mathcal{H}$. The operators D_T, J_T are defined as in (5.1), where $J = I_{\mathcal{H}}$ and $\widetilde{J} = I_{\mathcal{H}^d}$.

We recall that the canonical minimal isometric dilation $(\mathcal{V}, \mathcal{K})$ is defined by the relations (5.4) and (5.6). In addition, with respect to the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}^d$, we introduce the operators

$$J_{T^*} := \operatorname{sgn}(I - \sum_{j=1}^{d} T_j T_j^*), \quad D_{T^*} := |I - \sum_{j=1}^{d} T_j T_j^*|^{1/2}. \tag{6.1}$$

It is easy to see that J_{T^*} is a selfadjoint partial isometry which commutes with the positive operator D_{T^*}. In particular, the subspace

$$\mathcal{D}_{T^*} := \overline{D_{T^*}\mathcal{H}} = J_{T^*}\mathcal{H}, \tag{6.2}$$

is invariant under both J_{T^*} and D_{T^*}, and $J_{T^*}|\mathcal{D}_{T^*}$ is a symmetry on $\mathcal{D}_{T^*}$. Define the new inner product $[\cdot,\cdot]_{\mathcal{D}_{T^*}}$ on the Hilbert space $\mathcal{D}_{T^*}$ by setting

$$[\xi, \eta]_{\mathcal{D}_{T^*}} := \langle J_{T^*}\xi, \eta\rangle, \quad \xi, \eta \in \mathcal{D}_{T^*}. \tag{6.3}$$

Then $(\mathcal{D}_{T^*}, [\cdot,\cdot]_{\mathcal{D}_{T^*}})$ is a Kreĭn space called the *adjoint defect space* of $\mathcal{T}$. Recall that the following intertwining relation holds

$$TD_T = D_{T^*}T. \tag{6.4}$$

Define the subspaces

$$\mathcal{L} := \bigvee_{j=1}^{d} (V_j - T_j)\mathcal{H} \quad \text{and} \quad \mathcal{L}_* := \overline{\big(I_{\mathcal{H}} - \sum_{j=1}^{d} V_j T_j^*\big)\mathcal{H}}. \tag{6.5}$$

Lemma 6.1. (a) *The subspaces $\mathcal{L}$ and $\mathcal{L}_*$ are wandering for $\mathcal{V}$, unitarily isomorphic to the Kreĭn spaces $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$, respectively. In addition, $\mathcal{L} \cap \mathcal{L}_* = \{0\}$.*

(b) *The subspaces $\mathcal{M}_+(\mathcal{L})$ and $\mathcal{M}_+(\mathcal{L}_*)$, defined as in (3.3), are invariant under each $V_1, \ldots, V_d, V_1^\sharp, \ldots, V_d^\sharp$.*

(c) *The subspace $\mathcal{M}_+(\mathcal{L})$ is regular, $\mathcal{K} = \mathcal{H} \oplus \mathcal{M}_+(\mathcal{L})$, and the Fourier representation of the d-orthogonal shift $\mathcal{V}|\mathcal{M}_+(\mathcal{L}) := (V_1|\mathcal{M}_+(\mathcal{L}), \ldots, V_d|\mathcal{M}_+(\mathcal{L}))$ is bounded.*

Proof. Let us observe that

$$\sum_{j=1}^{d}(V_j - T_j)h_j = 0 \oplus \big(e_0 \otimes D_T(\bigoplus_{j=1}^{d} h_j)\big), \quad h_j \in \mathcal{H}, \ j = 1, \ldots, d. \tag{6.6}$$

This shows that $\mathcal{L}$ is invariant under the fundamental symmetry $J_{\mathcal{K}}$ defined by (5.5), hence regular. In addition, relation (6.6) shows that the mapping

$$\phi(\sum_{j=1}^{d}(V_j - T_j)h_j) = D_T(\bigoplus_{j=1}^{d} h_j), \quad h_j \in \mathcal{H},\ j = 1, \dots, d, \tag{6.7}$$

is a unitary (hence bounded) operator $\phi \in \mathcal{B}(\mathcal{L}, \mathcal{D}_T)$ of Kreĭn spaces. Actually, since $\phi J_{\mathcal{K}}|\mathcal{L} = J_T\phi$, the operator ϕ is a unitary operator with respect to the underlying Hilbert spaces $(\mathcal{L}, \langle\cdot,\cdot\rangle_{J_{\mathcal{K}}})$ and $(\mathcal{D}_T, \langle\cdot,\cdot\rangle_{\mathcal{H}})$.

As in [17], it can be verified that $\mathcal{L}$ is wandering for $\mathcal{V}$. Using (6.6) and (5.6), we can show that $\mathcal{M}_+(\mathcal{L})$ is invariant under the fundamental symmetry $J_{\mathcal{K}}$, by checking on finite linear combinations. Thus, $\mathcal{M}_+(\mathcal{L})$ is a regular subspace of $\mathcal{K}$. Since $\mathcal{V}|\mathcal{M}_+(\mathcal{L})$ is implemented by the canonical d-orthogonal shift with generating space $\mathcal{D}_T$, it follows that its Fourier representation is bounded, cf. Theorem 4.6.

To see that the subspace $\mathcal{L}_*$ is regular, we observe that

$$\mathcal{L}_* = (I - \sum_{j=1}^{d} V_j V_j^{\sharp})\mathcal{K}. \tag{6.8}$$

Indeed, taking into account that the isometries $V_1, \dots, V_d$ have mutual orthogonal ranges, for any $\alpha \in \mathbb{F}_d^+$ with $|\alpha| \geq 1$, we have

$$(I_{\mathcal{H}} - \sum_{j=1}^{d} V_j V_j^{\sharp})V_\alpha h = V_\alpha h - V_\alpha h = 0, \quad h \in \mathcal{H}.$$

Hence, (6.8) holds, if we take into account the minimality of $\mathcal{K} = \bigvee_{\alpha\in\mathbb{F}_d^+} V_\alpha\mathcal{H}$. The fact that $\mathcal{L}_*$ is a wandering subspace for $\mathcal{V}$ was mentioned in connection with (3.8).

Observe now that

$$\phi_*((I - \sum_{j=1}^{d} V_j T_j^*)h = J_{T^*}D_{T^*}h, \quad h \in \mathcal{H}, \tag{6.9}$$

is a well-defined linear isometry from $(I - \sum_{j=1}^{d} V_j T_j^*)\mathcal{H}$ to $\mathcal{D}_{T^*}$. To see this, assume that $J_{T^*}D_{T^*}h = 0$ for some $h \in \mathcal{H}$. Then $(I - \sum_{j=1}^{d} T_j T_j^*)h = D_{T^*}J_{T^*}D_{T^*}h = 0$ and, using (6.4), we also have $D_T T^* h = T^* D_{T^*}h = 0$. Hence, and taking into account (5.6), it follows that $(I - \sum_{j=1}^{d} V_j T^*)h = 0$. Let $h, h' \in \mathcal{H}$ be arbitrary. Then

$$\begin{aligned}
[(I - \sum_{j=1}^{d} V_j T_j^*)h, (I - \sum_{j=1}^{d} V_j T_j^*)h']_{\mathcal{K}} &= [h, h'] - \sum_{j=1}^{d}[T_j^*h, T_j^*h']_{\mathcal{K}} \\
&= \langle (I - \sum_{j=1}^{d} T_j T_j^*)h, h'\rangle_{\mathcal{H}} \\
&= [D_{T^*}h, D_{T^*}h']_{\mathcal{D}_{T^*}},
\end{aligned}$$

which shows that ϕ_* is isometric. Notice that ϕ_* is densely defined in $\mathcal{L}_*$ and has dense range in $\mathcal{D}_{T^*}$. Taking into account that $\mathcal{L}_*$ is regular, we infer that ϕ_* is injective. Therefore, $\kappa^{\pm}(\mathcal{L}_*) = \kappa^{\pm}(\mathcal{D}_{T^*})$ and the regular subspace $\mathcal{L}_*$ is isometrically isomorphic to the Kreĭn space $\mathcal{D}_{T^*}$. The invariance of the subspaces $\mathcal{M}_+(\mathcal{L})$ and $\mathcal{M}_+(\mathcal{L}_*)$ under $V_1, \ldots, V_d, V_1^\sharp, \ldots, V_d^\sharp$ follows essentially as in [17].

It remains to prove that $\mathcal{L} \cap \mathcal{L}_* = \{0\}$. For this, we first show that

$$\mathcal{L}_* \oplus (\bigoplus_{j=1}^{d} V_j \mathcal{H}) = \mathcal{H} \oplus \mathcal{L}. \tag{6.10}$$

Indeed, we will actually prove a stronger equality between linear manifolds, namely

$$(I - \sum_{j=1}^{d} V_j T_j^*)\mathcal{H} + \sum_{j=1}^{d} V_j \mathcal{H} = \mathcal{H} + \sum_{j=1}^{d} (V_j - T_j)\mathcal{H}. \tag{6.11}$$

To this end, let $h \in \mathcal{H}, h_1, \ldots, h_d \in \mathcal{H}$ be arbitrary. We have

$$\begin{aligned} h - \sum_{j=1}^{d} V_j T_j^* h + \sum_{j=1}^{d} V_j h_j &= \sum_{j=1}^{d} T_j h_j + (I - \sum_{j=1}^{d} T_j T_j^*) h \\ &\quad + \sum_{j=1}^{d} (V_j - T_j)(h_j - T_j^* h), \end{aligned}$$

which proves one inclusion in (6.11). For the converse, we use the fact that

$$\begin{aligned} h - \sum_{j=1}^{d} (V_j - T_j) h_j &= (I - \sum_{j=1}^{d} V_j T_j^*)(h - \sum_{j=1}^{d} T_j h_j) \\ &\quad + \sum_{j=1}^{d} V_j (T_j^* h + h_j - \sum_{i=1}^{d} T_j^* T_i h_j). \end{aligned}$$

Thus, (6.11) holds. Using the definitions of $\mathcal{L}$ and $\mathcal{L}_*$, we obtain (6.10) by an approximation argument. Furthermore, according to (5.6), it follows that $\mathcal{H} \vee \bigoplus_{j=1}^{d} V_j \mathcal{H} = \mathcal{H} \oplus \mathcal{L}$. By taking orthogonals and using (6.10), we deduce that $\mathcal{L} \cap \mathcal{L}_* = \{0\}$. □

The subspace

$$\mathcal{R} := \mathcal{M}_+(\mathcal{L}_*)^{\perp}, \tag{6.12}$$

is called the *residual space* of the minimal isometric dilation $\mathcal{V}$. The following characterization of the residual space $\mathcal{R}$ will play an important rôle in our investigation. We should remark that the case $d = 1$ was considered by B. McEnnis in [12].

Theorem 6.2. *A vector $k \in \mathcal{K}$ belongs to the residual space $\mathcal{R}$ if and only if there exists a family $\{h_\sigma\}_{\sigma \in \mathbb{F}_d^+}$ of vectors in $\mathcal{H}$ such that:*

(i) *$h_0 = P_{\mathcal{H}} k$;*

(ii) *$\sum_{j=1}^{d} T_j h_{\sigma g_j} = h_\sigma$ for all $\sigma \in \mathbb{F}_d^+$;*

(iii) $\{\sum_{j=1}^{d}(V_j - T_j)h_{\sigma g_j}\}_{\sigma\in\mathbb{F}_d^+}$ *is the sequence of Fourier coefficients of the vector* $P_{\mathcal{M}_+(\mathcal{L})}k$ *in* $\mathcal{M}_+(\mathcal{L})$.

Moreover, $\{h_\sigma\}_{\sigma\in\mathbb{F}_d^+}$ *and* k *uniquely determine each other, with the above properties.*

Proof. Let $k \in \mathcal{K}$ and assume that $\{h_\sigma\}_{\sigma\in\mathbb{F}_d^+}$ satisfies the conditions (i) through (iii). Since $\mathcal{K} = \mathcal{H} \oplus \mathcal{M}_+(\mathcal{L})$, the vector k has a unique respresentation $k = h_0 + s$ where $h_0 = P_{\mathcal{H}}k \in \mathcal{H}$ and $s = P_{\mathcal{M}_+(\mathcal{L})}k \in \mathcal{M}_+(\mathcal{L})$. Since $\{h_\sigma\}_{\sigma\in\mathbb{F}_d^+}$ is the sequence of Fourier coefficients of s in $\mathcal{M}_+(\mathcal{L})$, we have

$$s - \sum_{|\sigma|\le N} V_\sigma l_\sigma \in \bigvee_{|\beta|=N+1} V_\beta \mathcal{M}_+(\mathcal{L}), \quad N = 0, 1, \dots . \tag{6.13}$$

For any $l_* = (I - \sum_{j=1}^{d} V_j T_j^*)h$ and $\beta = g_{i_1}\cdots g_{i_N} \in \mathbb{F}_d^+$ with $|\beta| = N \ge 0$, let us prove that

$$[k, V_\beta l_*]_{\mathcal{K}} = 0. \tag{6.14}$$

Taking into account (6.13) and that $V_\sigma \mathcal{L}_* \perp V_\sigma \mathcal{M}_+(\mathcal{L}_*)$, we have

$$\begin{aligned}[k, V_\beta l_*]_{\mathcal{K}} &= [h_0 + s, V_\beta l_*]_{\mathcal{K}} \\ &= [h_0, V_\beta l_*]_{\mathcal{K}} + \sum_{|\sigma|\le N-1} [V_\sigma l_\sigma, V_\beta l_*]_{\mathcal{K}} + \sum_{|\alpha|=N} [V_\alpha l_\alpha, V_\beta l_*]_{\mathcal{K}}.\end{aligned}$$

Since $V_j^\sharp|\mathcal{H} = T_j^\sharp$, we deduce that

$$\begin{aligned}[h_0, V_\beta l_*]_{\mathcal{K}} &= [h_0, V_\beta(I - \sum_{j=1}^{d} V_j T_j^*)h]_{\mathcal{K}} = \langle h_0, T_\beta(I - \sum_{j=1}^{d} T_j T_j^*)h\rangle_{\mathcal{H}} \\ &= \langle T_\beta^* h_0, (I - \sum_{j=1}^{d} T_j T_j^*)h\rangle_{\mathcal{H}}.\end{aligned}$$

On the other hand, since $V_i^\sharp V_j = 0$ for $i \ne j$, and using (ii) and (iii), we obtain

$$\begin{aligned}\sum_{|\alpha|=N} [V_\alpha, V_\beta l_*]_{\mathcal{K}} &= [V_\beta l_\beta, V_\beta l_*]_{\mathcal{K}} = [l_\beta, l_*]_{\mathcal{K}} \\ &= [\sum_{j=1}^{d}(V_j - T_j)h_{\beta g_j}, (I - \sum_{j=1}^{d} V_j T_j^*)h]_{\mathcal{K}} \\ &= [\sum_{j=1}^{d} V_j h_{\beta g_j}, (I - \sum_{j=1}^{d} V_j T_j^*)h]_{\mathcal{K}} - [\sum_{j=1}^{d} T_j h_{\beta g_j}, (I - \sum_{j=1}^{d} V_j T_j^*)h]_{\mathcal{K}}.\end{aligned}$$

Since $\mathcal{L}_* \perp V_j\mathcal{K}$ for $j = 1, \dots, d$, and $h_\beta = \sum_{j=1}^{d} T_j h_{\beta g_j}$, we get

$$\sum_{|\alpha|=N} [V_\alpha l_\alpha, V_\beta l_*]_{\mathcal{K}} = -\langle h_\beta, (I - \sum_{j=1}^{d} T_j T_j^*)h\rangle_{\mathcal{H}}.$$

Now, fix $\sigma \in \mathbb{F}_d^+$, $|\sigma| \leq N-1$, and calculate $[V_\sigma l_\sigma, V_\beta l_*]_{\mathcal{K}}$. If $\sigma \not\leq \beta$ (see the definition in Remark 4.3), then $[V_\sigma l_\sigma, V_\beta l_*]_{\mathcal{K}} = 0$. When $\sigma \leq \beta$, that is $\beta = \sigma\gamma$ for some $\gamma \in \mathbb{F}_d^+$ with $|\sigma| = N - |\gamma|$, we have

$$\begin{aligned}
[V_\sigma l_\sigma, V_\beta l_*]_{\mathcal{K}} &= [\sum_{j=1}^d (V_j - T_j) h_{\sigma g_j}, V_\gamma (I - \sum_{j=1}^d V_j T_j^*) h]_{\mathcal{K}} \\
&= \sum_{j=1}^d [V_j h_{\sigma g_j}, V_\gamma (I - \sum_{j=1}^d V_j T_j^*) h]_{\mathcal{K}} - \langle \sum_{j=1}^d T_j h_{\sigma g_j}, T_\gamma (I - \sum_{j=1}^d T_j T_j^*) h \rangle_{\mathcal{H}} \\
&= \langle h_{\sigma g_p}, T_\alpha (I - \sum_{j=1}^d T_j T_j^*) h \rangle_{\mathcal{H}} - \langle h_\sigma, T_\gamma (I - \sum_{j=1}^d T_j T_j^*) h \rangle_{\mathcal{H}} \\
&= \langle T_\alpha^* h_{\sigma g_p} - T_\gamma^* h_\sigma, (I - \sum_{j=1}^d T_j T_j^*) h \rangle_{\mathcal{H}},
\end{aligned}$$

where $\gamma = g_p \alpha$ for some $p \in \{1, \ldots, d\}$, and $|\alpha| = N - |\sigma| - 1$. Using this result when $k = 1, \ldots, N$, $\sigma = g_{i_0} g_{i_1} \cdots g_{i_{k-1}}$, $g_p = g_{i_k}$ and $\alpha = g_{i_{k+1}} \cdots g_{i_N} g_{i_{N+1}}$ (here, $g_{i_0} = g_{i_{N+1}} = g_0$, the neutral element in $\mathbb{F}_d^+$), and summing up, we obtain

$$\begin{aligned}
\sum_{|\sigma| \leq N-1} [V_\sigma l_\sigma, V_\beta l_*]_{\mathcal{K}} &= \langle \sum_{k=1}^N (T^*_{g_{i_{k+1}} \cdots g_{i_{N+1}}} h_{g_{i_0} \cdots g_{i_k}} - T^*_{g_{i_k} \cdots g_{i_{N+1}}} h_{g_{i_0} \cdots g_{i_{k-1}}}), (I - \sum_{j=1}^d T_j T_j^*) h \rangle_{\mathcal{H}} \\
&= \langle h_{g_{i_0} \cdots g_{i_N}} - T^*_{g_{i_1} \cdots g_{i_{N+1}}} h_0, (I - \sum_{j=1}^d T_j T_j^*) h \rangle_{\mathcal{H}} \\
&= \langle h_\beta - T_\beta^* h_0, (I - \sum_{j=1}^d T_j T_j^*) h \rangle_{\mathcal{H}}
\end{aligned}$$

(notice the telescopic sum on the right hand side of the first equality). So far, we have proved that (6.14) holds for all $\beta \in \mathbb{F}_d^+$. Since the linear manifold $(I - \sum_{j=1}^d V_j T_j^*)\mathcal{H}$ is dense in $\mathcal{L}_*$, we infer that $k \perp \mathcal{M}_+(\mathcal{L}_*)$, and hence $k \in \mathcal{R}$.

Conversely, suppose that $k \in \mathcal{R}$. We will construct $\{h_\sigma\}_{\sigma \in \mathbb{F}_d^+}$ inductively. Let $h_0 = P_{\mathcal{L}_*} k$, $N \geq 0$ and assume that $\{h_\sigma\}_{|\sigma| \leq N}$ has been defined so that:

(ii)′ $\sum_{j=1}^d T_j h_{\sigma g_j} = h_\sigma$ for all $\sigma \in \mathbb{F}_d^+$ with $|\sigma| \leq N-1$;

(iii)′ $\sum_{j=1}^d (V_j - T_j) h_{\sigma g_j} = l_\sigma$ for $|\sigma| \leq N-1$, where $\{l_\sigma\}_{\sigma \in \mathbb{F}_d^+}$ are the Fourier coefficients of the vector $P_{\mathcal{M}_+(\mathcal{L})} k$ in $\mathcal{M}_+(\mathcal{L})$.

Let now $\beta \in \mathbb{F}_d^+$, $|\beta| = N$ and $l_* = (I - \sum_{j=1}^d V_j T_j^*)h$ be fixed. Since $k \in \mathcal{R}$, we have $[k, V_\beta l_*] = 0$. According to our previous calculations, we have

$$\begin{aligned} 0 &= [h_0, V_\beta l_*]_\mathcal{K} + \sum_{|\sigma|\leq N-1} [V_\alpha l_\alpha, V_\beta l_*]_\mathcal{K} \\ &= \langle T_\beta^* h_0, (I - \sum_{j=1}^d T_j T_j^*)h\rangle_\mathcal{H} + \langle h_\beta - T_\beta^* h_0, (I - \sum_{j=1}^d T_j T_j^*)h\rangle_\mathcal{H} + [l_\beta, l_*]_\mathcal{K} \\ &= \langle h_\beta, (I - \sum_{j=1}^d T_j T_j^*)h\rangle_\mathcal{H} + [l_\beta, l_*]_\mathcal{K}, \end{aligned}$$

and hence

$$[l_\beta, l_*]_\mathcal{K} = -\langle h_\beta, (I - \sum_{j=1}^d T_j T_j^*)h\rangle_\mathcal{H}. \tag{6.15}$$

On the other hand, since $\mathcal{L} \perp \mathcal{H}$ and $l_\beta \in \mathcal{L}$, we have

$$\begin{aligned} [l_\beta, l_*]_\mathcal{K} &= [l_\beta, (I - \sum_{j=1}^d V_j T_j^*)h]_\mathcal{K} \\ &= -[l_\beta, \sum_{j=1}^d (V_j T_j^* - T_j T_j^*)h]_\mathcal{K} \\ &= -[l_\beta, \sum_{j=1}^d (V_j - T_j)T_j^* h]_\mathcal{K}. \end{aligned}$$

Hence, and using (6.15), we deduce that

$$\langle h_\beta, (I - \sum_{j=1}^d T_j T_j^*)h\rangle_\mathcal{H} = [l_\beta, \sum_{j=1}^d (V_j - T_j)T_j^* h]_\mathcal{K}. \tag{6.16}$$

Define $Q \in \mathcal{B}(\mathcal{H}^d, \mathcal{L})$ by

$$Q(\bigoplus_{j=1}^d h_j) := \sum_{j=1}^d (V_j - T_j)h_j, \quad h_j \in \mathcal{H}.$$

It is easy to see that (6.16) implies

$$(I - \sum_{j=1}^d T_j T_j^*)h_\beta = [T_1 \quad \ldots \quad T_d]Q^\sharp l_\beta. \tag{6.17}$$

For each $j = 1, \ldots, d$, define

$$h_{\beta g_j} = T_j^* h_\beta + P_j Q^\sharp l_\beta, \tag{6.18}$$

where P_j denotes the orthogonal projection of $\mathcal{H}^d$ onto its j-th coordinate. Using (6.17) and (6.18), we get

$$\sum_{j=1}^{d} T_j h_{\beta g_j} = \sum_{j=1}^{d} T_j T_j^* h_\beta + [T_1 \ \dots \ T_d] Q^\sharp l_\beta = h_\beta. \tag{6.19}$$

Thus, the condition (ii)$'$ is satisfied for all $|\sigma| \le N$. According to (6.18), we have

$$\begin{aligned}
[l_\beta, \sum_{j=1}^{d} (V_j - T_j) h_j]_{\mathcal{K}} &= [l_\beta, Q(\bigoplus_{j=1}^{d} h_j)]_{\mathcal{K}} = \langle Q^\sharp l_\beta, \bigoplus_{j=1}^{d} h_j \rangle_{\mathcal{H}} \\
&= \sum_{j=1}^{d} \langle P_j Q^\sharp l_\beta, h_j \rangle_{\mathcal{H}} \\
&= \sum_{j=1}^{d} \langle h_{\beta g_j} - T_j^* h_\beta, h_j \rangle_{\mathcal{H}}. \qquad (6.20)
\end{aligned}$$

On the other hand, making use of (6.19) and that $V_j^\sharp | \mathcal{H} = T_j^\sharp$, we get

$$\begin{aligned}
[\sum_{j=1}^{d} (V_j - T_j) h_{\beta g_j}, \sum_{j=1}^{d} (V_j - T_j) h_j]_{\mathcal{K}} &= \sum_{i,j=1}^{d} [V_i h_{\beta g_i}, V_j h_j]_{\mathcal{K}} - \langle \sum_{j=1}^{d} T_j h_{\beta g_j}, \sum_{j=1}^{d} T_j h_j \rangle_{\mathcal{H}} \\
&= \sum_{j=1}^{d} \langle h_{\beta g_j}, h_j \rangle_{\mathcal{H}} - \langle h_\beta, \sum_{j=1}^{d} T_j h_j \rangle_{\mathcal{H}} \\
&= \sum_{j=1}^{d} \langle h_{\beta g_j} - T_j^* h_\beta, h_j \rangle_{\mathcal{H}}.
\end{aligned}$$

Comparing this with (6.20), we get

$$l_\beta = \sum_{j=1}^{d} (V_j - T_j) h_{\beta g_j}, \tag{6.21}$$

which shows that condition (iii) holds for any $\beta \in \mathbb{F}_d^+$, $|\beta| = N$. This completes the induction argument.

We now prove that k and $\{h_\sigma\}_{\sigma \in \mathbb{F}_d^+}$ uniquely determine each other, under the conditions (i) through (iii). To this end, let us first remark that this correspondence is linear. Then, if $h_\sigma = 0$ for all $\sigma \in \mathbb{F}_d^+$, it follows that $h_0 = 0$ and $k \in \mathcal{M}_+(\mathcal{L})$. Since $\{l_\sigma\}_{\sigma \in \mathbb{F}_d^+}$ are the Fourier coefficients of k in $\mathcal{M}_+(\mathcal{L})$, relation (iii) implies that $l_\sigma = 0$ for all $\sigma \in \mathbb{F}_d^+$. Since $\mathcal{M}_+(\mathcal{L})$ is regular, hence nondegenerate, Proposition 3.1 shows that $k = 0$. Thus, $\{h_\sigma\}_{\sigma \in \mathbb{F}_d^+}$ uniquely determines k.

To prove the converse statement, let us first recall that $\{l_\sigma\}_{\sigma \in \mathbb{F}_d^+}$ is uniquely determined by k. From (6.21) we get

$$l_\beta = \sum_{j=1}^{d} V_j h_{\beta g_j} - \sum_{j=1}^{d} T_j h_{\beta g_j}.$$

On the other hand, using the property (i), we obtain

$$\sum_{j=1}^{d} V_j h_{\beta g_j} = l_\beta h_\beta.$$

Taking into account that $V_i^\sharp V_j = 0$ whenever $i \neq j$, we get

$$h_{\beta h_j} = V_j^\sharp (l_\beta + h_\beta), \quad j = 1, 2, \ldots, d, \tag{6.22}$$

which shows that $\{l_\sigma\}_{\sigma \in \mathbb{F}_d^+}$ is uniquely determined by k. □

We now consider some important consequences of this theorem.

Corollary 6.3. *The residual space $\mathcal{R}$ is nonnegative, that is, $[k,k]_\mathcal{K} \geq 0$ for all $k \in \mathcal{R}$.*

Proof. Let $k \in \mathcal{R}$ and let $\{h_\sigma\}_{\sigma \in \mathbb{F}_d^+} \subset \mathcal{H}$ be the family of vectors associated with k, as in Theorem 6.2. According to Lemma 6.1, $\mathcal{M}_+(\mathcal{L})$ is regular and the Fourier representation associated with the d-orghonal shift $\mathcal{V}|\mathcal{M}_+(\mathcal{L})$ is bounded. Let $\{l_\alpha\}_{\alpha \in \mathbb{F}_d^+}$ be the sequence of Fourier coefficients of $P_{\mathcal{M}_+(\mathcal{L})}k$. Using Proposition 3.6, we infer that

$$\begin{aligned}
[k,k]_\mathcal{K} &= \|h_0\|^2 + \sum_{n=0}^{\infty} \sum_{|\alpha|=n} [l_\alpha, l_\alpha]_\mathcal{K} \\
&= \|h_0\| + \sum_{n=0}^{\infty} \sum_{|\sigma|=n} \big[\sum_{j=1}^{d} (V_j - T_j) h_{\sigma g_j}, \sum_{j=1}^{d} (V_j - T_j) h_{\sigma g_j}\big]_\mathcal{K} \\
&= \|h_0\|^2 + \lim_{N \to \infty} \sum_{|\sigma| \leq N-1} \big\{\sum_{j=1}^{d} \|h_{\sigma g_j}\|^2 - \sum_{i,j=1}^{d} \langle T_j^* T_i h_{\sigma g_i}, h_{\sigma g_j}\rangle_\mathcal{H}\big\} \\
&= \|h_0\|^2 + \lim_{N \to \infty} \sum_{|\sigma| \leq N-1} \big\{\sum_{j=1}^{d} \|h_{\sigma g_j}\|^2 - \sum_{j=1}^{d} \langle T_j^* h_\sigma, h_{\sigma g_j}\rangle_\mathcal{H}\big\} \\
&= \|h_0\|^2 + \lim_{N \to \infty} \sum_{|\sigma| \leq N-1} \big\{\sum_{j=1}^{d} \|h_{\sigma g_j}\|^2 - \|h_\sigma\|^2\big\} \\
&= \|h_0\|^2 + \lim_{N \to \infty} \big(\sum_{|\sigma| \leq N-1} \sum_{j=1}^{d} \|h_{\sigma g_j}\|^2 - \sum_{|\sigma| \leq N-1} \|h_\sigma\|^2\big\} \\
&= \lim_{N \to \infty} \sum_{|\beta|=N} \|h_\beta\|^2.
\end{aligned}$$

This proves that $[k,k] \geq 0$. □

We say that a d-tuple $\mathcal{T} := (T_1, \ldots, T_d)$ of operators is *power bounded* if there exists $C > 0$ such that

$$\|\sum_{|\alpha|=n} T_\alpha T_\alpha^*\| \leq C, \qquad \text{for any } n = 0, 1, 2, \ldots . \tag{6.23}$$

Corollary 6.4. *If $\mathcal{T} := (T_1, \ldots, T_d)$ is power bounded, then the residual space $\mathcal{R}$ is positive, that is, $[k,k]_\mathcal{K} > 0$ for all $k \in \mathcal{R} \setminus \{0\}$, and $\mathcal{M}_+(\mathcal{L})$ is nondegenerate.*

Proof. Since $\mathcal{R}$ is nonnegative, according to the Schwarz inequality, in order to prove that $\mathcal{R}$ is positive, it is enough to prove that the only neutral element in $\mathcal{R}$ is the zero vector. Let $k \in \mathcal{R}$ be such that $[k,k]_{\mathcal{K}} = 0$. With the notation from the proof of Corollary 6.3, we have

$$\lim_{N\to\infty} \sum_{|\beta|=N} \|h_\beta\|^2 = 0.$$

On the other hand, since for all $\sigma \in \mathbb{F}_d^+$ we have $\sum_{j=1}^{d} T_j h_{\sigma g_j} = h_\sigma$, it follows that

$$\sum_{|\mu|=n} T_\mu h_{\sigma\mu} = h_\sigma, \quad \sigma \in \mathbb{F}_d^+,\ n = 0,1,2,\dots . \tag{6.24}$$

Fix $\sigma \in \mathbb{F}_d^+$ with $|\sigma| \le N$. Using (6.24) and (6.23), we get

$$\begin{aligned}\|h_\sigma\|^2 = \Big\| \sum_{|\mu|=N-|\sigma|} T_\mu h_{\sigma\mu}\Big\|^2 &\le C \sum_{|\mu|=N-|\sigma|} \|h_{\sigma\mu}\|^2 \\ &\le C \sum_{|\alpha|=N} \|h_\alpha\|^2.\end{aligned}$$

Taking the limit as $N \to \infty$, we obtain $\|h_\sigma\| = 0$. Since σ was arbitrary in $\mathbb{F}_d^+$, Theorem 6.2 shows that $k = 0$. Therefore, the subspace $\mathcal{R}$ is positive. Notice that $\mathcal{M}_+(\mathcal{L}_*) = \mathcal{R}^\perp$ is nondegenerate since $\mathcal{R}$ has the same property. The proof is complete. $\square$

Corollary 6.5. *If*

$$\lim_{n\to\infty} \sum_{|\alpha|=n} \|T_\alpha^* h\|^2 = 0, \quad h \in \mathcal{H}, \tag{6.25}$$

then $\mathcal{M}_+(\mathcal{L}_*) = \mathcal{K}$.

Proof. Let $k \in \mathcal{R}$ and $\{h_\sigma\}_{\sigma\in\mathbb{F}_d^+} \subset \mathcal{H}$ be as in Theorem 6.2. For arbitrary $h \in \mathcal{H}$ and $\sigma \in \mathbb{F}_d^+$ with $|\sigma| \le N$, we have

$$\begin{aligned}|\langle h_\sigma, h\rangle_{\mathcal{H}}| &= \Big|\langle \sum_{|\mu|=N-|\sigma|} T_\mu h_{\sigma\mu}, h\rangle_{\mathcal{H}}\Big| = \Big|\langle \sum_{|\mu|=N-|\sigma|} h_{\sigma\mu}, T_\mu^* h\rangle_{\mathcal{H}}\Big| \\ &\le \sum_{|\mu|=N-|\sigma|} \|h_{\sigma\mu}\| \cdot \|T_\mu^* h\| \le \Big(\sum_{|\mu|=N-|\sigma|} \|h_{\sigma\mu}\|^2\Big)^{1/2} \Big(\sum_{|\mu|=N-|\sigma|} \|T_\mu^* h\|^2\Big)^{1/2} \\ &\le \Big(\sum_{|\beta|=N} \|h_\beta\|^2\Big)^{1/2} \Big(\sum_{|\mu|=N-|\sigma|} \|T_\mu^* h\|^2\Big)^{1/2}.\end{aligned}$$

Since $\lim\limits_{N\to\infty} \sum\limits_{|\beta|=N} \|h_\beta\|^2$ exists and $\lim\limits_{N\to\infty} \sum\limits_{|\mu|=N-|\sigma|} \|T_\mu^* h\|^2 = 0$, we have $\langle h_\sigma, h\rangle = 0$. Thus, $h_\sigma = 0$ for all $\sigma \in \mathbb{F}_d^+$ and, by Theorem 6.2, we conclude that $k = 0$. This shows that $\mathcal{R} = \{0\}$ and hence $\mathcal{M}_+(\mathcal{L}_*) = \mathcal{K}$. $\square$

Remark 6.6. The converse of Corollary 6.5 is not true, in general. However, as will be shown in the next section, this becomes true if we impose a boundedness condition on the "characteristic function" of $\mathcal{T}$, cf. Corollary 7.6.

Corollary 6.7. *If* $h \in \mathcal{H}$ *satisfies* $\lim\limits_{n\to\infty} \sum\limits_{|\alpha|=n} \|T_\alpha^* h\|^2 = 0$, *then* $h \in \mathcal{M}_+(\mathcal{L}_*)$.

Proof. Let $k \in \mathcal{K}$. As in Corollary 6.5, one can show that $[k,h] = \langle h_0, h\rangle = 0$, where $h_0 = P_{\mathcal{M}_+(\mathcal{L}_*)}k$. Therefore $h \perp \mathcal{R}$ and hence $h \in \mathcal{M}_+(\mathcal{L}_*)$. □

7. The Characteristic Function

We need to recall from [19], [20], and [21] a few facts concerning multi-analytic operators on Fock spaces.

The noncommutative analytic Toeplitz algebra F_d^∞ was introduced in [19] as the algebra of left multipliers of the full Fock space $\mathcal{F}_d^2$. This algebra can be identified with the weakly closed algebra generated by the left creation operators $S_1, \dots, S_n$ on the full Fock space $\mathcal{F}_d^2$, and the identity.

Let $\mathcal{K}, \mathcal{K}'$ be Hilbert spaces. As in [18], we say that a bounded linear operator $M \in B(\mathcal{F}_d^2 \otimes \mathcal{K}, \mathcal{F}_d^2 \otimes \mathcal{K}')$ is *multi-analytic* if $M(S_i \otimes I_\mathcal{K}) = (S_i \otimes I_{\mathcal{K}'})M$ for any $i = 1, \dots, d$. Notice that M is uniquely determined by the operator $\theta : \mathcal{K} \to \mathcal{F}_d^2 \otimes \mathcal{K}'$, $\theta k := M(e_0 \otimes k)$, $k \in \mathcal{K}$, which is called the *symbol* of M, and we denote $M = M_\theta$. Moreover, M_θ is uniquely determined by the "coefficients" of θ, i.e., the operators $\theta_\alpha \in B(\mathcal{K}, \mathcal{K}')$ given by

$$\langle \theta_\alpha k, k'\rangle := \langle \theta k, e_\alpha \otimes k'\rangle = \langle M_\theta(e_0 \otimes k), e_\alpha \otimes k'\rangle, \quad k \in \mathcal{K},\ k' \in \mathcal{K}',\ \alpha \in \mathbb{F}_d^+. \tag{7.1}$$

Notice that $\sum_{\alpha \in \mathbb{F}_d^+} \theta_\alpha^* \theta_\alpha \leq \|M_\theta\| I_\mathcal{K}$. We can associate with M_θ a unique formal Fourier expansion

$$M_\theta \sim \sum_{\alpha \in \mathbb{F}_d^+} U^* S_\alpha U \otimes \theta_\alpha, \tag{7.2}$$

where U is the (flipping) unitary operator on $\mathcal{F}_d^2$ mapping $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ into $e_{i_k} \otimes \cdots \otimes e_{i_2} \otimes e_{i_1}$. Since M_θ acts like its Fourier representation on "polynomials", we will identify them for simplicity. As in [20], using the noncommutative von Neumann inequality [19], one can show that if $0 < r < 1$, then

$$M_\theta = \mathrm{SOT} - \lim_{r \nearrow 1} \sum_{\alpha \in \mathbb{F}_d^+} r^{|\alpha|} U^* S_\alpha U \otimes \theta_\alpha,$$

where the series converges in the uniform norm and the limit is taken in the strong-operator topology (SOT). According to [21], when $\mathcal{K} = \mathcal{K}'$, the algebra of all multi-analytic operators acting on $\mathcal{F}_d^2 \otimes \mathcal{K}$ can be identified with $F_d^\infty \bar{\otimes} B(\mathcal{K})$, the weakly closed algebra generated by the spatial tensor product of the two algebras. A similar result holds in our more general setting. The set of multi-analytic operators in $B(\mathcal{F}_d^2 \otimes \mathcal{K}, \mathcal{F}_d^2 \otimes \mathcal{K}')$ can be identified with $F_d^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$, which is equal to the weakly closed operator space generated by $S_\alpha \otimes Z$, $\alpha \in \mathbb{F}_d^+$, $Z \in B(\mathcal{K}, \mathcal{K}')$.

A strong connection between the algebra F_d^∞ and the function theory on the open unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$ was established through the noncommutative von Neumann inequality [19] (see also [20]). Consequently, if M_θ is a multi-analytic operator with Fourier expansion given by (7.2), then

$$f(\lambda_1, \dots, \lambda_d) := \sum_{\alpha \in \mathbb{F}_d^+} \lambda_\alpha \theta_\alpha$$

is a $B(\mathcal{K}, \mathcal{K}')$-valued bounded analytic function in the unit ball $\mathbb{B}_d$.

Throughout this section we assume that $\mathcal{H}$ is a *Hilbert space* and that the sequence $\mathcal{T} := (T_1, \ldots, T_d)$ consists of operators in $\mathcal{B}(\mathcal{H})$. When $d = \infty$, we assume that the series $\sum_{i=1}^{d} T_i T_i^*$ is strongly convergent. We keep the notation of the previous section. For each $\alpha \in \mathbb{F}_d^+$, let $\theta_\alpha \in B(\mathcal{D}_T, \mathcal{D}_{T^*})$ be given by

$$\theta_\alpha := \begin{cases} -\sum_{i=1}^{d} T_i P_i J_T, & \text{if } \alpha = g_0 \\ J_{T^*} D_{T^*} T_\sigma^* P_i D_T J_T, & \text{if } \alpha = g_i \sigma, \end{cases} \tag{7.3}$$

where P_i stands for the orthogonal projection of $\mathcal{H}^d = \bigoplus_{n=1}^{d} \mathcal{H}$ onto its j-component, and $\mathcal{S} := (S_1, \ldots, S_d)$ is the canonical d-orthogonal shift. We introduce the *characteristic function* of $\mathcal{T}$ to be the family of multi-analytic operators $(M_{\theta_\mathcal{T}})_r : \mathcal{F}_d^2 \otimes \mathcal{D}_T \to \mathcal{F}_d^2 \otimes \mathcal{D}_{T^*}$ defined by

$$(M_{\theta_\mathcal{T}})_r := \sum_{n=0}^{\infty} \sum_{|\alpha|=n} r^{|\alpha|} U^* S_\alpha U \otimes \theta_\alpha \tag{7.4}$$

for those $r \in (0,1)$ for which the series is uniform convergent (e.g., if $r < \| \sum_{i=1}^{d} T_i T_i^* \|^{-1/2}$). We say that $\mathcal{T}$ has *bounded characteristic function* if

$$\sup_{0<r<1} \|(M_{\theta_\mathcal{T}})_r\| < \infty. \tag{7.5}$$

In this case, the bounded characteristic function associated with $\mathcal{T}$ is the multi-analytic operator $M_{\theta_\mathcal{T}} : \mathcal{F}_d^2 \otimes \mathcal{D}_T \to \mathcal{F}_d^2 \otimes \mathcal{D}_{T^*}$ defined by

$$M_{\theta_\mathcal{T}} := \mathrm{SOT} - \lim_{r \nearrow 1} (M_{\theta_\mathcal{T}})_r.$$

The existence of this limit can be proved as in [20]. Moreover, the characteristic function has the Fourier expansion

$$M_{\theta_\mathcal{T}} \sim -I_{\mathcal{F}_d^2} \otimes (\sum_{i=1}^{d} T_i P_i J_T) + \sum_{n=1}^{\infty} \sum_{i=1}^{d} \sum_{|\alpha|=n} S_{g_i\sigma} \otimes J_{T^*} D_{T^*} T_\sigma^* P_i D_T J_T.$$

In what follows, to simplify our notation, we set $\Theta_\mathcal{T} := M_{\theta_\mathcal{T}}$. We recall from [16] the definition of the *spectral radius* of a sequence of operators, i.e.,

$$\rho(\mathcal{T}) := \lim_{m \to \infty} \| \sum_{|\alpha|=m} T_\alpha T_\alpha^* \|^{1/m}.$$

Notice that, if $s\mathcal{T} := (sT_1, \ldots, sT_d)$, $(s > 0)$, then $\rho(s\mathcal{T}) = s\rho(\mathcal{T})$. The following result shows that the class of d-tuples of operators with bounded characteristic function is pretty large.

Proposition 7.1. *Let $\mathcal{T} := \{T_1, \ldots, T_d\}$ be a d-tuple of operators on a Hilbert space. If $\rho(\mathcal{T}) < 1$, then $\mathcal{T}$ has bounded characteristic function.*

Proof. Let $s > 1$ such that $\rho(s\mathcal{T}) < 1$ and denote $\mathcal{A} := \{A_1, \dots, A_d\}$, where $A_i = sT_i, i = 1, \dots, d$. According to [16], $\rho(\mathcal{A}) < 1$ if and only if $\sum_{n=1}^{\infty} \sum_{|\alpha|=n} A_\alpha A_\alpha^*$ is strongly convergent. Now, we prove that if $0 < r < 1$, then the series

$$\sum_{n=1}^{\infty} \sum_{i=1}^{d} \sum_{|\alpha|=n} r^{n+1} S_{g_i\alpha} \otimes A_\alpha P_i \tag{7.6}$$

is convergent in the uniform norm. Indeed, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} r^{n+1} \|\sum_{i=1}^{d} \sum_{|\alpha|=n} S_{g_i\alpha} \otimes A_\alpha P_i\| &\leq \sum_{n=1}^{\infty} r^{n+1} \|\sum_{i=1}^{d} \sum_{|\alpha|=n} P_i A_\alpha A_\alpha^* P_i\|^{1/2} \\
&= \sum_{n=1}^{\infty} r^{n+1} \| \sum_{|\alpha|=n} \sum_{i=1}^{d} P_i A_\alpha A_\alpha^* P_i\|^{1/2} \\
&\leq \sum_{n=1}^{\infty} r^{n+1} \| \sum_{|\alpha|=n} A_\alpha A_\alpha^*\|^{1/2} \\
&\leq \left(\sum_{n=1}^{\infty} r^{n+1}\right) \| \sum_{\alpha \in \mathbb{F}_d^+} A_\alpha A_\alpha^*\|^{1/2}.
\end{aligned}$$

In particular, (7.6) converges in the uniform norm if $r = \frac{1}{s}$. Therefore, the series

$$\sum_{n=1}^{\infty} \sum_{i=1}^{d} \sum_{|\alpha|=n} S_{g_i\alpha} \otimes T_\alpha P_i \quad \text{and} \quad \sum_{n=1}^{\infty} \sum_{i=1}^{d} \sum_{|\alpha|=n} S_{g_i\alpha} \otimes J_{T^*} D_{T^*} T_\alpha P_i D_T J_T$$

are also convergent in the uniform norm. This shows that the characteristic function $\Theta_{\mathcal{T}}$ is a bounded multi-analytic operator. □

Let $(\mathcal{V}, \mathcal{K})$ be the canonical minimal isometric dilation of $\mathcal{T}$ as defined in (5.4) and (5.6), and let $\mathcal{L}$ and $\mathcal{L}_*$ be the wandering subspaces defined by (6.5). Define the operator $M\colon \mathcal{M}_+(\mathcal{L}) \to \mathcal{F}_d^2 \otimes \mathcal{D}_T$ by

$$M := (I_{\mathcal{F}_d^2} \otimes \phi) \circ \Phi_{\mathcal{L}}, \tag{7.7}$$

where $\phi\colon \mathcal{L} \to \mathcal{D}_T$ is the unitary operator defined by (6.7) and $\Phi_{\mathcal{L}}\colon \mathcal{M}_+(\mathcal{L}) \to \mathcal{F}_d^2 \otimes \mathcal{L}$ is the Fourier representation associated with $\mathcal{V}$ and the wandering space $\mathcal{L}$. Recall that, by Lemma 6.1, the Fourier representation $\Phi_{\mathcal{L}}$ is a bounded unitary operator. Therefore, M has the same property. Note that the operator M can be described explicitly by

$$M(V_\alpha l_\alpha) := e_\alpha \otimes \phi l_\alpha, \quad l_\alpha \in \mathcal{L},\ \alpha \in \mathbb{F}_d^+, \tag{7.8}$$

and can be extended by linearity and continuity to the whole space.

Les us recall from the proof of Lemma 6.1 that the linear operator ϕ_* (see (6.9)), is isometric, densely defined in the regular subspace $\mathcal{L}_*$, and has dense range in the Kreĭn space $\mathcal{D}_{T^*}$. Now, consider the operator M_* defined in $\mathcal{M}_+(\mathcal{L}_*)$ and valued in $\mathcal{F}_d^2 \otimes \mathcal{D}_{T^*}$ by

$$M_*\big(\sum_{|\alpha| \leq N} V_\alpha (I - \sum_{j=1}^{d} V_j T_j) h_\alpha\big) := \sum_{|\alpha| \leq N} \big(e_\alpha \otimes \phi_* (I - \sum_{j=1}^{d} V_j T_j) h_\alpha\big), \tag{7.9}$$

where $N \in \mathbb{N}$, $h_\alpha \in \mathcal{H}$. Taking into account the properties of ϕ_*, we note that M_* is densely defined, isometric, and has dense range. In particular, it is injective and its inverse M_*^{-1} is also densely defined, isometric, and has dense range.

Theorem 7.2. *If $\mathcal{T} := (T_1, \ldots, T_d)$ has bounded characteristic function, then $\mathcal{M}_+(\mathcal{L}_*)$ is a regular subspace of $\mathcal{K}$, the operator defined by (7.9) extends uniquely to a bounded operator $M_*\colon \mathcal{M}_+(\mathcal{L}_*) \to \mathcal{F}_d^2 \otimes \mathcal{D}_{T^*}$, and*

$$\Theta_{\mathcal{T}} = M_*\big(P_{\mathcal{M}_+(\mathcal{L}_*)}|\mathcal{M}_+(\mathcal{L})\big)M^{-1}, \tag{7.10}$$

where $P_{\mathcal{M}_+(\mathcal{L}_*)}$ denotes the selfadjoint projection of the Kreĭn space $\mathcal{K}$ onto the regular subspace $\mathcal{M}_+(\mathcal{L}_*)$.

Proof. We will first prove that, for all vectors

$$u := \sum_{|\beta|\leq N} V_\beta(I - \sum_{j=1}^d V_jT_j)h_\beta, \quad v := \sum_{|\alpha|\leq N} V_\alpha l_\alpha, \tag{7.11}$$

where $N \geq 0$, $(l_\alpha)_{|\alpha|\leq N} \subset \mathcal{L}$ and $(h_\beta)_{|\beta|\leq N} \subset \mathcal{H}$, we have

$$[\Theta_{\mathcal{T}} M v, M_* u]_{\mathcal{F}_d^2\otimes\mathcal{D}_{T^*}} = [v, u]_{\mathcal{K}}. \tag{7.12}$$

Since $\Theta_{\mathcal{T}}$ is a multi-analytic operator, we have

$$\Theta_{\mathcal{T}}(S_j \otimes I_{\mathcal{D}_T}) = (S_j \otimes I_{\mathcal{D}_{T^*}})\Theta_{\mathcal{T}}, \quad j = 1, \ldots, d.$$

From the definition of the operator M, we also have

$$MV_j|\mathcal{M}_+(\mathcal{L}) = (S_j \otimes I_{\mathcal{D}_T})M, \quad j = 1, \ldots, d.$$

On the other hand, using the same reasoning as in Lemma 4.1, we infer that

$$M_*V_j|\mathrm{Dom}\,(M_*) = (S_j \otimes I_{\mathcal{D}_{T^*}})M_*, \quad j = 1, \ldots, d.$$

Now, one can easily see that

$$[\Theta_{\mathcal{T}} M V_\alpha l, M_* V_\beta l_*]_{\mathcal{F}_d^2\otimes\mathcal{D}_{T^*}} = [(S_\alpha \otimes I_{\mathcal{D}_{T^*}})\theta_{\mathcal{T}}\phi l, (S_\beta \otimes I_{\mathcal{D}_{T^*}})(e_0 \otimes \phi_* l_*)]_{\mathcal{F}_d^2\otimes\mathcal{D}_{T^*}},$$

where $l_* := (I - \sum\limits_{j=1}^d V_jT_j)h$ and $h \in \mathcal{H}$. This shows that, if $\alpha \not\leq \beta$, then

$$[\Theta_{\mathcal{T}} M V_\alpha l, M_* V_\beta l_*]_{\mathcal{F}_d^2\otimes\mathcal{D}_{T^*}} = 0 = [V_\alpha l, V_\beta l_*]_{\mathcal{K}}.$$

Thus, in order to prove (7.12), it is sufficient to show that

$$[(S_\gamma^* \otimes I_{\mathcal{D}_{T^*}})\theta_{\mathcal{T}}\phi l, e_0 \otimes \phi_* l_*]_{\mathcal{F}_d^2\otimes\mathcal{D}_{T^*}} = [V_\gamma^\sharp l, l_*]_{\mathcal{K}}, \tag{7.13}$$

for all $\gamma \in \mathbb{F}_d^+$, $l \in \mathcal{L}$, and $l_* := (I - \sum\limits_{j=1}^d V_jT_j)h$, $h \in \mathcal{H}$. Furthermore, (7.13) is equivalent to

$$[P_{\mathcal{D}_{T^*}}(S_\gamma^* \otimes I_{\mathcal{D}_{T^*}})\theta_{\mathcal{T}}\phi l, \phi_* l_*]_{\mathcal{D}_{T^*}} = [P_{\mathcal{L}_*}V_\gamma^\sharp l, l_*]_{\mathcal{K}}. \tag{7.14}$$

Performing essentially the same calculation as in the proof of Theorem 3.1 from [18], one can obtain

$$P_{\mathcal{L}_*}V_\alpha^\sharp V_i^\sharp l = (I_{\mathcal{K}} - \sum_{j=1}^d V_jT_j^*)T_\alpha^* P_i J_T D_T^2(\bigoplus_{j=1}^d h_j), \tag{7.15}$$

and

$$P_{\mathcal{L}_*}l = -(I_{\mathcal{K}} - \sum_{j=1}^{d} V_j T_j^*)(\sum_{j=1}^{d} T_j h_j). \tag{7.16}$$

On the other hand, using the definition of the characteristic function, we have

$$P_{\mathcal{D}_{T^*}} \theta_{\mathcal{T}} \phi l = -\sum_{j=1}^{d} T_j P_j J_T \phi l, \tag{7.17}$$

and

$$P_{\mathcal{D}_{T^*}}(S_\alpha^* S_i^* \otimes I_{\mathcal{D}_{T^*}})\theta_{\mathcal{T}} \phi l = J_{T^*} D_{T^*} T_\alpha^* P_i D_T J_T \phi l. \tag{7.18}$$

Now, from (7.16), (7.17), and (6.4), we obtain relation (7.14) when $\gamma = g_0$. Then, using (7.15) and (7.18), we obtain (7.14) for all $\gamma \in \mathbb{F}_d^+$, $|\gamma| \geq 1$. Thus, (7.12) is proved.

Since M and $\Theta_{\mathcal{T}}$ are bounded, it follows that (7.12) holds for all $v \in \mathcal{M}_+(\mathcal{L})$ and u as in (7.11), that is,

$$[\Theta_{\mathcal{T}} M v, z]_{\mathcal{F}_d^2 \otimes \mathcal{D}_{T^*}} = [v, M_*^{-1} z]_{\mathcal{K}}, \quad v \in \mathcal{M}_+(\mathcal{L}),\ z \in \operatorname{Ran}(M_*), \tag{7.19}$$

that is, the densely defined operator M_*^{-1} has a bounded adjoint. Therefore, M_*^{-1} has a bounded extension and hence, it extends uniquely to a bounded operator $X\colon \mathcal{F}_d^2 \otimes \mathcal{D}_{T^*} \to \mathcal{K}$. The operator X is isometric and it necessarily has the range $\mathcal{M}_+(\mathcal{L}_*)$. Hence, $\mathcal{M}_+(\mathcal{L}_*)$ is a regular subspace of $\mathcal{K}$. In addition, $X^\sharp | \mathcal{M}_+(\mathcal{L}_*) \supset M_*$ and hence M_* has a unique extension to a bounded operator $M_*\colon \mathcal{M}_+(\mathcal{L}_*) \to \mathcal{F}_d^2 \otimes \mathcal{D}_{T^*}$. Now, relation (7.19) shows that (7.10) holds true. The proof is complete. ☐

Corollary 7.3. *If $\mathcal{T}$ has bounded characteristic function, then the Fourier representation $\Phi_{\mathcal{L}_*}$ of $\mathcal{V}$, with respect to the wandering space $\mathcal{L}_*$, is a bounded unitary operator from $\mathcal{M}_+(\mathcal{L}_*)$ to $\mathcal{F}_d^2 \otimes \mathcal{L}_*$, and the operator ϕ_*, defined by (6.9), is continuous and extends uniquely to a unitary operator of Kreĭn spaces from $\mathcal{L}_*$ to $\mathcal{D}_{T^*}$.*

Proof. Indeed, relation (7.9) implies

$$M_*(I - \sum_{j=1}^{d} V_j T_j) h = e_0 \otimes \phi_*(I - \sum_{j=1}^{d} V_j T_j) h, \quad h \in \mathcal{H}.$$

By Theorem 7.2, M_* is continuous, hence ϕ_* has the same property. Since ϕ_* is isometric, densely defined in the regular space $\mathcal{L}_*$ and with values in the Kreĭn space $\mathcal{D}_{T^*}$, it extends uniquely to a unitary operator $\phi_* \in \mathcal{B}(\mathcal{L}_*, \mathcal{D}_{T^*})$. Therefore, relation (7.9) implies

$$M_*(\sum_{|\alpha| \leq N} V_\alpha l_\alpha) = \sum_{|\alpha| \leq N} (e_\alpha \otimes \phi_* l_\alpha), \quad N \in \mathbb{N},\ l_\alpha \in \mathcal{L}_*. \tag{7.20}$$

Hence, we obtain $M_* = \Phi_{\mathcal{L}_*} \circ (I_{\mathcal{F}_d^2} \otimes \phi_*)$ and therefore, $\Phi_{\mathcal{L}_*} = M_* \circ (I_{\mathcal{F}_d^2} \otimes \phi_*^{-1})$ is a bounded operator. ☐

We can now prove one of the main results of this paper.

Theorem 7.4. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{T} := (T_1, \ldots, T_d)$ be a d-tuple of bounded operators on $\mathcal{H}$. If $\mathcal{T}$ has bounded characteristic function, then it is jointly similar to a row contraction, that is, there exists a boundedly invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that the operator defined by the row matrix $[ST_1S^{-1} \ \ldots \ ST_dS^{-1}]$ is a contraction.*

We have two proofs for this theorem.

First Proof. We actually prove that there exists a unitary norm on $\mathcal{K}$ with respect to which $T = [T_1 \ \dots \ T_d]$ is a contraction.

By Theorem 7.2, the subspace $M_+(\mathcal{L}_*)$ is regular, therefore $\mathcal{R} = M_+(\mathcal{L}_*)^\perp$ has the same property. In particular, the Wold-von Neumann decomposition holds (see Theorem 3.5). Since $\mathcal{R}$ is a nonnegative subspace (see Corollary 6.3), we infer that it is uniformly positive, that is, $[r,r]_\mathcal{K} \geq C\|r\|^2$ for some $C > 0$ and all $r \in \mathcal{R}$. In addition, by Corollary 7.3, the Fourier representation $\Phi_{\mathcal{L}_*}\colon \mathcal{M}_+(\mathcal{L}_*) \to \mathcal{F}_d^2 \otimes \mathcal{L}_*$ is a bounded unitary operator of Kreĭn spaces. Therefore, the norm defined by

$$\|u + r\| := \sqrt{\|\Phi_{\mathcal{L}_*} u\|^2_{\mathcal{F}_d^2 \otimes \mathcal{L}_*} + [r,r]_\mathcal{K}}, \quad u \in \mathcal{M}_+(\mathcal{L}_*),\ r \in \mathcal{R}, \tag{7.21}$$

is a unitary norm on $\mathcal{K}$. Note that, with respect to this new norm, $\Phi_{\mathcal{L}_*}$ becomes unitary also with respect to the Hilbert space underlying the scalar products of $\mathcal{M}_+(\mathcal{L}_*)$ and $\mathcal{F}_d^2 \otimes \mathcal{L}_*$, respectively.

We claim that, with respect to the unitary norm $\|\cdot\|$ defined by (7.21), we have

$$\sum_{j=1}^d \|V_j^\sharp k\|^2 \leq \|k\|^2, \quad k \in \mathcal{K}. \tag{7.22}$$

Indeed, any $k \in \mathcal{K}$ has a decomposition $k = u + r$ with $u \in \mathcal{M}_+(\mathcal{L}_*)$ and $r \in \mathcal{R}$. Then, taking into account that $I - \sum_{j=1}^d V_j V_j^\sharp$ is the selfadjoint projection onto $\mathcal{L}_*$ and that $\mathcal{R} \perp \mathcal{L}_*$, we have

$$\sum_{j=1}^d \|V_j^\sharp r\|^2 = \sum_{j=1}^d [V_j^\sharp r, V_j^\sharp r]_\mathcal{K} = [\sum_{j=1}^d V_j V_j^\sharp r, r]_\mathcal{K} = [r,r]_\mathcal{K} = \|r\|^2. \tag{7.23}$$

On the other hand, since $\Phi_{\mathcal{L}_*}$ is bounded, relation (4.11) becomes

$$\Phi_{\mathcal{L}_*} \circ V_j^\sharp = (S_j^* \otimes I_{\mathcal{L}_*}) \circ \Phi_{\mathcal{L}_*}, \quad j = 1, \dots, d. \tag{7.24}$$

In addition, for any $x \in \mathcal{F}_d^2 \otimes \mathcal{L}_*$, we have

$$\sum_{j=1}^d \|(S_j^* \otimes I_{\mathcal{L}_*})x\| = \|(I - P_{\mathbb{C}e_0}^{\mathcal{F}_d^2 \otimes \mathcal{L}_*} \otimes I_{\mathcal{L}_*})x\|^2 \leq \|x\|^2.$$

Hence, taking into account (7.24), we have

$$\begin{aligned}
\sum_{j=1}^d \|V_j^\sharp u\|^2 &= \sum_{j=1}^d \|\Phi_{\mathcal{L}_*} V_j^\sharp u\|^2 \\
&= \sum_{j=1}^d \|(S_j^* \otimes I_{\mathcal{L}_*}) \Phi_{\mathcal{L}_*} u\|^2 \\
&\leq \|\Phi_{\mathcal{L}^*} u\|^2 = \|u\|^2.
\end{aligned}$$

Now, using the definition of the unitary norm in (7.21), and that $V_j^\sharp$ leaves invariant both $\mathcal{R}$ and $\mathcal{M}_+(\mathcal{L}_*)$, we obtain

$$\sum_{j=1}^d \|V_j^\sharp k\|^2 = \sum_{j=1}^d \|V_j^\sharp u + V_j^\sharp r\|^2 = \sum_{j=1}^d \|V_j^\sharp u\|^2 + \sum_{j=1}^d \|V_j^\sharp r\|^2.$$

Hence, using (7.23), we get (7.22). Since $\mathcal{V}$ is an isometric dilation of $\mathcal{T}$, this yields

$$\sum_{j=1}^d \|T_j^* h\|^2 \leq \|h\|^2, \quad h \in \mathcal{H}, \tag{7.25}$$

that is, the operator $[T_1 \ \dots \ T_d]$ is a contraction with respect to the unitary norm $\|\cdot\|$ restricted to $\mathcal{H}$. $\square$

Second Proof. By Theorem 7.2, the operator M_* is bounded and the subspace $\mathcal{M}_+(\mathcal{L}_*)$ is regular in $\mathcal{K}$. Therefore, we can define the bounded operator

$$F := M_* P_{\mathcal{M}_+(\mathcal{L}_*)} \colon \mathcal{K} \to \mathcal{F}_d^2 \otimes \mathcal{D}_{T^*}, \tag{7.26}$$

where $P_{\mathcal{M}_+(\mathcal{L}_*)}$ is the selfadjoint projection of the Kreĭn space $\mathcal{K}$ onto its regular subspace $\mathcal{M}_+(\mathcal{L}_*)$. Then

$$Fk = \sum_{\alpha \in \mathbb{F}_d^+} e_\alpha \otimes \phi_* P_{\mathcal{L}_*} V_\alpha^\sharp k, \quad k \in \mathcal{K}. \tag{7.27}$$

Indeed, if $k \in \mathcal{M}_+(\mathcal{L}_*)$, then relation (7.27) coincides with (7.20). For arbitray $k \in \mathcal{K}$, we decompose $k = r + P_{\mathcal{M}_+(\mathcal{L}_*)}k$, $r \in \mathcal{R}$, and take into account that the residual space $\mathcal{R}$ is invariant under $V_j^\sharp$, $j = 1, \dots, d$. Since $V_\alpha^\sharp r \in \mathcal{R}$ for all $\alpha \in \mathbb{F}_d^+$, and $\mathcal{R} \perp \mathcal{M}_+(\mathcal{L}_*)$, we have $\mathcal{R} \perp \mathcal{L}_*$ and $P_{\mathcal{L}_*} V_\alpha^\sharp r = 0$ for all $\alpha \in \mathbb{F}_d^+$. Thus (7.27) is proved.

We claim now that

$$Fh = \sum_{\alpha \in \mathbb{F}_d^+} e_\alpha \otimes D_{T^*} T_\alpha^* h, \quad h \in \mathcal{H}. \tag{7.28}$$

To see this, recall that $P_{\mathcal{L}_*} = I - \sum_{j=1}^d V_j V_j^\sharp$ and hence, for any $\alpha \in \mathbb{F}_d^+$ and $h \in \mathcal{H}$, we have

$$P_{\mathcal{L}_*} V_\alpha^\sharp h = \big(I - \sum_{j=1}^d V_j V_j^\sharp\big) V_\alpha^\sharp h = \big(I - \sum_{j=1}^d V_j T_j^*\big) T_\alpha^* h.$$

Thus, by (6.9) we have

$$\phi_* P_{\mathcal{L}_*} V_\alpha^\sharp h = D_{T^*} T_\alpha^* h,$$

and (7.28) follows.

Now, using (7.27), (7.28), and taking into account that the operator F is bounded, we deduce that there exists $C > 0$ such that

$$\sum_{\alpha \in \mathbb{F}_d^+} \|D_{T^*} T_\alpha^*\|^2 \leq C \|h\|^2, \quad h \in \mathcal{H}. \tag{7.29}$$

Using Corollary 2.8 and Proposition 2.6 from [16], we conclude that $\mathcal{T}$ is jointly similar to a row contraction. $\square$

Let us recall, from [22], that if $\mathcal{T}$ is jointly similar to a row contraction then $\rho(\mathcal{T}) \leq 1$. On the other hand, Proposition 7.1 and Theorem 7.4 imply the following Rota type similarity result obtained in [16]: if $\rho(\mathcal{T}) < 1$, then $\mathcal{T}$ is jointly similar to a row contraction. We also remark that if $\lim_{n\to\infty} \sum_{|\alpha|=n} T_\alpha T_\alpha^* = 0$, then, using [20], we can extend the completely bounded representation $\pi_{\mathcal{T}}$ (see the introduction) from $\mathcal{A}_d$ to the noncommutative analytic Toeplitz algebra F_d^∞.

In case the characteristic function of $\mathcal{T}$ is bounded, more results on the geometry of the canonical minimal isometric dilation can be obtained.

Theorem 7.5. *If $\mathcal{T}$ has bounded characteristic function, then, for any $h \in \mathcal{H}$, we have*

$$P_{\mathcal{R}} h = \lim_{n\to\infty} \sum_{|\alpha|=n} V_\alpha T_\alpha^* h, \tag{7.30}$$

and

$$[P_{\mathcal{R}} h, h]_{\mathcal{K}} = \lim_{n\to\infty} \sum_{|\alpha|=n} \|T_\alpha^* h\|^2, \quad P_{\mathcal{H}} P_{\mathcal{R}} h = \lim_{n\to\infty} \sum_{|\alpha|=n} T_\alpha T_\alpha^* h. \tag{7.31}$$

Proof. Indeed, $\sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp$ is the selfadjoint projection onto the subspace $\bigoplus_{|\alpha|=n} V_\alpha \mathcal{M}_+(\mathcal{L}_*)$ and hence,

$$0 = P_{\mathcal{R}}(h - \sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp h) = P_{\mathcal{R}}(h - \sum_{|\alpha|=n} V_\alpha T_\alpha^* h), \quad h \in \mathcal{H}.$$

Since the sequence $(\sum_{|\alpha|=n} V_\alpha V_\alpha^\sharp)_{n\geq 0}$ converges strongly, we obtain (7.30) by passing to the limit as $n \to \infty$. The relations from (7.31) follow from here. □

Corollary 7.6. *If the characteristic function of $\mathcal{T}$ is bounded, then*

(a) *A vector $h \in \mathcal{H}$ is in $\mathcal{M}_+(\mathcal{L}_*)$ if and only if* $\lim_{n\to\infty} \sum_{|\alpha|=n} T_\alpha T_\alpha^* h = 0$;

(b) $\mathcal{M}_+(\mathcal{L}_*) = \mathcal{K}$ *if and only if* $\lim_{n\to\infty} \sum_{|\alpha|=n} T_\alpha T_\alpha^* = 0$.

Proof. (a) This is a direct consequence of the second relation in (7.31).

(b) If $\mathcal{M}_+(\mathcal{L}_*) = \mathcal{K}$, then $P_{\mathcal{R}} = 0$. Again by (7.31), we get $0 = P_{\mathcal{H}} P_{\mathcal{R}} h = \lim_{n\to\infty} \sum_{|\alpha|=n} T_\alpha T_\alpha^* h$ for all $h \in \mathcal{H}$. The converse was proved in Corollary 6.5. □

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